Time covariant continua

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Plan

In her famous paper entitled *Invariant Variational Problems* [1], E. Noether concludes (as translated in English in [2 p. 64]) :
"the term *relativity* as it is used in physics should be replaced by *invariance with respect to a group*".

We wish to explore the possibilities offered by requiring such an invariance when applied to a physical media.

[1] Noether, E. : Invariante Variationsprobleme. Nachrichten von der Königlichen Gesellschaft der Wissenshcafter zu Göttingen, Mathematisch-physikalische Klasse, pp. 235-257, (1918). [2] Kosmann-Schwarzbach Y., The Noether Theorems, Springer (2018).

Plan

A Riemannian manifold

- $\mathcal{M}$: four-dimensional pseudo-Riemannian manifold endowed with a Lorentzian metric $\boldsymbol{g}$ of signature $(+,-,-,-)$
- Event P : A point in $\mathcal{M}$ identified by a set of four coordinates noted x with $(x^{\mu})_{\mu\in\{0,1,2,3\}} \in \mathbb{R}^4$
- Tangent space and its dual
- Tensor fields
- Derivative operators
  - Lie derivative along a vector field X noted $\underset{X}{\mathscr{L}}$
  - Covariant derivative with a symmetric and metric connection noted ∇

Energy is distributed in the universe

Given N function fields $\phi_{\mathcal{I}}$, define the energy $\mathscr{E}_{\Omega}$ of a compact four-volume $\Omega \subset \mathcal{M}$:

$$\mathscr{E}_{\Omega}(A(\phi_{\mathcal{I}})) = \int_{\Omega} A(\phi_{\mathcal{I}}(x)) \sqrt{g}\, dx^4$$

- $A > 0$.
- $\sqrt{g}\, dx^4$ is the volume form associated with the metric $\boldsymbol{g}$, $g = -\det(\boldsymbol{g})$.

Physical quantities are independent of the choice of a coordinate system

A is a *scalar* field

Plan

Principle ?

Explore the possibilities given by imposing that

the possible solutions are diffeomorphisms
that leave the energy $\mathscr{E}_\Omega$ invariant.

Remarks

Invariance of $\mathscr{E}_\Omega$: [1, 2, 3, 4, 5]

- Is it possible ?
- What is the group ?
  - The group of all the diffeomorphisms ("general covariance") ?
  - Another group ?

There is a difference between [5]

- The invariance of $\mathscr{E}_\Omega$ with respect to coordinate systems ("covariance")
- The invariance of $\mathscr{E}_\Omega$ with respect to the action of the group of all diffeomorpisms ("general covariance").

The least action principle

In this proposition the least action principle is disregarded
in favor of an invariance of $\mathscr{E}_\Omega$.

[1] Noether, E. : Invariante Variationsprobleme. Nachrichten von der Königlichen Gesellschaft der Wissenshcafter zu Göttingen, Mathematisch-physikalische Klasse, pp. 235-257, (1918). [2] Kosmann-Schwarzbach Y., The Noether Theorems, Springer (2018). [3] Soper D., Classical field theory, (1975). [4] Dirac, P. A. M., An action principle for the motion of particles, GRG (1974). [5] Rakotomanana L., Covariance and Gauge Invariance in continuum Physics (2018).

Local transformation φ_ϵ

$$\varphi_\epsilon : x \in \Omega \mapsto \tilde{x} \in \widetilde{\Omega} \qquad \tilde{x} = \varphi_\epsilon(x)$$

with $\varphi_0(x) = x$, $\varphi_\epsilon(x) = x + \epsilon X(x) + o(\epsilon)$, X a vector field, $X \in \mathfrak{X}$, $\mathfrak{X}$ the set of vector fields generating the transformations φ_ϵ, $\varphi_\epsilon \in G$ where G is a group of transformations.

Definition

Given an integral of the form

$$\mathscr{E}_\Omega(A(\phi_\mathcal{I})) = \int_\Omega A(\phi_\mathcal{I}(x)) \sqrt{g}\, dx^4,$$

$\mathscr{E}_\Omega$ is said to be invariant with respect to the action of G, a group of local transformations φ_ϵ, if for all $\varphi_\epsilon \in G$,

$$\mathscr{E}_{\varphi_\epsilon(\Omega)}\left(A(\varphi_\epsilon \phi_\mathcal{I})\right) = \mathscr{E}_\Omega\left(A(\phi_\mathcal{I})\right)$$

where $\varphi_\epsilon \phi$ is the transformed value of ϕ due to the action of φ_ϵ.

