Generalized Continuum with Application to Cosmology : Importance of Geometry[1]

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[1] Main reference: *Continuum Physics with Application to Cosmology, May 2026, World Scientific.*

Outline

I. Introduction and Motivation

1. Dark energy
2. Dark matter
3. Vacuum polarization

Acceleration of the Universe Expansion (1998)

1. **Experimental observations** : **Hubble** discovered that the Universe is in expansion (1929). Universe expansion is **accelerating** . (Perlmutter et al 1998, and Riess et al 1998)

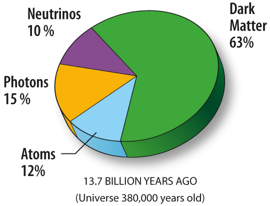
2. **Λ-CDM model** : Assuming homogeneous and isotropic Universe, and filled with barotropic fluid (density ρ and pressure p), a **constant** Λ is introduced fo. Einstein equation implies Friedmann (1922) :

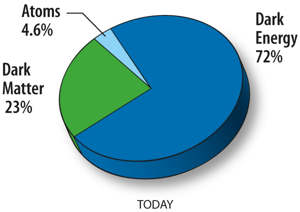
$$\begin{cases} \left(\dfrac{\dot{R}}{R}\right)^2 = \dfrac{8\pi G}{3}\rho - \dfrac{\Lambda}{3} - \dfrac{k}{R^2} \\ \dfrac{\ddot{R}}{R} = -\dfrac{4\pi G}{3}(\rho + 3p) - \dfrac{\Lambda}{3} \end{cases}$$

with Λ = cosmological constant, and equation of state $p = p(\rho)$.

3. **Hypothesis**: It is generally accepted that an unknown **Dark Energy** Λ is the cause of acceleration (e.g. Peebles & Ratra 2003).

Dark Energy & Dark Matter : Data fitting





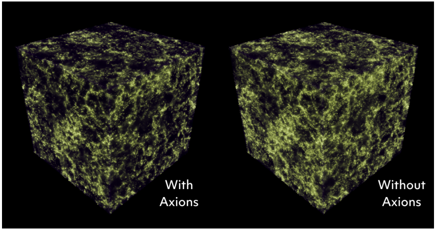
Based on Λ-**CDM and its variants**, all current research estimates that unknown **Dark Energy constitutes** $\simeq 70\%$ **of the Universe** (Abbott et al. 2025).

- **Phenomenological**: Data fittings : $\Lambda_{\rm obs} \simeq 1.09 \times 10^{-52} m^{-2}$
- **Quantum Physics**: Calculus from quantum vacuum: $\dfrac{\Lambda_{\rm qua}}{\Lambda_{\rm obs}} \geq 10^{120}$
- **Other approaches**: Electromagnetic, Spacetime Defects, Exotic Fluid

QUESTION 1 : Could the geometric structure of spacetime provide indications for Dark Energy nature?

Dark Matter and Axion Model

1. **Axion** : (Hypothetic) particle to provide **Dark Matter (DM)** $\simeq 25\%$ assumed to "hold" the Universe. Density $\simeq$, 1 proton / few cm^3.
2. **Dark matter vs. Clumpiness** : Numerical simulation by (Rogers et al 2023) https://www.artsci.utoronto.ca/news/astronomers-discover-new-link-between-dark-matter-and-clumpiness-universe



3. **Origin** : Axions and Axion-Like-Particles (ALP) are thought to come from Active Galactic Nuclei (e.g. Malyshev et al 2025).

QUESTION 2 : **Axions and ALP are leading candidates for Dark Matter: What would be their geometrical roots ?**

Vacuum Polarization

Vacuum birefringence : Most relevant results on **vacuum birefringence** were highlighted by studying the **light** emitted by the very dense and strongly **magnetic neutron star** RX J1856.5-3754, (Mignani et al 2017).

QED : Ligth-(Strong) Magnetic field interaction is not reachable with classical Maxwell electromagnetism in **vacuum**. Light-by-light scattering was predicted by **Quantum ElectroDynamics** nearly 90 years ago.

QUESTION 3 : How about electromagnetism within GCM as vacuum model without using QED ?

