Nonsmooth continuation and bifurcation in contact problems

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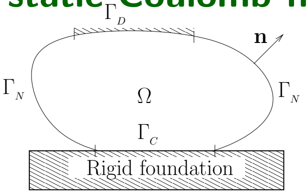
Outline

1. The static problem with “static” Coulomb friction
2. Continuation and branching methods in discrete problems

1. The static problem with “static” Coulomb friction

1) Small deformation static contact problem with static Coulomb friction



$$-\operatorname{div}\sigma(u) = f \qquad \text{in } \Omega, \tag{1}$$

$$\sigma(u) = \mathcal{A}\varepsilon(u) \qquad \text{in } \Omega, \tag{2}$$

$$\sigma(u)\mathrm{n} = F \qquad \text{on } \Gamma_N, \tag{3}$$

$$u = 0 \qquad \text{on } \Gamma_D, \tag{4}$$

Contact condition on Γ_C:

$$u_N \leq 0, \ \sigma_n(u) \leq 0, \ u_N\sigma_n(u) = 0. \tag{5}$$

"Static" friction condition on Γ_C:

$$\text{if } \ u_T = 0 \quad \text{then} \quad |\sigma_T(u)| \leq -\mathcal{F}\sigma_n(u), \tag{6}$$

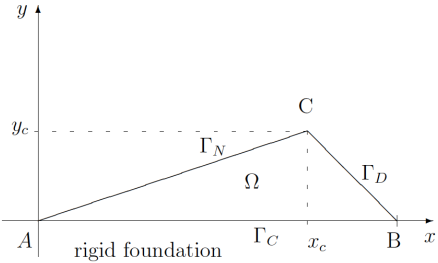
$$\text{if } \ u_T \neq 0 \quad \text{then} \quad \sigma_T(u) = \mathcal{F}\sigma_n(u)\frac{u_T}{|u_T|}. \tag{7}$$

2) Existence result for the static frictional problem

- Without friction, the solution is unique by Stampacchia theorem.
- With friction : Existence for small $\mathscr{F}$ on a infinite strip : [Necas, Jarusek, Haslinger 1980] Bollettino U.M.I.
- Existence for small $\mathscr{F}$ on a regular bounded domain: [Jarusek 1983] Czechoslovak Mathematical Journal.
- Existence for any $\mathscr{F}$ for isotropic elasticity and example of non-existence for large friction coefficient in anisotropic elasticity: P. Ballard & F. Iurlano, preprint 2023 "Optimal existence results for the 2d elastic contact problem with Coulomb friction".

3) Non-Uniqueness examples

Non uniqueness **for large friction coefficients ($\mathscr{F} > 1$)**. From Patrick Hild, "Multiple solutions of stick and separation type in the Signorini model with Coulomb friction", ZAMM, vol 85:9, 2005.



With $A = (0,0), B = (1,0), C = (x_c, y_c),\ x_c \in (0,1),\ y_c = \sqrt{\dfrac{1-x_c}{1+x_c}}$.

Polynomial solutions for linear honogeneous isotropic elasticity.

For $\mathscr{F} \geq \sqrt{\dfrac{1+x_c}{1-x_c}} > 1$, P. Hild exhibits two solutions of the Coulomb problem: one in grasing contact and one in stick contact.
(also some other multiplicity examples for a quadrilateral domain).

4) A Uniqueness criterion

From Y. Renard, "A uniqueness criterion for the Signorini problem with Coulomb friction", SIAM j. on math. analysis 38 (2), 452-467

If a solution satifies $\lambda_T = \mathscr{F}\lambda_N\xi$, with ξ a multiplier satisfying $\xi = \dfrac{u_T}{\|u_T\|}$ if $u_T \neq 0, |\xi| \leq 1$ if $u_T = 0$, and with $\|\xi\|_M = \sup_{v_t \in X_T} \dfrac{\|\xi v_t\|_{X_T}}{\|v_t\|_{X_T}}$, if there exists $C_0 > 0$ such that

$$\mathscr{F} < \frac{C_0}{\|\xi\|_M},$$

Then, the solution is unique.