Local transformation φ_ϵ

$$\varphi_\epsilon : x \in \Omega \mapsto \tilde{x} \in \widetilde{\Omega} \qquad \tilde{x} = \varphi_\epsilon(x)$$

with $\varphi_0(x) = x$, $\varphi_\epsilon(x) = x + \epsilon X(x) + o(\epsilon)$, X a vector field, $X \in \mathfrak{X}$, $\mathfrak{X}$ the set of vector fields generating the transformations φ_ϵ, $\varphi_\epsilon \in G$ where G is a group of transformations.

Definition

The integral

$$\mathscr{E}_\Omega(A(\phi_\mathcal{I})) = \int_\Omega A(\phi_\mathcal{I})\ \sqrt{g}dx^4 = \int_\Omega \mathcal{A}(\phi_\mathcal{I})\ dx^4$$

with $\mathcal{A} = A\,\sqrt{g}$ is said to be invariant with respect to the action of G, a group of local transformations noted φ_ϵ, if there exists a vector density field $\mathcal{B}$ such that

$$\sum_\mathcal{I} \frac{\partial \mathcal{A}}{\partial \phi_\mathcal{I}} \underset{X}{\mathscr{L}} \phi_\mathcal{I} = \nabla \cdot \mathcal{B} \qquad \forall X \in \mathfrak{X}$$

Principle : conclusion

Process

1. Constitutive choice (definition of physics) : Choose the fields $\phi_{\mathcal{I}}$
2. Constitutive choice (definition of physics) : Define the *scalar* function $A\left(\phi_{\mathcal{I}}\right)$
3. Look for the conditions that leave invariant the integral

$$\mathscr{E}_{\Omega}(A\left(\phi_{\mathcal{I}}\right)) = \int_{\Omega} A\left(\phi_{\mathcal{I}}\right)\ \sqrt{g}dx^{4} = \int_{\Omega} \mathcal{A}\left(\phi_{\mathcal{I}}\right)\ dx^{4}.$$

   In other words, look for the set $\mathfrak{X}$ of vector fields X and for the conditions on the fields $\phi_{\mathcal{I}}$ such that there exists a vector density field $\mathcal{B}$ such that

$$\sum_{\mathcal{I}} \frac{\partial \mathcal{A}}{\partial \phi_{\mathcal{I}}} \underset{X}{\mathscr{L}} \phi_{\mathcal{I}} = \nabla \cdot \mathcal{B} \qquad \forall X \in \mathfrak{X}$$

Plan

1. Choose the fields :
   - the metric $\boldsymbol{g}$
   - the twice covariant Ricci curvature tensor $\boldsymbol{R}$

2. Define the integral $\mathscr{E}^C$ as

$$\mathscr{E}^C_\Omega(A^C(\boldsymbol{R},\boldsymbol{g})) = \int_\Omega A^C(\boldsymbol{R},\boldsymbol{g})\sqrt{g}\,\mathrm{d}x^4 = \int_\Omega \frac{1}{2\kappa}\boldsymbol{R}\boldsymbol{g}^{-1}\sqrt{g}\,\mathrm{d}x^4$$

where κ is a constant scalar ; $\mathcal{A}^C = A^C\sqrt{g}$

3. Invariance wrt a group of transformations :

$$\frac{\partial\mathcal{A}^C}{\partial\boldsymbol{g}}\underset{X}{\mathscr{L}}\boldsymbol{g} + \frac{\partial\mathcal{A}^C}{\partial\boldsymbol{R}}\underset{X}{\mathscr{L}}\boldsymbol{R} = \frac{1}{2\kappa}\left(\frac{\partial(\sqrt{g}\boldsymbol{R}\boldsymbol{g}^{-1})}{\partial\boldsymbol{g}}\underset{X}{\mathscr{L}}\boldsymbol{g} + \sqrt{g}\boldsymbol{g}^{-1}\underset{X}{\mathscr{L}}\boldsymbol{R}\right)$$
$$= \frac{1}{2\kappa}\left(\frac{R}{2}\boldsymbol{g}^{\boldsymbol{-1}} - \boldsymbol{R}^\sharp\right)\underset{X}{\mathscr{L}}\boldsymbol{g} + \nabla\cdot\mathcal{B}$$