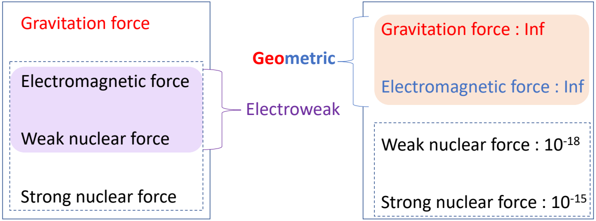
Goal of the present work

To develop a Generalized Continuum Model (GCM) in an attempt to provide answers to these three questions.

- **GCM** is a manifold $\mathscr{M}$ compacted and connected (Whyburn 1935), based on an action:

$$\mathscr{S}_{GCM} = \int_{\mathscr{M}} \mathscr{L}\,(\mathrm{Geometry}, \mathrm{Physics})\ dv$$

- **Physics in vacuum**: Consider only **actions-at-a-distance**: **Gravitation** and **Electromagnetism**.



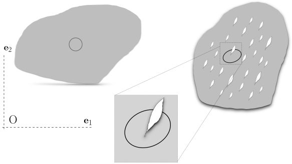
II. Generalized Continuum Model as Vacuum Spacetime

Nucleation of defects in the Universe:

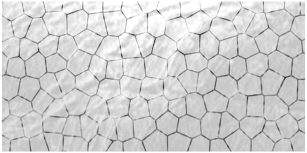
- **Universe cooling** (light release) during early (dense) phase of inflation creates **mismatches** at higher density (e.g. Ivanov & Wellenzohn 2016, Katanaev 2021) & **Symmetry breaking** of the continuum background (e.g. Fujikura et al 2025).
- **Types of defects** are mainly point, **line**, and **surface** defects corresponding to non-metricity, **torsion**, and **curvature** respectively (e.g. Puntigam & Soleng 1997).
- **Contemporary tendency:** Suggestion of "Crystal spacetime" ! (e.g. Wilczek 2012, Sacha & Zakrevski 2018, Moon et al 2025, Mäkinen et al 2025, Zhao et al 2025)

Vacuum Spacetime Model (1)

GCM $\mathscr{M}$: allows us to model distributions of discontinuity - **dislocations** and **disclinations** - (R 1998, Kleinert 1999) [2]



$\longrightarrow$



GCM = Set of Connected Microcosms (Gonseth 1929).

- **Transport of $\mathbf{f}_\beta$ along $d\mathbf{x}$**: (linear with respect to $d\mathbf{x} := dx^\alpha \mathbf{f}_\alpha$)

$$\mathbf{f}_\beta(\mathbf{x} + d\mathbf{x}) - \mathbf{f}_\beta(\mathbf{x}) := \nabla_{d\mathbf{x}} \mathbf{f}_\beta = \nabla_{\mathbf{f}_\alpha} \mathbf{f}_\beta \; dx^\alpha$$

- **Connection coefficients**: Projecting onto $\mathbf{f}^\gamma$: $\Gamma^\gamma_{\alpha\beta} := \mathbf{f}^\gamma \left(\nabla_{\mathbf{f}_\alpha} \mathbf{f}_\beta \right)$.

[2] **Planck-Kleinert Crystal** with characteristic length scale $\ell_P = 10^{-35}[m]$

GCM is a **Riemann-Cartan** manifold $(\mathscr{M}, \mathbf{g}, \nabla)$ with:

- **Metric $g_{\alpha\beta}$** (satisfying Hilbert causality: locally $[+,-,-,-]$) measures length, angle and distance,
- **Connection $\Gamma^{\lambda}_{\alpha\beta}$** measures discontinuity of (scalar & vector) fields:
  1. **Torsion**: $\aleph^{\lambda}_{\alpha\beta} := \Gamma^{\lambda}_{\alpha\beta} - \Gamma^{\lambda}_{\beta\alpha}$ $\quad (\text{Here}: \ \aleph^{\lambda}_{\alpha\beta} = \aleph_{\alpha}\delta^{\lambda}_{\beta} - \aleph_{\beta}\delta^{\lambda}_{\alpha})$
  2. **Curvature** : $\Re^{\gamma}_{\alpha\beta\lambda} = (\partial_{\alpha}\Gamma^{\gamma}_{\beta\lambda} + \Gamma^{\gamma}_{\alpha\mu}\,\Gamma^{\mu}_{\beta\lambda}) - (\partial_{\beta}\Gamma^{\gamma}_{\alpha\lambda} + \Gamma^{\gamma}_{\beta\mu}\,\Gamma^{\mu}_{\alpha\lambda})$

GCM is **not Euclidean** as the Cauchy Continuum (symmetric stress and first-gradient model). Other GCM examples exist (e.g. Maugin 2011):

1. **Cosserat** continuum (rigid micro-structure)
2. **Micromorphic** continuum (deformable micro-structure)
3. **Higher-gradient** and **Nonlocal** models ...