In another words, if the solution is "regular enough" (for instance, sliding in a unique direction) and the friction coefficient small, this solution is unique.

5) Finite element approximation of the static Coulomb friction problem

Mixed weak formulation:

$$
\left\{
\begin{array}{l}
\text{Find } u \in V, \lambda_N \in \Lambda_N \text{ and } \lambda_T \in \Lambda_T(-\mathscr{F}\lambda_N) \text{ with} \\
a(u,v) = \ell(v) + \langle \lambda_N, v_N \rangle_{X'_N, X_N} + \langle \lambda_T, v_T \rangle_{X'_T, X_T} \quad \forall\, v \in V, \\
\langle \mu_N - \lambda_N, u_N \rangle_{X'_N, X_N} \geq 0, \quad \forall \mu_N \in \Lambda_N, \\
\langle \mu_T - \lambda_T, u_T \rangle_{X'_N, X_N} \geq 0, \quad \forall \mu_T \in \Lambda_T(-\mathscr{F}\lambda_N),
\end{array}
\right.
$$

A mixed finite element approximation is a choice of:

- A finite element space V^h for the displacement
- Two finite element space $(X'_N)^h$ and $(X'_T)^h$ satisfying an inf-sup condition (or the addition of a stabilization term
- Two approximations of admissibles sets of stresses $\Lambda_N^h \subset (X'_N)^h$ and $\Lambda_T^h - \mathscr{F}\lambda_N^h) \subset (X'_T)^h$

The finite element approximation reads:

$$\left\{\begin{array}{l}\text{Find } u^h \in V^h, \lambda_N^h \in \Lambda_N^h \text{ and } \lambda_T^h \in \Lambda_T^h(-\mathscr{F}\lambda_N^h) \text{ with}\\ a(u^h, v^h) = \ell(v^h) + \langle \lambda_N^h, v_N^h \rangle_{X_N', X_N} + \langle \lambda_T^h, v_T^h \rangle_{X_T', X_T} \quad \forall\, v^h \in V^h,\\ \langle \mu_N^h - \lambda_N^h, u_N^h \rangle_{X_N', X_N} \geq 0, \quad \forall \mu_N^h \in \Lambda_N^h,\\ \langle \mu_T^h - \lambda_T^h, u_T^h \rangle_{X_N', X_N} \geq 0, \quad \forall \mu_T^h \in \Lambda_T^h(-\mathscr{F}\lambda_N^h),\end{array}\right.$$

Existence and uniqueness of a solution:

The discrete problem admits at least a solution for all values of $\mathscr{F}$. There exists $C > 0$ such that for

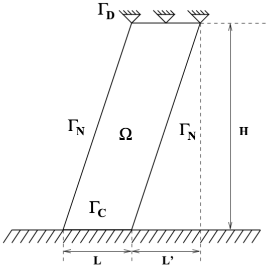
$$\mathscr{F} \leq Ch^{1/2},$$

The solution is unique.

(See for instance the result in Chouly, Hild Renard, "Finite Element Approximation of Contact and Friction in Elasticity", Springer 2023)

6) Non-uniqueness in finite element approximation

Non uniqueness for different geometries in Patrick Hild, R. Hassani and I. Ionescu "Sufficient conditions of non-uniqueness of the elastic equilibrium with Coulomb friction", Math. Methods Appl. Sci. (M2AS), Vol. 27, No. 1, 47-67, 2004.



With convergence of the critical friction coefficient when the mesh is refined (no theoretical proof, but this tends to indicates there is no multiplicity for small coefficients in the continuous case)
Some non-uniqueness for a friction coefficient which tends to zero when H/L tends to infinity.