$\mathscr{E}^C_\Omega$ is *general* covariant if

$$\boldsymbol{R}^\sharp - \frac{R}{2}\boldsymbol{g}^{\boldsymbol{-1}} = 0 \qquad \text{(Einstein tensor)}$$

[1] Noether, E. : Invariante Variationsprobleme. Nachrichten von der Königlichen Gesellschaft der Wissenshcafter zu Göttingen, Mathematisch-physikalische Klasse, pp. 235-257, (1918). [2] Kosmann-Schwarzbach Y., The Noether Theorems, Springer (2018).

Example of application : mass

World-tube

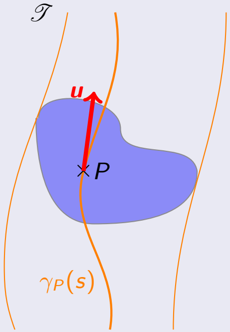

Figure 1 – Representation of a world-tube $\mathscr{T}$ with world-lines.

4D manifold

- Continuously distributed matter : rest mass density ρ
- Event $P \in \mathcal{M}$ identified by a set x of four coordinates $(x^\mu)_{\mu\in\{0,1,2,3\}}$
- Worldline $\gamma_P(s)$ curve in $\mathcal{M}$,

$$\gamma_P : \mathbb{R} \to \mathcal{M}, s \mapsto \gamma_P(s) = \{\phi_s(P)\}.$$

- 4-velocity noted $\boldsymbol{u}$:

$$u^\mu = \frac{dx^\mu}{ds} \quad \text{with} \quad ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu .$$

time-like future oriented unit vector

Example of application : mass

1. Choose the mechanical fields :
   - the Lorentz metric $\boldsymbol{\eta}$
   - the momentum $p = f_\rho \sqrt{g} u$, vector density field
     with the mass density $\rho = f_\rho \sqrt{g} = \sqrt{p \cdot p}$
     and the velocity field u with $u \cdot u = 1$, the field u never vanishes
2. Define the integral $\mathscr{E}^p$ as

$$\mathscr{E}^p_\Omega(A^p(p, \boldsymbol{\eta})) = \int_\Omega c^2 \sqrt{p \cdot p}\, \mathrm{d}x^4 = \int_\Omega c^2 \rho\, \mathrm{d}x^4 \qquad \mathcal{A}^p = A^p \sqrt{g} = c^2 \sqrt{p \cdot p}$$

3. Invariance wrt to a group of transformations :

$$\frac{\partial \mathcal{A}^p}{\partial \boldsymbol{\eta}} \underset{X}{\mathscr{L}} \boldsymbol{\eta} + \frac{\partial \mathcal{A}^p}{\partial p} \underset{X}{\mathscr{L}} p$$

$$= \frac{\rho u \otimes u}{2} \underset{X}{\mathscr{L}} \boldsymbol{\eta} + X^\flat\ \nabla \cdot (\rho u \otimes u) + \nabla \cdot \mathcal{B}$$

$A^p > 0$, the transformation generated by u has to be in the set of transformations

$\mathscr{E}^p_\Omega$ is invariant wrt the group of Killing fields if

$$\text{if } \nabla \cdot (\rho u \otimes u) = 0 \iff$$
$$\text{if } a = u \cdot \nabla u = 0 \text{ and } \underset{u}{\mathscr{L}} \rho = 0$$

Example of application : mass

Given the integral $\mathscr{E}^p$:

$$\mathscr{E}^p_\Omega(A^p(p,\boldsymbol{\eta})) = \int_\Omega \mathcal{A}^p\,(p,\boldsymbol{\eta})\ \mathrm{d}x^4 = \int_\Omega c^2\sqrt{p\cdot p}\,\mathrm{d}x^4$$

$\mathscr{E}^p_\Omega$ is invariant wrt changes of coordinates

Unconditionally : *scalar* product.

