# Differential Complexes for Displacement and Stress Potential Problems

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# Introduction

We are going to talk about the displacement and stress potentials problems.

We propose to:

- revisit and reformulate these problems
- establish necessary and sufficient integrability/compatibility conditions
- explicitly calculate solutions
- formulate the dual of the Elasticity complex, providing a natural formulation of Airy and Beltrami potentials

Corner stone of these reformulations: generalized differential complexes.

# Outline

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# The diplacement problem

**Context:** Infinitesimal strain

**Compatibility**: A strain $\varepsilon$ is said to be compatible if it is induced by a displacement vector field $\boldsymbol{\xi}$

$$\boldsymbol{\varepsilon} = \frac{1}{2}\left(\nabla\boldsymbol{\xi} + (\nabla\boldsymbol{\xi})^t\right).$$

**Displacement problem:** Given a strain $\varepsilon$

- find a compatibility condition.
- calculate $\xi$ form the data of a compatible $\varepsilon$.

# The Saint-Venant tensor [Cesàro-Volterra 1906]

Built $\boldsymbol{\xi}$ at $\mathbf{x} \in \Omega \subset \mathbb{R}^3$ using the formula

$$\boldsymbol{\xi}(\mathbf{x}) = \boldsymbol{\xi}(\mathbf{x}_0) + \int_{\mathbf{x}_t \in \mathcal{C}} \underbrace{\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\xi}(\mathbf{x}_t)}_{\text{depends only on } \boldsymbol{\varepsilon}} \longrightarrow \text{must not depend on } \mathcal{C}$$


- **Compatibility condition:**

taking $\mathcal{C}$ as a closed path

$\Downarrow$

the integral vanishes

$\Downarrow$

Saint-Venant 4-tensor $\mathbf{W}$

$$\boxed{\mathbf{W}_{ijkl} := \partial_l \partial_k \varepsilon_{ij} - \partial_l \partial_i \varepsilon_{jk} + \partial_i \partial_j \varepsilon_{kl} - \partial_k \partial_j \varepsilon_{il} = 0}$$

- **Local Integrator (convex subset):** Then, taking for $\mathcal{C}$ the straight line $\mathbf{x}_t = t\mathbf{x} \subset \Omega$

$$\boxed{\boldsymbol{\xi}(\mathbf{x}) = \boldsymbol{\xi}(0) + \boldsymbol{\Omega}(0)\mathbf{x} + \int_0^1 \boldsymbol{\varepsilon}(t\mathbf{x})\mathbf{x}dt + \int_0^1 (1-t)\left[\nabla\boldsymbol{\varepsilon}(t\mathbf{x}) : (\mathbf{x} \otimes \mathbf{x}) - (\mathbf{x} \otimes \mathbf{x}) : \nabla\boldsymbol{\varepsilon}(t\mathbf{x})\right] dt}$$

where $\boldsymbol{\xi}(0) + \boldsymbol{\Omega}(0)\mathbf{x}$ is an element of $\mathfrak{se}_3 := \left\{\boldsymbol{\xi} + \boldsymbol{\Omega}\mathbf{x} \mid \boldsymbol{\xi} \in \mathbb{R}^3, \ \boldsymbol{\Omega} \in \mathfrak{so}_3(\mathbb{R})\right\}$ (rigid motion).

## ... a lot of other incompatibility tensors

Linearizing the deformation gradient, we get

$$\mathbf{F} = \mathbf{Id}_3 + \underbrace{\boldsymbol{\varepsilon}}_{\text{strain}} + \underbrace{\boldsymbol{\omega} \wedge \bullet}_{\text{vorticity}},$$

- Taking $\mathbf{rot} \circ \mathbf{rot}^t$, it remains:

$$\underbrace{\mathbf{rot}\,\mathbf{rot}^t\,\mathbf{F}}_{=0 \text{ if icompatibility}} = \underbrace{\mathbf{rot}\,\mathbf{rot}^t\,\boldsymbol{\varepsilon}}_{:=\mathbf{Inc}}.$$

- Taking $\mathrm{tr} \circ \mathbf{rot}^t$, it remains:

$$\underbrace{\mathrm{tr}\,\mathbf{rot}^t\,\mathbf{F}}_{=0 \text{ if incompatibility}} = 3 \underbrace{\mathbf{div}\,\boldsymbol{\omega}}_{\text{another one}}$$

**But:** ad hoc procedure, only necessary conditions and many different tensors!

## Outline

# Construction of De Rham complex

To understand the elasticity complex, we need to understand more deeply the construction of the De Rham complex.