Classical: Relative gravitation is defined by **Einstein-Hilbert** action:

$$\mathscr{S}_{EH} := \frac{1}{2\chi} \int_{\mathscr{M}} \overline{\mathcal{R}} \, d\overline{v}$$

Overline means that torsion is equal to zero $\aleph_\alpha \equiv 0$!

GCM: Gravitation action is slightly extended with $\aleph_\alpha \neq 0$ to obtain:

1. **Einstein-Hilbert-Palatini** action (1925)

$$\boxed{\mathscr{S}_{EHP} := \frac{1}{2\chi} \int_{\mathscr{M}} \mathcal{R} \, dv} \quad \text{with} \quad \mathcal{R} := \mathbf{g}^{\beta\lambda} \, \Re_{\beta\lambda} = \mathbf{g}^{\beta\lambda} \delta_\gamma^\alpha \, \Re_{\alpha\beta\lambda}^\gamma$$

2. **Volume-element** dv is necessary for GCM (e.g. Saa 1995, R 2026)

$$dv = e^{\vartheta(\tau)} \, d\overline{v}, \quad \tau := \mathbf{g}^{\mu\nu} \, \aleph_\mu \aleph_\nu$$

with the Riemann volume-element $d\overline{v} := \sqrt{|\mathrm{Det}\mathbf{g}|} \; dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3$.

Electromagnetism (1) : Standard Yang-Mills action

3D formulation for **flat vacuum spacetime**:

$$\left\{ \begin{array}{lcl} \mathbf{D} & = & \epsilon_0 \; \mathbf{E} \\ \mathbf{H} & = & (1/\mu_0)\, \mathbf{B} \end{array} \right. \longrightarrow \mathscr{S}_{EM} := \frac{1}{2} \int_{\mathcal{V}} (\mathbf{D}.\mathbf{E} - \mathbf{B}.\mathbf{H})\, d\mathcal{V}$$

Electric **D** (induction), **E** (field) & Magnetic **B** (induction), **H** (field).

GCM 4D formulation with Faraday strength $\boxed{\mathcal{F}_{\mu\nu} := \nabla_\mu A_\nu - \nabla_\nu A_\mu}$, where A_μ **electromagnetic potential** & Excitation tensor: $\mathcal{H}^{\mu\nu} = \mathbb{H}^{\mu\nu\alpha\beta}\; \mathcal{F}_{\alpha\beta}$.

$$\begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B^3 & -B^2 \\ E_2 & -B^3 & 0 & B^1 \\ E_3 & B^2 & -B^1 & 0 \end{pmatrix} \to \mathcal{F}_{\mu\nu}, \begin{pmatrix} 0 & D^1 & D^2 & D^3 \\ -D^1 & 0 & H_3 & -H_2 \\ -D^2 & -H_3 & 0 & H_1 \\ -D^3 & H_2 & -H_1 & 0 \end{pmatrix} \to \mathcal{H}^{\mu\nu}$$

Standard Yang-Mills action (1953) defines the electromagnetic field:

$$\boxed{\mathscr{S}_{YM} := -\frac{1}{4} \int_{\mathscr{B}} \mathcal{H}^{\mu\nu} \mathcal{F}_{\mu\nu}\; dv \leftarrow \mathscr{S}_{EM}}$$

Electromagnetism (2) : Dual Yang-Mills action

4D formulation with dual Faraday & Excitation (Hodge operator + **g**-Dual):

$$\begin{pmatrix} 0 & H_1 & H_2 & H_3 \\ -H_1 & 0 & D^3 & -D^2 \\ -H_2 & -D^3 & 0 & D^1 \\ -H_3 & D^2 & -D^1 & 0 \end{pmatrix} \to \tilde{\mathcal{F}}_{\mu\nu}, \quad \begin{pmatrix} 0 & -B^1 & -B^2 & -B^3 \\ B^1 & 0 & E_3 & -E_2 \\ B^2 & -E_3 & 0 & E_1 \\ B^3 & E_2 & -E_1 & 0 \end{pmatrix} \to \tilde{\mathcal{H}}^{\mu\nu}$$