$\mathscr{E}^p_\Omega$ is invariant wrt the group of Killing fields if

if $\nabla\cdot(\frac{p\otimes p}{\sqrt{p\cdot p}}) = 0$

Example of application : inertia [1]

1. Choose the fields :
   - the metric $\boldsymbol{g}$
   - the twice covariant Ricci curvature tensor $\boldsymbol{R}$
   - the vector density field $p = f_\rho \sqrt{g} u$ with $\rho = \sqrt{p \cdot p}$ $\rho = f_\rho \sqrt{g}$ $u \cdot u = 1$
2. Define the integral $\mathscr{E}^{Cp}$ as

$$\mathscr{E}_\Omega^{Cp}(A^{Cp}(\boldsymbol{R}, \boldsymbol{g}, p)) = \int_\Omega \left[\frac{1}{2\kappa} \boldsymbol{R} \boldsymbol{g}^{-1} \sqrt{g} + c^2 \sqrt{p \cdot p} \right] \mathrm{d}x^4$$

   where κ and c are constant scalars.
3. Invariance wrt a group of transformations :

$$\frac{\partial \mathcal{A}^{Cp}}{\partial \boldsymbol{g}} \underset{X}{\mathscr{L}} \boldsymbol{g} + \frac{\partial \mathcal{A}^{Cp}}{\partial p} \underset{X}{\mathscr{L}} p + \frac{\partial \mathcal{A}^{Cp}}{\partial \boldsymbol{R}} \underset{X}{\mathscr{L}} \boldsymbol{R}$$

$$= \left[\frac{1}{2\kappa} \left(\frac{R}{2} \boldsymbol{g}^{-1} - \boldsymbol{R}^\sharp \right) + \frac{c^2 \rho u \otimes u}{2} \right] \underset{X}{\mathscr{L}} \boldsymbol{g} + X^\flat \nabla \cdot (c^2 \rho u \otimes u) + \nabla \cdot \mathcal{B}$$

$\mathscr{E}_\Omega^{Cp}$ is *general* covariant if

$$\boldsymbol{R}^\sharp - \frac{R}{2} \boldsymbol{g}^{-1} = \kappa \boldsymbol{T} \quad \text{and} \quad \nabla \cdot \boldsymbol{T} = 0 \quad \text{with} \quad \boldsymbol{T} = c^2 \rho u \otimes u \iff$$
$$\text{if } \boldsymbol{R}^\sharp - \frac{R}{2} \boldsymbol{g}^{-1} = \kappa \boldsymbol{T} \quad \text{and } a = u \cdot \nabla u = 0 \text{ and } \underset{u}{\mathscr{L}} \rho = 0$$

[1] Dirac, P. A. M., An action principle for the motion of particles, GRG (1974)

Plan

Example of application : elasticity

1. Choose the fields :
   - the metric $\boldsymbol{g}$
   - the twice covariant Ricci curvature tensor $\boldsymbol{R}$
   - the vector density field $p = f_\rho \sqrt{g} u$ with $\rho = \sqrt{p \cdot p}$ $\rho = f_\rho \sqrt{g}$ $u \cdot u = 1$
   - the elasticity tensor density field $\boldsymbol{E}$, forth time contravariant
   - a symmetric twice covariant tensor $\boldsymbol{b}$

2. Define the integral $\mathscr{E}_\Omega^{CpE}$ as

$$\mathscr{E}_\Omega^{CpE}(A^{CpE}(\boldsymbol{R}, \boldsymbol{g}, p, \boldsymbol{E}, \boldsymbol{b})) =$$
$$\int_\Omega \left[\frac{1}{2\kappa} \boldsymbol{R} \boldsymbol{g}^{-1} \sqrt{g} + c^2 \sqrt{p \cdot p} + \frac{1}{2} (\boldsymbol{E} : \frac{1}{2} (\boldsymbol{g} - \boldsymbol{b})) : \frac{1}{2} (\boldsymbol{g} - \boldsymbol{b}) \right] \, \mathrm{d}x^4$$