2. Continuation and branching methods in discrete problems

1) Motivation and framework

We assume we have to find $y \in \mathbb{R}^{N+1}$ satisfying

$$H(y) = 0,$$

for a map

$$H : \mathbb{R}^{N+1} \mapsto \mathbb{R}^N.$$

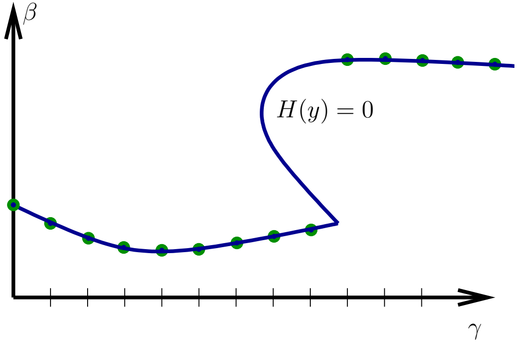
For instance, in the finite element approximation of the contact problem, y contains the degrees of freedom of the displacement, the mulipliers and an additionnal continuation parameter :

$$y = (\gamma, U, \lambda_N, \lambda_T),$$

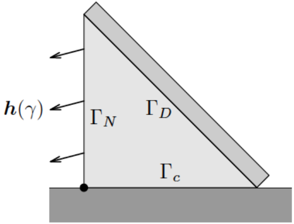
where γ is a continuation parameter.

- Often γ is a loading parameter such that the solution is know for $\gamma = 0$ and we search for the solution for instance for $\gamma = 1$
- This could be to solve the static problem, but also to solve a quasistatic problem (γ being an abstract time) or a time step of a dynamic problem.
- It is also possible to take the friction coefficient as a continuation parameter.

A possible situation for a quasi-static evolution for $y = (\gamma, \beta)$



Very Basic Example



Single linear finite element. Only **one moving node**. Here $y = (\gamma, u_N, u_T, \lambda_N, \lambda_T)$ and

$$H(y) = \begin{pmatrix} au_N - bu_T + h_2(\gamma) - \lambda_N \\ -bu_N + au_T - h_1(\gamma) - \lambda_T \\ \lambda_N + (\lambda_N - u_N)_- \\ \lambda_T - P_{[-\mathscr{F}(\lambda_N - u_N)_-, \mathscr{F}(\lambda_N - u_N)_-]}(\lambda_T - u_T) \end{pmatrix}$$

where $a = (\lambda + 3\mu)/2$, $b = (\lambda + \mu)/2$, $h(\gamma) = (h_1(\gamma), h_2(\gamma))$ the load vector on Γ_N, (u_T, u_N) the displacement of the moving node.

2) The Moore-Penrose continuation for smooth cases ($H(y)$ is a C^1 map)

- Compute a sequence $\{y_j\}$ satisfaying $H(y_j) = 0$, a sequence of steps $\{h_j\}$ and a sequence $\{t_j\}$ of unit tangent vectors:

$$\nabla H(y_j)t_j = 0, \quad \|t_j\| = 1.$$

- Consists of two steps:
  prediction – an initial approximation of y_{i+1}, t_{j+1} is given by

$$Y_0 = y_j + h_j t_j, \quad T_0 = t_j,$$

  correction – y_{i+1}, t_{j+1} are obtained from a Newton-like procedure

- h_{j+1} is adapted from the convergence of Newton correction:

$$h_{j+1} = \begin{cases} \alpha_{dec} h_j \text{ if not converged }, \\ \alpha_{inc} h_j \text{ if converged in small number of iterations }, \\ h_j \text{ otherwise} \end{cases}$$

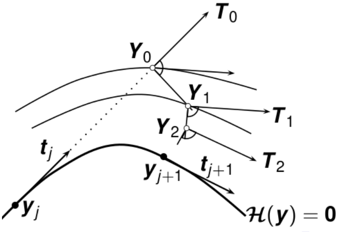
where $0 < \alpha_{dec} < 1 < \alpha_{inc}$ chosen experimentally.