Dual Yang-Mills action (Bliokh et al 2013) defines the electromagnetic field:

$$\boxed{\tilde{\mathscr{S}}_{YM} := \frac{1}{4}\int_{\mathscr{B}} \tilde{\mathcal{H}}^{\mu\nu}\tilde{\mathcal{F}}_{\mu\nu}\, dv \leftarrow \mathscr{S}_{EM}}, \quad \tilde{\mathcal{F}}_{\mu\nu} := \nabla_\mu \tilde{A}_\nu - \nabla_\nu \tilde{A}_\mu$$

$\tilde{A}_\mu$: **dual electromagnetic potential** and $\tilde{\mathcal{H}}^{\mu\nu} = \tilde{\mathbb{H}}^{\mu\nu\alpha\beta}\ \tilde{\mathcal{F}}_{\alpha\beta}$.

Definition

Vacuum Spacetime action (GCM) is defined by $\mathscr{S}_{GCM}$ (R 2026):

$$\boxed{\mathscr{S}_{GCM} := \mathscr{S}_{EHP} + \frac{1}{2}\left(\mathscr{S}_{YM} + \tilde{\mathscr{S}}_{YM}\right)}$$

Field Equations are obtained by its variation:

$$\delta\mathscr{S}_{GCM} = 0, \quad \forall\ \delta g^{\mu\nu}, \forall\ \delta\aleph_\nu, \forall\ \delta A_\nu, \forall\ \delta\tilde{A}_\nu$$

Proposition

For a GCM $\mathscr{M}$ with action $\mathscr{S}_{GCM}$, coupled **field equations (4 Rows)** *are deduced, owing that $dv = e^{\vartheta(\tau)}\ d\overline{v}$,:*

$$\begin{cases} \frac{1}{2\chi} G_{\mu\nu} + T^{\mathrm{em}}_{\mu\nu} + \mathscr{L}_{GCM}\ \vartheta'(\tau)\, \aleph_\mu \aleph_\nu & = & 0 & (1) \\ \frac{1}{2\chi}\mathcal{D}^\mu + \frac{1}{2}\left(\mathcal{H}^{\mu\nu} A_\nu - \tilde{\mathcal{H}}^{\mu\nu}\tilde{A}_\nu\right) + \frac{2}{3}\mathscr{L}_{GCM}\ \vartheta'(\tau)\, \mathcal{D}^\mu & = & 0 & (2) \\ \nabla_\mu \mathcal{H}^{\mu\nu} & = & 0 & (3) \\ -\nabla_\mu \tilde{\mathcal{H}}^{\mu\nu} & = & 0 & (4) \end{cases}$$

with the **Einstein tensor** *and* **distortion vector***:*

$$G_{\mu\nu} := \mathcal{R}_{\mu\nu} - (\mathcal{R}/2)\ g_{\mu\nu}, \quad \mathcal{D}^\mu := 3 g^{\mu\nu} \aleph_\nu,$$

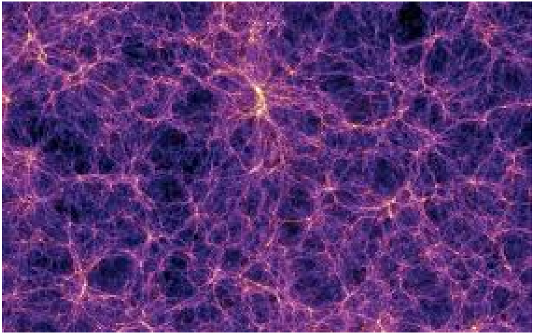
with Einstein-Maxwell energy-momentum:

$$T^{\mathrm{em}}_{\mu\nu} := (1/8)\mathcal{H}^{\alpha\beta}\mathcal{F}_{\alpha\beta}\ g_{\mu\nu} + (1/4)\mathcal{H}^{\lambda\rho}\left(g_{\mu\lambda}\mathcal{F}_{\rho\nu} + \mathcal{F}_{\mu\lambda} g_{\rho\nu}\right)$$

Proof: The variation $\delta\mathscr{S}_{GCM} := \int_{\mathscr{M}} \delta\mathscr{L}\ dv + \mathscr{L}\ \delta dv = 0$ induces

$$\delta\mathscr{S}_{GCM} = \int_{\mathscr{M}} \left(\mathbb{E}_{\mu\nu}\ \delta g^{\mu\nu} + \mathbb{L}^{\mu\nu}_{\nu}\ \delta\aleph_\mu + \mathbb{M}^{\nu}\ \delta A_\nu + \tilde{\mathbb{M}}^{\nu}\ \delta\tilde{A}_\nu\right) dv + \underbrace{\mathrm{B.\ Term}}_{=0} = 0 \quad \square$$

III a. Application to Dark Energy



Cosmic web : Clusters of galaxies via filaments (e.g. **Vernstrom et al. 2021** "Discovery of **Magnetic Fields** Along Stacked Cosmic Filaments as Revealed by Radio and X-Ray Emission").