3. Invariance wrt a group of transformations

$$\frac{\partial \mathcal{A}^{CpE}}{\partial \boldsymbol{g}} \underset{X}{\mathscr{L}} \boldsymbol{g} + \frac{\partial \mathcal{A}^{CpE}}{\partial \boldsymbol{R}} \underset{X}{\mathscr{L}} \boldsymbol{R} + \frac{\partial \mathcal{A}^{CpE}}{\partial p} \underset{X}{\mathscr{L}} p + \frac{\partial \mathcal{A}^{CpE}}{\partial \boldsymbol{E}} \underset{X}{\mathscr{L}} \boldsymbol{E} + \frac{\partial \mathcal{A}^{CpE}}{\partial \boldsymbol{b}} \underset{X}{\mathscr{L}} \boldsymbol{b}$$
$$= \dots$$

2. Define the integral $\mathscr{E}_\Omega^{CpE}$ as

$$\mathscr{E}_\Omega^{CpE}(A^{CpE}(\boldsymbol{R},\boldsymbol{g},p,\boldsymbol{E},\boldsymbol{b})) = \int_\Omega \left[\frac{1}{2\kappa}\boldsymbol{R}\boldsymbol{g}^{-1}\sqrt{g} + c^2\sqrt{p\cdot p} + \frac{1}{2}(\boldsymbol{E}:\frac{1}{2}(\boldsymbol{g}-\boldsymbol{b})) : \frac{1}{2}(\boldsymbol{g}-\boldsymbol{b})\right]\,\mathrm{d}x^4$$

3. Invariance wrt a group of transformations

$$\frac{\partial\mathcal{A}^{CpE}}{\partial\boldsymbol{g}}\underset{X}{\mathscr{L}}\boldsymbol{g} + \frac{\partial\mathcal{A}^{CpE}}{\partial\boldsymbol{R}}\underset{X}{\mathscr{L}}\boldsymbol{R} + \frac{\partial\mathcal{A}^{CpE}}{\partial p}\underset{X}{\mathscr{L}}p + \frac{\partial\mathcal{A}^{CpE}}{\partial\boldsymbol{E}}\underset{X}{\mathscr{L}}\boldsymbol{E} + \frac{\partial\mathcal{A}^{CpE}}{\partial\boldsymbol{b}}\underset{X}{\mathscr{L}}\boldsymbol{b}$$
$$= \left[\frac{1}{2\kappa}\left(\frac{R}{2}\boldsymbol{g}^{-1} - \boldsymbol{R}^\sharp\right) + c^2\frac{p\otimes p}{2\sqrt{p\cdot p}} + \frac{1}{2}\boldsymbol{E}:\frac{1}{2}(\boldsymbol{g}-\boldsymbol{b})\right]\underset{X}{\mathscr{L}}\boldsymbol{g}$$
$$+\frac{\partial\mathcal{A}^{CpE}}{\partial p}\underset{X}{\mathscr{L}}p + \frac{\partial\mathcal{A}^{CpE}}{\partial\boldsymbol{E}}\underset{X}{\mathscr{L}}\boldsymbol{E} + \frac{\partial\mathcal{A}^{CpE}}{\partial\boldsymbol{b}}\underset{X}{\mathscr{L}}\boldsymbol{b} + \nabla\cdot\mathcal{B}$$

3. Invariance wrt a group of transformations

$$\frac{\partial \mathcal{A}^{CpE}}{\partial \boldsymbol{g}} \underset{X}{\mathscr{L}} \boldsymbol{g} + \frac{\partial \mathcal{A}^{CpE}}{\partial \boldsymbol{R}} \underset{X}{\mathscr{L}} \boldsymbol{R} + \frac{\partial \mathcal{A}^{CpE}}{\partial p} \underset{X}{\mathscr{L}} p + \frac{\partial \mathcal{A}^{CpE}}{\partial \boldsymbol{E}} \underset{X}{\mathscr{L}} \boldsymbol{E} + \frac{\partial \mathcal{A}^{CpE}}{\partial \boldsymbol{b}} \underset{X}{\mathscr{L}} \boldsymbol{b}$$
$$= \left[\frac{1}{2\kappa} \left(\frac{R}{2} \boldsymbol{g}^{-1} - \boldsymbol{R}^\sharp \right) + c^2 \frac{p \otimes p}{2\sqrt{p \cdot p}} + \frac{1}{2} \boldsymbol{E} : \frac{1}{2} (\boldsymbol{g} - \boldsymbol{b}) \right] \underset{X}{\mathscr{L}} \boldsymbol{g}$$
$$+ \frac{\partial \mathcal{A}^{CpE}}{\partial p} \underset{X}{\mathscr{L}} p + \frac{\partial \mathcal{A}^{CpE}}{\partial \boldsymbol{E}} \underset{X}{\mathscr{L}} \boldsymbol{E} + \frac{\partial \mathcal{A}^{CpE}}{\partial \boldsymbol{b}} \underset{X}{\mathscr{L}} \boldsymbol{b} + \nabla \cdot \mathcal{B}$$