Correction step

From Y_ℓ, T_ℓ known, $Y_{\ell+1}, T_{\ell+1}$ are calculated by one Newton iteration applied to the $2(N+1)\times 2(N+1)$ system

$$H_\ell(Y,T) = \left(\begin{array}{c} H(Y) \\ T_\ell\cdot(Y-Y_\ell) \\ J_\ell T \\ T_\ell\cdot T - T_\ell\cdot T_\ell \end{array}\right) = \left(\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \end{array}\right),$$

where $J_\ell = \nabla H(Y_\ell)$.



Simple bifurcation detection

Bifurcation point : At least two branches of solution cross at this point.
Simple bifurcation point : only two branches with $\nabla H(y)$ has a kernel of dimension with two tangents of the two branches.
Detection : change of sign in the continuation of $\det(J(y_j))$ with

$$J(y_j) = \begin{pmatrix} \nabla H(y_j) \\ t_j \end{pmatrix}.$$

Large scale problems: (finite element discretization) the determinant is not significative. With B, C and d constant but randomly chosen, the change of sign is detected by $\tau_{BP}(Y)$ defined by

$$\begin{pmatrix} J(Y) & B \\ C^T & d \end{pmatrix} \begin{pmatrix} Y \\ \tau_{BP}(Y) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

Simple bifurcation branching

Possible algorithm:

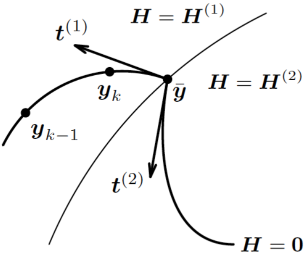
- Determine precisely the bifurcation point with a bisection-like procedure on $\tau_{BP}(Y)$.
- Compute the eigenvector corresponding to the smallest eigenvalue of $J(Y)$.
- Start a continuation procedure in that direction

3) The Moore-Penrose continuation for non-smooth cases (with T. Ligursky)

For contact and friction problem (even regularized):

$H(y)$ **is only piecewise** C^1

The branches of solutions are also piecewise C^1:



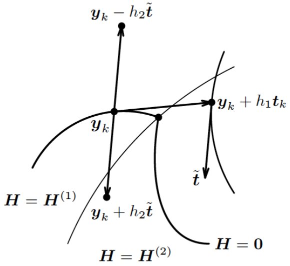
More precisely in the transition point $\bar{y}$, there is a generalized multivalued gradient in the Clarke sense:

$$\nabla H(\bar{y}) = \underset{i \in I_{\bar{y}}}{ConvexHull}(\nabla H^{(i)}(\bar{y})),$$

where $H^{(i)}$ are C^1 maps: smooth parts of $H(y)$ around $\bar{y}$.

Proposed strategy for passing the irregular point

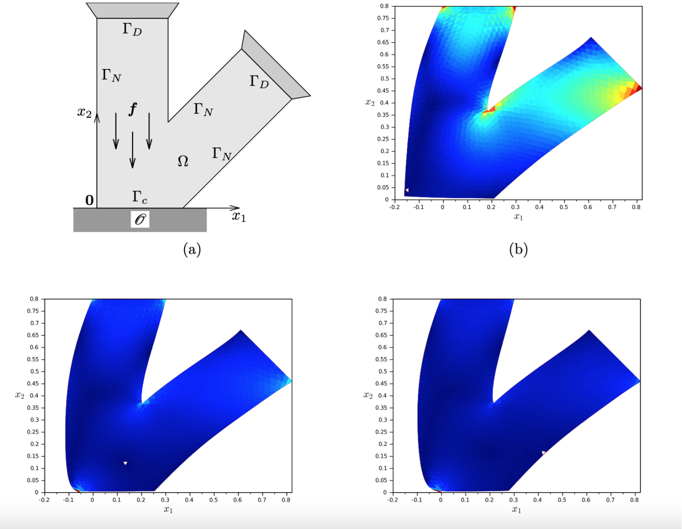
- Problem occurs when **continuation failed**, i.e. when $y_k + ht_k$ far enough from the solution branch that correction failed. In that case, h **is the minimal step** allowed.