Einstein-(Cartan)-Maxwell equations

For a **swift analysis**, for very small $\vartheta'(\tau) \simeq 0$ and $\overline{\nabla}\aleph \simeq 0$.

- **GCM** : From **Row 1**, we deduce:

$$\boxed{\frac{1}{2\chi}\left(\overline{G}_{\mu\nu} + \Lambda_{\aleph}\, g_{\mu\nu} + 2\,\aleph_\mu \aleph_\nu\right) + T^{\mathrm{em}}_{\mu\nu} = 0}$$

where $\Lambda_{\aleph} := g^{\alpha\beta}\, \aleph_\alpha \aleph_\beta$ is a **Dark Energy Field** **deduced** from torsion !

- **Classical** : Remind the **Einstein-Maxwell** equation for Λ-CDM models:

$$\frac{1}{2\chi}\left(\overline{G}_{\mu\nu} + \Lambda\, g_{\mu\nu}\right) + T^{\mathrm{em}}_{\mu\nu} = 0$$

where Λ is a constant **introduced** as **Dark Energy** *ad hoc*.

Remark

1. $\Lambda_{\aleph} < 0$ *acts as* **Dark Energy Field** *for* **space-like** *torsion.*
2. *Tensor* $2\,\aleph_\mu \aleph_\nu$ *creates* **local anisotropy** *of the vacuum.*
3. *No need to assume ad hoc cosmological constant* Λ.

Electromagnetic source of Dark Energy

1. **Torsion and EM field** : From **Row 2**, we deduce:

$$\boxed{\aleph_\alpha = -\frac{1}{3}\chi \, g_{\mu\alpha}\left(\mathcal{H}^{\mu\nu}A_\nu - \tilde{\mathcal{H}}^{\mu\nu}\tilde{A}_\nu\right)}$$

Only known physical constants χ, ϵ_0 and μ_0 are involved !

2. **Physical interpretation** : **Chern-Simons current** is better preserved than electromagnetic energy over the time:

$$\frac{1}{2}\left(\mathcal{H}^{\mu\nu}A_\nu - \tilde{\mathcal{H}}^{\mu\nu}\tilde{A}_\nu\right) \leftarrow \frac{1}{2}\begin{pmatrix} \mathbf{B}\cdot\tilde{\mathbf{A}} - \mathbf{D}\cdot\mathbf{A} \\ \mathbf{E}\times\tilde{\mathbf{A}} + \mathbf{H}\times\mathbf{A} \end{pmatrix} \begin{array}{l} \text{Optical Helicity} \\ \text{Spin Angular Momentum} \end{array}$$

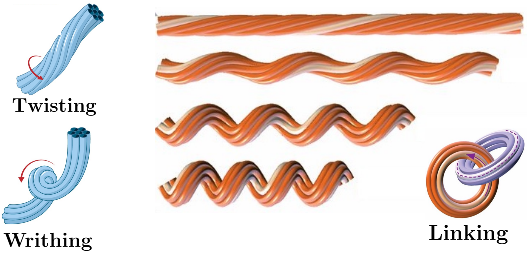
Remark

- *Optical Helicity and Spin Angular Momentum generate Torsion.*
- *Torsion may be a source of Dark Energy.*
- **Conclusion** *: Optical Helicity and SAM of electromagnetic field contribute to* **acceleration** *of Universe expansion.*

Physical interpretation of Optical Helicity

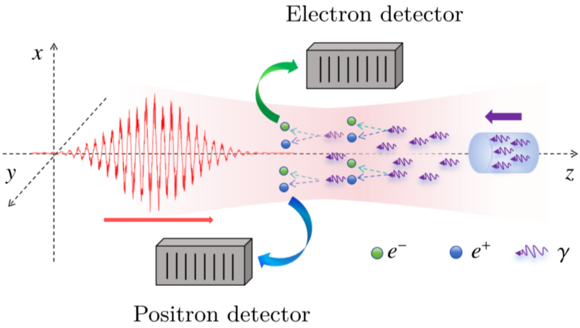
Optical Helicity $\mathcal{H}_{em}$ is the time-component ($\mu = 0$) of the Chern-Simons current for electromagnetic field.