Conditions for $\mathscr{E}_\Omega^{CpE}$ to be invariant : I

$$\boldsymbol{R}^\sharp - \tfrac{R}{2} \boldsymbol{g}^{-1} = \kappa \boldsymbol{T} \quad \text{and} \quad \nabla \cdot \boldsymbol{T} = 0 \quad \text{with}$$

$$\boldsymbol{T} = c^2 \frac{p \otimes p}{\sqrt{p \cdot p}} + \boldsymbol{E} : \frac{1}{2} (\boldsymbol{g} - \boldsymbol{b})$$

Example of application : elasticity

2. Define the integral $\mathscr{E}_\Omega^{CpE}$ as

$$\mathscr{E}_\Omega^{CpE}(A^{CpE}(\boldsymbol{R},\boldsymbol{g},p,\boldsymbol{E},\boldsymbol{b})) = \int_\Omega \left[\frac{1}{2\kappa}\boldsymbol{R}\boldsymbol{g}^{-1}\sqrt{g} + c^2\sqrt{p\cdot p} + \frac{1}{2}(\boldsymbol{E}:\frac{1}{2}(\boldsymbol{g}-\boldsymbol{b})):\frac{1}{2}(\boldsymbol{g}-\boldsymbol{b})\right]\,\mathrm{d}x^4$$

3. Invariance wrt a group of transformations

$$\frac{\partial\mathcal{A}^{CpE}}{\partial\boldsymbol{g}}\underset{X}{\mathscr{L}}\boldsymbol{g} + \frac{\partial\mathcal{A}^{CpE}}{\partial\boldsymbol{R}}\underset{X}{\mathscr{L}}\boldsymbol{R} + \frac{\partial\mathcal{A}^{CpE}}{\partial p}\underset{X}{\mathscr{L}}p + \frac{\partial\mathcal{A}^{CpE}}{\partial\boldsymbol{E}}\underset{X}{\mathscr{L}}\boldsymbol{E} + \frac{\partial\mathcal{A}^{CpE}}{\partial\boldsymbol{b}}\underset{X}{\mathscr{L}}\boldsymbol{b}$$
$$= \left[\frac{1}{2\kappa}\left(\frac{R}{2}\boldsymbol{g}^{-1} - \boldsymbol{R}^\sharp\right) + c^2\frac{p\otimes p}{2\sqrt{p\cdot p}} + \frac{1}{2}\boldsymbol{E}:\frac{1}{2}(\boldsymbol{g}-\boldsymbol{b})\right]\underset{X}{\mathscr{L}}\boldsymbol{g}$$
$$+\frac{\partial\mathcal{A}^{CpE}}{\partial p}\underset{X}{\mathscr{L}}p + \frac{\partial\mathcal{A}^{CpE}}{\partial\boldsymbol{E}}\underset{X}{\mathscr{L}}\boldsymbol{E} + \frac{\partial\mathcal{A}^{CpE}}{\partial\boldsymbol{b}}\underset{X}{\mathscr{L}}\boldsymbol{b} + \nabla\cdot\mathcal{B}$$

Conditions for $\mathscr{E}_\Omega^{CpE}$ to be invariant : II

Conditions for $\underset{Y}{\mathscr{L}}p = \underset{Y}{\mathscr{L}}(\rho u) = 0$: $\underset{Y}{\mathscr{L}}\rho = 0$ and $Y = Ku$