## De Rham 1-complex

A graded sequence of differential forms spaces (alternate tensor fields):

$$\begin{array}{ccccccccc}
\Omega_1^0(\mathbb{R}^3) & \xrightarrow{\mathbf{grad}} & \Omega_1^1(\mathbb{R}^3) & \xrightarrow{\mathbf{rot}} & \Omega_1^2(\mathbb{R}^3) & \xrightarrow{\mathbf{div}} & \Omega_1^3(\mathbb{R}^3) & \longrightarrow & \{0\} \\
f & \longmapsto & \alpha & \longmapsto & \beta & \longmapsto & \omega & \longmapsto & 0 \\
\bullet & \longrightarrow & \boxed{1} & \longrightarrow & \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array} & \longrightarrow & \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline\end{array} & \longrightarrow & \bullet
\end{array}$$

Explicit calculations:

- $\alpha_i = \partial_i f$
- $\beta_{ij} = \partial_i \alpha_j - \partial_j \alpha_i$
- $\omega_{ijk} = \partial_k \beta_{ij} - \partial_i \beta_{kj} - \partial_j \beta_{ik} + \partial_i \beta_{jk} + \partial_j \beta_{ki} - \partial_k \beta_{ji} = \partial_k \beta_{ij} + \partial_j \beta_{ki} + \partial_i \beta_{jk}$

Fundamental property:

$$\begin{cases} \mathbf{rot} \circ \mathbf{grad} = 0 \\ \mathbf{div} \circ \mathbf{rot} = 0 \end{cases}$$

# From De Rham complex to the elasticity complex

**De Rham complex for alternate forms**

$\Downarrow$

but elasticity theorie has other indicials symmetries ($\xi$, $\varepsilon$, $\mathbf{W}$)

$\Downarrow$

Dubois-Violette [2002] generalizes De Rham's theory (other indicial symmetries)

$\Downarrow$

produces a linear elasticity complex naturaly provided by symmetries

# A differential complex for elasticity

**Elasticity 2-complex:** naturally adapted to the symmetries of elasticity tensors.

- Dubois-Violette theory on $\mathbb{R}^3$:

$$\begin{array}{ccccc} \Omega_2^1(\mathbb{R}^3) & \xrightarrow{\mathrm{d}_1 = D_1} & \Omega_2^2(\mathbb{R}^3) & \xrightarrow{\mathrm{d}_2\,\mathrm{d}_3 = D_2} & \Omega_2^4(\mathbb{R}^3) \\ \boldsymbol{\xi}^\flat & \longmapsto & \boldsymbol{\varepsilon} & \longmapsto & \mathbf{W} \\ \text{(displacement)} & & \text{(strain)} & & \text{(Saint-Venant)} \end{array} \qquad D_2 D_1 = \mathrm{d}_3\,\mathrm{d}_2\,\mathrm{d}_1 = 0.$$

- Differentials operators defined by Young symmetrization:

$$\boxed{i \mid j} \qquad (D_1\boldsymbol{\xi}^\flat)_{ij} = \frac{1}{2}\left(\partial_i\boldsymbol{\xi}_j + \partial_j\boldsymbol{\xi}_i\right) := \boldsymbol{\varepsilon}_{ij}$$

$$\begin{array}{|c|c|}\hline i & j \\ \hline k & l \\ \hline\end{array} \qquad (D_2\boldsymbol{\varepsilon})_{ijkl} = \frac{1}{3}\left(\partial_i\partial_j\boldsymbol{\varepsilon}_{kl} - \partial_k\partial_j\boldsymbol{\varepsilon}_{il} - \partial_i\partial_l\boldsymbol{\varepsilon}_{kj} + \partial_k\partial_l\boldsymbol{\varepsilon}_{ij}\right) := \frac{1}{3}\mathbf{W}_{ijkl}$$

**Proposition (Saint-Venant compatibility condition)**

*If the strain $\varepsilon$ is generated by a displacement $\boldsymbol{\xi}^\flat$ we have:*

$$\mathbf{W} := D_2\boldsymbol{\varepsilon} = D_2 D_1 \boldsymbol{\xi}^\flat = 0.$$

*In coordinates:* $W_{ijkl} = \partial_i\partial_j\varepsilon_{kl} - \partial_k\partial_j\varepsilon_{il} - \partial_i\partial_l\varepsilon_{kj} + \partial_k\partial_l\varepsilon_{ij} = 0.$