$$\frac{1}{2}\left(\mathcal{H}^{0\nu} A_\nu - \tilde{\mathcal{H}}^{0\nu} \tilde{A}_\nu\right) = \frac{1}{2}\left(\mathbf{B} \cdot \tilde{\mathbf{A}} - \mathbf{D} \cdot \mathbf{A}\right) := \mathcal{H}_{em}$$



Topological quantity $\mathcal{H}_{em}$ measures **twisting**, shearing, and **writhing**, potentially leading to **linking** (knots = particles ?) of electromagnetic tube (**Moreau 1961**, Afanasiev & Stepanovsky 1996, Arrays et al 2017, Berger 1984, McKinnon et al 2025, ...). $\simeq$ On Vortex Atoms (Kelvin 1867)

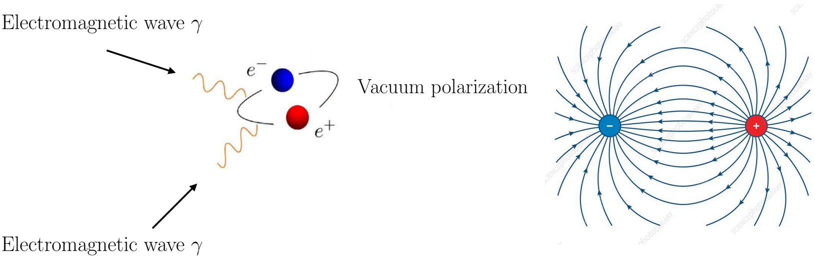
III b. Application to Vacuum Polarization



Schematic of nonlinear Breit–Wheeler pair production driven by a quantum squeezed-light pulse (Ge et al 2026).

Vacuum Polarization

Breit-Wheeler process (1934) : Collision of 2 high-energy photons (γ rays) **split virtual particles** of the vacuum - QED framework - (Adam et al. *Phy Rev Lett* 2021, $>$ 300 authors).



Vacuum Polarization : Formation of **electric dipoles** $\left[e^- \quad e^+\right]$ affects surrounding electromagnetic field, leading to **vacuum polarization**.[3]

[3] This cannot be described by classical Maxwell electromagnetism.

Semi-classical approach of vacuum polarization

- **Semi-classical model**: Extension of vacuum model by **nonlinear electrodynamics** (QED) (e.g. Euler & Köckel 1936, Weisskopf 1936):

$$\mathscr{S}_{EKHW} = \int_{\mathscr{E}} \mathscr{L}_{EKHW} \, dV, \quad \mathscr{L}_{EKHW} = \mathscr{L}_{YM} + \left(\alpha_0 \; \mathscr{L}_{YM}^2 + \beta_0 \; \mathscr{L}_2^2\right) + \cdots$$

with the **two invariants** (in terms of primal variables, 3D formulation):

$$\begin{aligned} \mathscr{L}_{YM} &:= -\frac{1}{4}\mathcal{H}^{\mu\nu}\mathcal{F}_{\mu\nu} = \frac{1}{2}\left(\epsilon_0 \mathbf{E}\cdot\mathbf{E} - \frac{\mathbf{B}}{\mu_0}\cdot\mathbf{B}\right) \\ \mathscr{L}_2 &:= \frac{1}{4}\tilde{\mathcal{H}}^{\mu\nu}\mathcal{F}_{\mu\nu} = \mathbf{B}\cdot\mathbf{E} \end{aligned}$$

where α_0 and β_0 are constants from **quantum physics** $(m_e, e, \hbar, c, \alpha)$