Choose $\boldsymbol{b}$ such that $\underset{u}{\mathscr{L}}\boldsymbol{b} = 0$

Impose $\underset{u}{\mathscr{L}}\boldsymbol{E} = 0$

Example of application : elasticity

3. Invariance wrt a group of transformations

$$\frac{\partial \mathcal{A}^{CpE}}{\partial \boldsymbol{g}} \underset{x}{\mathscr{L}} \boldsymbol{g} + \frac{\partial \mathcal{A}^{CpE}}{\partial \boldsymbol{R}} \underset{x}{\mathscr{L}} \boldsymbol{R} + \frac{\partial \mathcal{A}^{CpE}}{\partial p} \underset{x}{\mathscr{L}} p + \frac{\partial \mathcal{A}^{CpE}}{\partial \boldsymbol{E}} \underset{x}{\mathscr{L}} \boldsymbol{E} + \frac{\partial \mathcal{A}^{CpE}}{\partial \boldsymbol{b}} \underset{x}{\mathscr{L}} \boldsymbol{b}$$
$$= \left[\frac{1}{2\kappa} \left(\frac{R}{2} \boldsymbol{g}^{-1} - \boldsymbol{R}^\sharp \right) + c^2 \frac{p \otimes p}{2\sqrt{p \cdot p}} + \frac{1}{2} \boldsymbol{E} : \frac{1}{2} (\boldsymbol{g} - \boldsymbol{b}) \right] \underset{x}{\mathscr{L}} \boldsymbol{g}$$
$$+ \frac{\partial \mathcal{A}^{CpE}}{\partial p} \underset{x}{\mathscr{L}} p + \frac{\partial \mathcal{A}^{CpE}}{\partial \boldsymbol{E}} \underset{x}{\mathscr{L}} \boldsymbol{E} + \frac{\partial \mathcal{A}^{CpE}}{\partial \boldsymbol{b}} \underset{x}{\mathscr{L}} \boldsymbol{b} + \nabla \cdot \mathcal{B}$$

Conditions for $\mathscr{E}_\Omega^{CpE}$ to be invariant

$$\boldsymbol{R}^\sharp - \tfrac{R}{2} \boldsymbol{g}^{-1} = \kappa \boldsymbol{T} \quad \text{and} \quad \nabla \cdot \boldsymbol{T} = 0 \quad \text{with}$$

$$\boldsymbol{T} = c^2 \frac{p \otimes p}{\sqrt{p \cdot p}} + \boldsymbol{E} : \frac{1}{2} (\boldsymbol{g} - \boldsymbol{b}) = c^2 \rho u \otimes u + \boldsymbol{E} : \boldsymbol{e}$$

$$\underset{u}{\mathscr{L}} \rho = 0$$

Choose b such that $\underset{u}{\mathscr{L}} \boldsymbol{b} = 0$

Impose $\underset{u}{\mathscr{L}} \boldsymbol{E} = 0$

World-tube

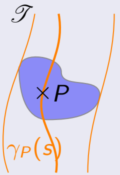

Figure 2 – Event P of coordinates x or $\underset{\sim}{x}$ or $\hat{x}$.

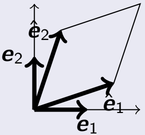

Figure 3 – Example for current basis vectors $\boldsymbol{e}_\mu$ and proper basis vectors $\hat{\boldsymbol{e}}_\mu$ for sliding.

Time-space deformation $\boldsymbol{b}$

Second order covariant tensor

$$\boldsymbol{b} = \eta_{\mu\nu} \hat{\boldsymbol{e}}^{\mu} \otimes \hat{\boldsymbol{e}}^{\nu}$$

with $\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

Time-space strain $\boldsymbol{e}$

Second order covariant tensor :

$$\boldsymbol{e} = \frac{1}{2}(\boldsymbol{g} - \boldsymbol{b})$$

$$\underset{u}{\mathscr{L}} \boldsymbol{b} = 0$$

$$\underset{u}{\mathscr{L}} \boldsymbol{e} = \frac{1}{2}\left(\underset{u}{\mathscr{L}} \boldsymbol{g} - \underset{u}{\mathscr{L}} \boldsymbol{b}\right) = \frac{1}{2} \underset{u}{\mathscr{L}} \boldsymbol{g}$$