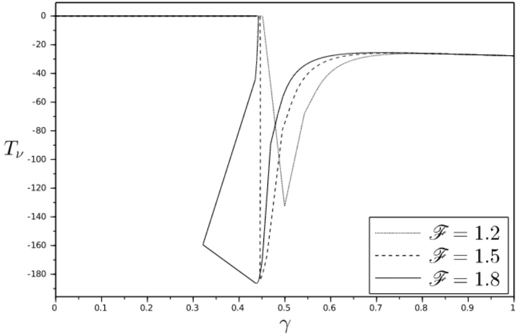
- Then compute $\tilde{t}$ such that $\nabla H(y_k + h_1 t_k)\tilde{t} = 0$, $\|\tilde{t}\| = 1$. With eventually h_1 a bit greater than h.
- Try the correction procedure with the prediction $y_k \pm h_2\tilde{t}$

Tricky test: be sure not to go backward on the previous branch.

Example in large deformation elasticity



γ : horizontal displacement to the right for the top edge. Here three solutions found by our the continuation method for $\gamma = 0.38$ and $\mathscr{F} = 1.8$

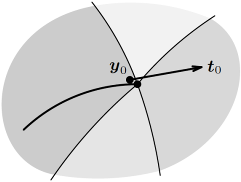


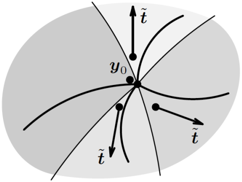
The non-smooth continuation branch, T_ν the normal contact stress for the left contact node. Multi-solutions for $\mathscr{F} = 1.8$ only.

4) The bifurcation-detection of branches for non-smooth cases (with T. Ligursky)

Nonsmooth point may concentrate the bifurcations

Simple bifurcation point in a non-smooth point : multivalued $\nabla H(\bar{y})$ contains one or more elements having a kernel of dimension 2. Direction depends on the selection of the smooth parts ... !





The proposed strategy:

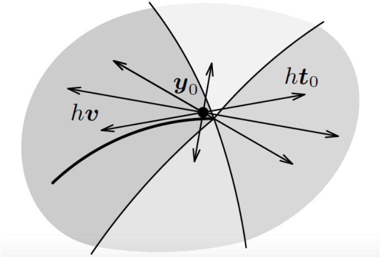
- The bifurcation detection lies on the change of sign of the determinant of

$$J(\alpha) = (1-\alpha)\begin{pmatrix} \nabla H(y_j) \\ t_j \end{pmatrix} + \alpha \begin{pmatrix} \nabla H(y_{j+1}) \\ t_{j+1} \end{pmatrix}.$$

- A bisection procedure is used to approximate $\bar{y}$, still denoting y_j and y_{j+1} the two points before and after $\bar{y}$
- Computation of tangents $\tilde{t}$ from

$$\nabla H(y_j + hv)\tilde{t} = 0, \quad \|\tilde{t}\| = 1,$$

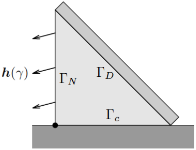
  with v some linear combinations of t_j, t_{j+1}.
- Try a continuation for each $\pm\tilde{t}$ found.



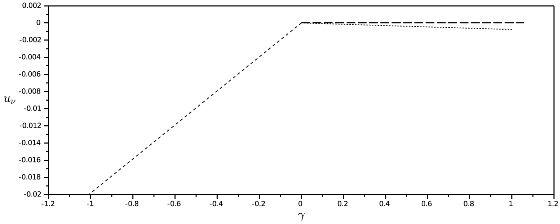
Basic example in linear elasticity

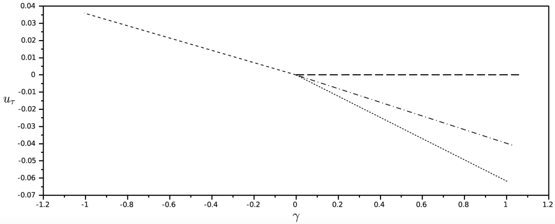
$$\lambda = 100\,GPa, \mu = 82\,GPa, \mathscr{F} = 1.7,$$