# Outline

1. The displacement problem
   - Elasticity complex
   - Calculation of integrated elasticity law
   - Link between Saint-Venant and incompatibility tensors

2. What is new in my work? Natural formulation for stress potentials
   - Dual elasticity complex
   - Reformulation of stress potentials problem

# Sufficient condition and calculation of the displacement $\xi$

**Question:** Is the compatibility condition sufficient? Can we calculate a closed form for $\xi$?

Each Dubois–Violette complex comes with an homotopy formula:

$$\underbrace{D_1 K_1 \varepsilon}_{\text{integrator}} + \underbrace{K_2 D_2 \varepsilon}_{\text{obstruction}} = \varepsilon$$

Compatibility $\Longrightarrow$ $\mathbf{W} := D_2\varepsilon = 0 \Longrightarrow D_1(K_1\varepsilon) = \varepsilon$

We shall construct integrators $K_1$, $K_2$ on a convex domain:

- **(1)** $K_1$ **local intégrator:** corresponds to the Cesàro-Volterra formula

$$K_1\varepsilon(\mathbf{x}) = \boldsymbol{\xi}(0) + \boldsymbol{\Omega}(0)\mathbf{x} + \int_0^1 \varepsilon(t\mathbf{x})\,\mathbf{x}\,dt + \int_0^1 (1-t)\left[\nabla\varepsilon(t\mathbf{x}) : (\mathbf{x}\otimes\mathbf{x}) - (\mathbf{x}\otimes\mathbf{x}) : \nabla\varepsilon(t\mathbf{x})\right] dt$$

- **(2)** $K_2$ **obstruction term:** a measure of the incompatibility degree

$$K_2\mathbf{W}(\mathbf{x}) := \int_0^1 dt \int_0^t s\mathbf{W}(s\mathbf{x}) : (\mathbf{x}\otimes\mathbf{x})\,ds$$

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## Link between $\mathbf{W} = 0$ and $\operatorname{rot}\operatorname{rot}^t \varepsilon = 0$

**What is the link between $\mathbf{Inc}$ ($2$-tensor) and $\mathbf{W}$ ($4$-tensor)?**

**Proposition (Equivalence between $\mathbf{W}$ and $\mathbf{Inc}$)**

*In dimension* $3$*:*

$$\mathbf{Inc}(\varepsilon) = -\operatorname{tr}_{12}\mathbf{W}(\varepsilon) + \frac{1}{2}\operatorname{tr}\left[\operatorname{tr}_{12}\mathbf{W}(\varepsilon)\right]\mathbf{q} \quad \text{and} \quad \underbrace{\mathbf{W}(\varepsilon) = -\mathbf{Inc}(\varepsilon) \owedge \mathbf{q} + \frac{1}{2}\operatorname{tr}\left[\mathbf{Inc}(\varepsilon)\right]\mathbf{q}\owedge\mathbf{q}}_{\textit{miracle of dimension } 3},$$

*where* $(\mathbf{a}\owedge\mathbf{b})_{ijkl} := a_{ij}b_{kl} - a_{kj}b_{il} - \left(a_{il}b_{kj} - a_{kl}b_{ij}\right)$

| | |
|---|---|
| $i$ | $j$ |
| $k$ | $l$ |

- The linear map $\mathbf{h} \longmapsto \mathbf{h}\owedge\mathbf{q}$ is injective (in all dimensions) and surjective in dimension $3$
- Given $\mathbf{W}$ there exists a unique $\mathbf{h}$ such that $\mathbf{W} = \mathbf{h}\owedge\mathbf{q}$
- Calculation of $\mathbf{h}$ by harmonic decomposition of $\mathbf{W}$.

# Outline

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# De Rham dual complex

The dual of the Elasticity complex provide a natural formulation of the Airy (2D) and Beltrami (3D) stress potentials. We need to understand more deeply the De-Rham dual complex.