- **Polarizations** induced by $\mathscr{L}_{EKHW}$ derivatives (e.g. Zavattini et al 2022):

$$\left\{\begin{aligned} \mathbf{D}_{\mathcal{P}} &= \epsilon_0 \mathbf{E} + \mathbf{P} \\ \mathbf{H}_{\mathcal{P}} &= \frac{\mathbf{B}}{\mu_0} - \mathbf{M} \end{aligned}\right. \qquad \boxed{\left\{\begin{aligned} \mathbf{P} &= \epsilon_0\alpha_0\left(\epsilon_0 E^2 - \frac{B^2}{\mu_0}\right)\mathbf{E} + 2\beta_0\left(\mathbf{B}\cdot\mathbf{E}\right)\mathbf{B} \\ \mathbf{M} &= -\frac{\alpha_0}{\mu_0}\left(\epsilon_0 E^2 - \frac{B^2}{\mu_0}\right)\mathbf{B} + 2\beta_0\left(\mathbf{B}\cdot\mathbf{E}\right)\mathbf{E} \end{aligned}\right.}$$

Maxwell equations in GCM

- **Maxwell equations in GCM** are obtained from **Rows 3 & 4** (the same for $\tilde{A}^\mu$) (R 2018, 2026)

$$\boxed{-g^{\alpha\beta}\nabla_\alpha\nabla_\beta A^\mu + g^{\mu\alpha}\nabla_\alpha\nabla_\beta A^\beta - g^{\mu\alpha}\aleph^\gamma_{\beta\alpha}\nabla_\gamma A^\beta + g^{\mu\alpha}\Re_{\alpha\beta}A^\beta = 0}$$

1. 1^{st} term : Classical wave propagation (Maxwell in flat vacuum)
2. 2^{nd} term : Lorenz gauge (possibly magnetic monopoles)
3. 3^{rd} term : Torsion crosses wave : Polarizations ?
4. 4^{th} term : Curvature bends wave (Gravitational Lensing 1919).

- **Pointing out polarizations** is cumbersome with Maxwell equations in GCM. **Instead we re-write the GCM action !**

Polarizations in GCM (vacuum)

Faraday strength in GCM is decomposed as:

$$\mathcal{F}_{\mu\nu} := \nabla_\mu A_\nu - \nabla_\nu A_\mu = \overline{\mathcal{F}}_{\mu\nu} + (\aleph_\mu A_\nu - \aleph_\nu A_\mu)$$

Electric polarization is pointed out from constitutive equations:

$$\mathcal{H}^{\mu\nu} = \mathbb{H}^{\mu\nu\alpha\beta}\ \mathcal{F}_{\alpha\beta} = \overline{\mathcal{H}}^{\mu\nu} + \mathbb{H}^{\mu\nu\alpha\beta}(\aleph_\alpha A_\beta - \aleph_\beta A_\alpha) = \overline{\mathcal{H}}^{\mu\nu} + \mathcal{P}^{\mu\nu}$$

Magnetic polarization is obtained from: $\tilde{\mathcal{F}}_{\mu\nu} = \overline{\tilde{\mathcal{F}}}_{\mu\nu} + (\aleph_\mu \tilde{A}_\nu - \aleph_\nu \tilde{A}_\mu)$

$$\tilde{\mathcal{H}}^{\mu\nu} = \overline{\tilde{\mathcal{H}}}^{\mu\nu} + \tilde{\mathcal{M}}^{\mu\nu}, \quad \text{with} \quad \tilde{\mathcal{M}}^{\mu\nu} := \tilde{\mathbb{H}}^{\mu\nu\alpha\beta}\left(\aleph_\alpha \tilde{A}_\beta - \aleph_\beta \tilde{A}_\alpha\right)$$

Electric and magnetic polarizations expressed in 4D are generated by the **geometric structure** of vacuum $\mathscr{M}$:

$$\boxed{\left\{\begin{array}{lcl} \mathcal{P}^{\mu\nu} & = & \epsilon_0\,(\aleph^\mu\ A^\nu - \aleph_\nu\ A_\mu) \\ \tilde{\mathcal{M}}^{\mu\nu} & = & \dfrac{1}{\mu_0}\left(\aleph^\mu\ \tilde{A}^\nu - \aleph_\nu\ \tilde{A}_\mu\right) \end{array}\right.} \quad \leftarrow \quad \begin{pmatrix} \mathbf{P} \\ \\ \mathbf{M} \end{pmatrix}$$

Dark Matter and Polarization in GCM

Instead of torsion-vector $\aleph_\mu$, use of **axion $\phi := e^{\vartheta(\Lambda_\aleph)}$** ($\simeq$ cosmic string) is sometimes more familiar among cosmologists (e.g. Dror et al 2021). It is observed that a **link exists between them** (e.g. Gao et al 2025).