$$h(\gamma) = -\gamma(26, 7.5)\,GPa, \gamma \in [-1, 1].$$

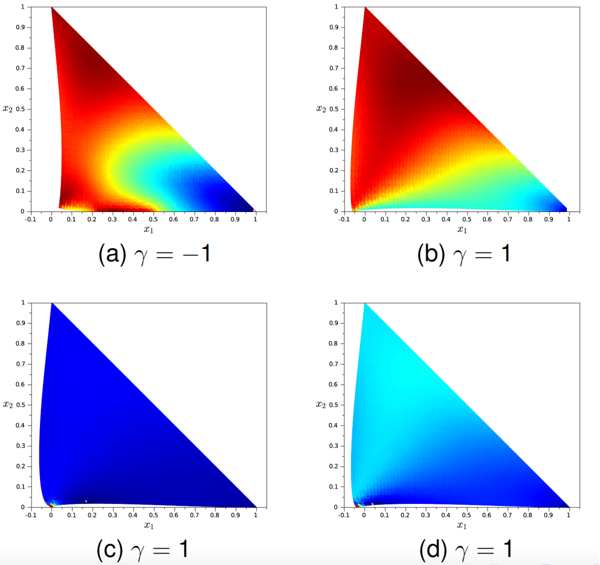


Uniform Mesh with 4096 linear element and 64 contact nodes.





Basic example in linear elasticity



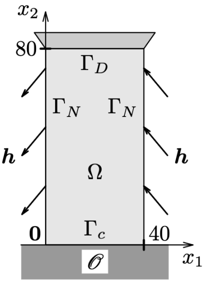
Basic example in non-linear elasticity

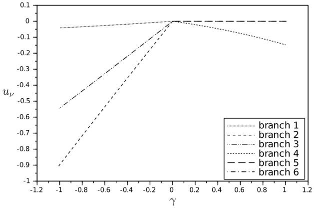
Ciarlet-Geymonat constitutive law

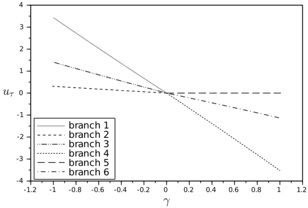
$\lambda = 4GPa, \mu = 0.12GPa, a = 0.03GPa, \mathscr{F} = 1,$

$h(\gamma) = -\gamma(2, 0.12(20 - x_1))GPa, \gamma \in [-1, 1].$

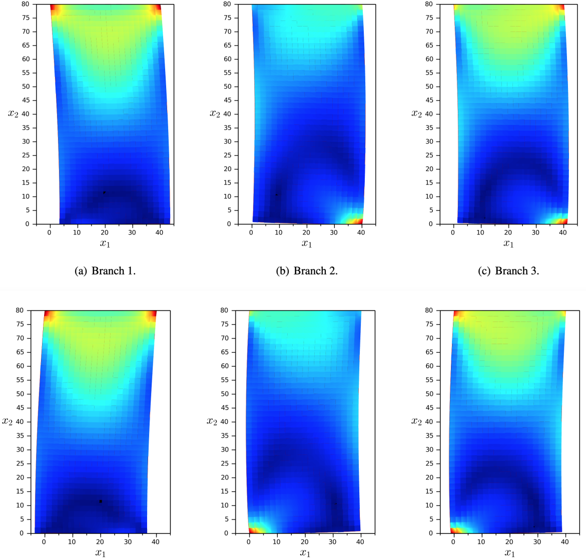
Uniform Mesh with 800 bilinear elements and 21 contact nodes.







Basic example in non-linear elasticity



(a) Branch 1. (b) Branch 2. (c) Branch 3.

(d) Branch 4. (e) Branch 5. (f) Branch 6.

5) Concluding remarks

- Friction induces non-uniqueness and non-regularities of the set of solutions for static or quasi-static problems.
- Nonsmooth point may concentrate bifurcations and turning points
- A goal : develop tools to detect/monitor/characterize these bifurcations.