## De Rham dual 1-complex

The Hoge operator

$$\star_k : \begin{array}{ccc} \Omega^k(\mathbb{R}^n) & \longrightarrow & \Omega^{n-k}(\mathbb{R}^n)^* \\ \omega_{i_1\cdots i_k} & \longmapsto & \epsilon^{i_1\cdots i_k j_1\cdots j_{n-k}}\omega_{i_1\cdots i_k} \end{array}, \qquad \epsilon^{i_1\cdots i_n} \text{ Levi-Civita symbol,}$$

allows to define the co-differential $\mathrm{d}^*$ form the exterior differential $\mathrm{d}$

$$\mathrm{d}_k^* = \star_{n-k}\,\mathrm{d}_{n-k-1}\,\star_{n-k-1}^{-1} : \Omega^{k+1}(\mathbb{R}^n)^* \longrightarrow \Omega^k(\mathbb{R}^n)^*,$$

constructing the De-Rham dual complex

$$\begin{tikzcd}
\Omega^0(\mathbb{R}^n) \arrow[r, "\mathrm{d}_0"] & \cdots \arrow[r, "\mathrm{d}_{k-1}"] & \Omega^k(\mathbb{R}^n) \arrow[r, "\mathrm{d}_k"] \arrow[drr, "\star_k"'] & \cdots \arrow[r, "\mathrm{d}_{n-k-1}"] & \Omega^{n-k}(\mathbb{R}^n) \arrow[r, "\mathrm{d}_{n-k}"] \arrow[dll, "\star_{n-k}"] & \cdots \arrow[r, "\mathrm{d}_{n-1}"] & \Omega^n(\mathbb{R}^n) \\
\Omega^0(\mathbb{R}^n)^* & \cdots \arrow[l, "\mathrm{d}_0^*"'] & \Omega^k(\mathbb{R}^n)^* \arrow[l, "\mathrm{d}_{k-1}^*"'] & \cdots \arrow[l, "\mathrm{d}_k^*"'] & \Omega^{n-k}(\mathbb{R}^n)^* \arrow[l, "\mathrm{d}_{n-k-1}^*"'] & \cdots \arrow[l, "\mathrm{d}_{n-k}^*"'] & \Omega^n(\mathbb{R}^n)^* \arrow[l, "\mathrm{d}_{n-1}^*"']
\end{tikzcd}$$

# Generalized Hodge operator

Dubois-Violette extends to generalized complexes $\star_k \colon \Omega_2^k(\mathbb{R}^n) \longrightarrow \Omega_2^{2n-k}(\mathbb{R}^n)^*$, defined by

$$(\star_k \mathbf{T})^{j_1 \cdots j_{2n-k}} := \begin{cases} \epsilon^{i_1 i_3 \cdots i_{k-1} j_1 j_3 \cdots j_{2n-k-1}} \epsilon^{i_2 i_4 \cdots i_k j_2 j_4 \cdots j_{2n-k}} \boldsymbol{T}_{i_1 \cdots i_k}, & \text{if } k \text{ is even;} \\ \epsilon^{i_1 i_3 \cdots i_k j_1 j_3 \cdots j_{2n-k}} \epsilon^{i_2 i_4 \cdots i_{k-1} j_2 j_4 \cdots j_{2n-k-1}} \boldsymbol{T}_{i_1 \cdots i_k}, & \text{if } k \text{ is odd.} \end{cases}$$

By composition we define the generalized co-differential

$$\mathrm{d}_k^* \colon \Omega_2^{k+1}(\mathbb{R}^n)^* \longrightarrow \Omega_2^k(\mathbb{R}^n)^*, \qquad \mathrm{d}_k^* := \star_{2n-k}\, \mathrm{d}_{2n-k-1}\, {\star_{2n-k-1}}^{-1},$$

and the corresponding co-integrators

$$K_k^* := \star_{2n-k-1}\, K_{2n-k-1}\, \star_{2n-k}^{-1} \colon \Omega_2^k(\mathbb{R}^n)^* \to \Omega_2^{k+1}(\mathbb{R}^n)^*.$$

We get the dual complex

$$\begin{tikzcd}
\Omega_2^0(\mathbb{R}^n) \arrow[r, "\mathrm{d}_0"] & \cdots \arrow[r, "\mathrm{d}_{k-1}"] & \Omega_2^k(\mathbb{R}^n) \arrow[r, "\mathrm{d}_k"] \arrow[drr, "\star_k"'] & \cdots \arrow[r, "\mathrm{d}_{2n-k-1}"] & \Omega_2^{2n-k}(\mathbb{R}^n) \arrow[r, "\mathrm{d}_{2n-k}"] \arrow[dll, "\star_{2n-k}"] & \cdots \arrow[r, "\mathrm{d}_{2n-1}"] & \Omega_2^{2n}(\mathbb{R}^n) \\
\Omega_2^0(\mathbb{R}^n)^* & \cdots \arrow[l, "\mathrm{d}_0^*"'] & \Omega_2^k(\mathbb{R}^n)^* \arrow[l, "\mathrm{d}_{k-1}^*"'] & \cdots \arrow[l, "\mathrm{d}_k^*"'] & \Omega_2^{2n-k}(\mathbb{R}^n)^* \arrow[l, "\mathrm{d}_{2n-k-1}^*"'] & \cdots \arrow[l, "\mathrm{d}_{2n-k}^*"'] & \Omega_2^{2n}(\mathbb{R}^n)^* \arrow[l, "\mathrm{d}_{2n-1}^*"']
\end{tikzcd}$$