Proposition

Let $\mathscr{M}$ be an Einstein-Cartan vacuum. The action (integration on $\mathscr{B}$) can be re-written to give, with volume-element $dv := e^{\vartheta(\Lambda_\aleph)}\, d\overline{v}$, :

$$
\begin{aligned}
\mathscr{S}_{GCM} \;=\; & \int_{\mathscr{B}} \left[\frac{1}{2\chi} \left(\overline{\mathcal{R}} - 6\, g^{\mu\nu} \frac{\nabla_\mu\phi\nabla_\nu\phi}{\phi^2} \right) - \frac{1}{8} \left(\overline{\mathcal{H}}^{\mu\nu} \overline{\mathcal{F}}_{\mu\nu} - \tilde{\overline{\mathcal{H}}}^{\mu\nu} \tilde{\overline{\mathcal{F}}}_{\mu\nu} \right) \right] \phi\, d\overline{v} \\
& - \frac{1}{4} \int_{\mathscr{B}} \left[\overline{\mathcal{H}}^{\mu\nu} (A_\nu \nabla_\mu\phi - A_\mu \nabla_\nu\phi) - \tilde{\overline{\mathcal{H}}}^{\mu\nu} (\tilde{A}_\nu \nabla_\mu\phi - \tilde{A}_\mu \nabla_\nu\phi) \right] d\overline{v} \\
& - \frac{1}{8} \int_{\mathscr{B}} \left[\overline{\mathcal{P}}^{\mu\nu} (A_\nu \nabla_\mu\phi - A_\mu \nabla_\nu\phi) - \tilde{\overline{\mathcal{M}}}^{\mu\nu} (\tilde{A}_\nu \nabla_\mu\phi - \tilde{A}_\mu \nabla_\nu\phi) \right] d\overline{v}
\end{aligned}
$$

Unknowns: *metric $g_{\mu\nu}$, axion ϕ, potentials A_μ and $\tilde{A}_\mu$.*

Proof: Start with $\mathscr{S}_{GCM}$ and proceed to change of variables $\square$.

Comments on Vacuum Polarization

1. **GCM vacuum $\mathscr{S}_{GCM}$** includes the **metric-scalar gravitation** (+ electromagnetism) in curved spacetime (Brans & Dicke 1961, Horndeski 1974, Kobayashi 2019, Dournac 2024) and additionnally (incomplete) **axions** for Dark Matter. Nowadays, **axion concept** is the leader candidate for Dark Matter ...

2. **Semi-classical vacuum $\mathscr{S}_{EKHW}$** (with QED) of Euler, Köckel, Heisenberg (1936) and Weisskopf (1936) $\mathscr{S}_{EKHW}$ **was designed** to include **Electric and Magnetic Polarization** by means of **nonlinear electrodynamics** (e.g. Ahmadiniaz et al 2025, Zhang et al, 2025).[4]

3. **GCM vacuum $\mathscr{L}_{GCM}$** **geometrically includes** by construction **AN Electric and Magnetic Polarization** engendered by **torsion** ($\simeq$ helicity and spin angular momentum of electromagnetic waves and appropriate **volume-element** $dv := \phi \, d\overline{v}$).

[4] QED cannot include gravitation by itself then $\mathscr{L}_{EH}$ should be added.

IV. Concluding Remarks and Outlook

1. **Dark Energy Field** (ex: $\Lambda_{\aleph} := g^{\mu\nu}\aleph_{\mu}\aleph_{\nu}$) is proposed (instead of cosmological constant Λ) by using a GCM with torsion, identified as **optical helicity** and **spin angular momentum** of magnetized Universe.

2. **Electric** and **Magnetic Polarization** are intrinsically included in **GCM** (as does the semi-classical QED). GCM also recovers **Horndeski** gravitation (**with axion** ϕ for **Dark Matter**).

3. **Outlook**: Despite promising insights, this model is **to be completed**.
   1. An immediate **next step** would be the **matching of polarization tensors** for the two models $\mathscr{S}_{EKHW}$ and $\mathscr{S}_{GCM}$?
   2. It would be worth to address the three aspects (DE, DM, VP) in relation with **experimental data** (namely about the continuum limit - Ehlers -) in the long term ....

Thank you for your attention !

But then again, maybe the axion does'nt exist ! (*Cosmology of axion dark matter*, Ciaran AJ O'Hare, Lecce, Italy, 179 pages, 2024)[5]

[5] Dare I say that everything is **geometry** ?