Time-covariance

The integral

$$\mathscr{E}_\Omega(A(\phi_\mathcal{I})) = \int_\Omega A(\phi_\mathcal{I})\ \sqrt{g} dx^4 = \int_\Omega \mathcal{A}(\phi_\mathcal{I})\ dx^4$$

is said to be time-covariant if there exists a vector density field $\mathcal{B}$ such that

$$\sum_\mathcal{I} \frac{\partial \mathcal{A}}{\partial \phi_\mathcal{I}} \underset{Y}{\mathscr{L}} \phi_\mathcal{I} = \nabla \cdot \mathcal{B} \qquad \forall Y \in \mathfrak{X}$$

Y is a time-like vector field and $U \in \mathfrak{X}$ where U is the vector field that generates the diffeomorphism that is the transformation undergone by the media.

Plan

Conclusion

	Energy density	Transformations	Conditions
Curvature	$\mathcal{A}^{C}(\boldsymbol{R}, \boldsymbol{g}) = \frac{1}{2\kappa} R \boldsymbol{g}^{-1} \sqrt{g}$	General covariance	$\boldsymbol{R}^{\sharp} - \frac{R}{2}\boldsymbol{g}^{-1} = 0$
Mass	$\mathcal{A}^{C}(p, \boldsymbol{\eta}) = c^2 \sqrt{p \cdot p} = c^2 \rho$	Killing fields Special relativity	$a = 0$ $\mathscr{L}_{u} \rho = 0$
Inertia	$\mathcal{A}^{Cp}(\boldsymbol{R}, \boldsymbol{g}, p) = \frac{1}{2\kappa} R \boldsymbol{g}^{-1} \sqrt{g}$ $+ c^2 \sqrt{p \cdot p}$	General covariance	$\boldsymbol{R}^{\sharp} - \frac{R}{2}\boldsymbol{g}^{-1} = \kappa \boldsymbol{T}$ $\boldsymbol{T} = c^2 \rho u \otimes u$ $\nabla \boldsymbol{T} = 0 \quad \mathscr{L}_{u} \rho = 0$
Elasticity	$\mathcal{A}^{CpE}(\boldsymbol{R}, \boldsymbol{g}, p, \boldsymbol{E}, \boldsymbol{b}) = \frac{1}{2\kappa} R \boldsymbol{g}^{-1} \sqrt{g}$ $+ c^2 \sqrt{p \cdot p}$ $+ \frac{1}{2}(\boldsymbol{E} : \frac{1}{2}(\boldsymbol{g} - \boldsymbol{b})) : \frac{1}{2}(\boldsymbol{g} - \boldsymbol{b})$	Time-covariance	$\boldsymbol{R}^{\sharp} - \frac{R}{2}\boldsymbol{g}^{-1} = \kappa \boldsymbol{T}$ $\boldsymbol{T} = \ldots \quad \nabla \boldsymbol{T} = 0$ $\mathscr{L}_{u} \rho = 0$ $\mathscr{L}_{u} \boldsymbol{E} = 0 \quad \mathscr{L}_{u} \boldsymbol{b} = 0$
Electro-magnetism		General covariance	
Thermo-dynamics		Time-covariance	

Conclusion

Principle

The energy should be time-covariant

Time-covariance

The integral

$$\mathscr{E}_\Omega(A(\phi_\mathcal{I})) = \int_\Omega A(\phi_\mathcal{I}) \ \sqrt{g} dx^4 = \int_\Omega \mathcal{A}(\phi_\mathcal{I}) \ dx^4$$

is said to be time-covariant if there exists a vector density field $\mathcal{B}$ such that

$$\sum_\mathcal{I} \frac{\partial \mathcal{A}}{\partial \phi_\mathcal{I}} \underset{Y}{\mathscr{L}} \phi_\mathcal{I} = \nabla \cdot \mathcal{B} \qquad \forall Y \in \mathfrak{X}$$

Y is a time-like vector field and $U \in \mathfrak{X}$ where U is the vector field that generates the diffeomorphism that is the transformation undergone by the media.