# Dual elasticity complex

In dimension $2$ or $3$ we obtain the dual elasticity complex:

$$
\begin{tikzcd}[column sep=large]
\boldsymbol{\xi}^\flat \in \Omega^1_2(\mathbb{R}^3) \arrow[r, shift left, "d_1 = D_1"] \arrow[d, dotted, no head] & \boldsymbol{\varepsilon} \in \Omega^2_2(\mathbb{R}^3) \arrow[l, shift left, "K_1"] \arrow[r, shift left, "d_3\, d_2 = D_2"] \arrow[d, phantom, "\star"] & \mathbf{W} \in \Omega^4_2(\mathbb{R}^3) \arrow[l, shift left, "K_2"] \arrow[d, dotted, no head] \\
\boldsymbol{\xi} \in \Omega^1_2(\mathbb{R}^3)^* \arrow[r, shift right, "K_1^*"'] & \boldsymbol{\sigma} \in \Omega^2_2(\mathbb{R}^3)^* \arrow[l, shift right, "d_1^* = D_1^*"'] \arrow[r, shift right, "K_2^*"'] & \mathbf{A} \in \Omega^4_2(\mathbb{R}^3)^* \arrow[l, shift right, "d_2^*\, d_3^* = D_2^*"']
\end{tikzcd}
$$

The dual homotopy formula:

$$d_k^* \, K_k^* + K_{k-1}^* d_{k-1}^* = \mathrm{id}, \qquad \text{on } \Omega^k_2(\mathbb{R}^n)^*,$$

allows to obtain

- dual necessary and sufficent compatibility condition
- explicit integrators $K_1^*$, $K_2^*$
- existence of stress potentials

# Outline

# Stress potentials in dimension 2 and 3

- In dimension 2 the Airy potential (1863) simplifies the resolution of plane stress problems in 2D.

$$(D_2^*\mathbf{A})^{ij} = (\star_2\ \mathrm{d}_1\ \mathrm{d}_0\ \underbrace{\star_0^{-1}\mathbf{A}}_{:=\varphi})^{ij} = (\star_2\ \mathrm{d}_1\ \mathrm{d}_0\ \ \varphi)^{ij} = \epsilon^{ik}\epsilon^{jl}\partial_k\partial_l\varphi.$$

> The exactness of the elasticity complex gives the existence of the Airy potential:
>
> $D_1^*\boldsymbol{\sigma} = 0 \Longrightarrow$ there exists $\varphi$, such that $\sigma^{ij} = (D_2^*\mathbf{A})^{ij} = \epsilon^{ik}\epsilon^{jl}\partial_k\partial_l\varphi.$

- In dimension 3: generalisations (Beltrami potential) merges compatibility conditions with the constitutive law into the formulation of the problem. Pommaret clarified using the *Spencer cohomology*:

$$(D_2^*\mathbf{A})^{ij} = (\star_4\ \mathrm{d}_3\ \mathrm{d}_2\ \underbrace{\star_2^{-1}\mathbf{A}}_{:=\frac{3}{4}\boldsymbol{\phi}})^{ij} = (\star_4\ \mathrm{d}_3\ \mathrm{d}_2\ \ \boldsymbol{\phi})^{ij} = \epsilon^{imk}\epsilon^{jnl}\partial_k\partial_l\phi_{mn}.$$

> The exactness of the elasticity complex gives the existence of the Beltrami potential:
>
> $D_1^*\boldsymbol{\sigma} = 0 \Longrightarrow$ there exists $\boldsymbol{\phi} \in \Omega_2^2(\mathbb{R}^3)$, such that $\sigma^{ij} = \epsilon^{imk}\epsilon^{jnl}\partial_k\partial_l\phi_{mn}.$

## Summary

**Summary:**

- Reformulate the displacement problem by a complex naturally adapted to symmetries of elasticity
- Retrieve compatibility condition $\mathbf{W} = 0$ wich becomes necessary and sufficient
- Recover the Cesàro-Volterra integral formula
- Give a measure of the incompatibility degree
- Interprete geometrically the equivalence in dimension $2$ and $3$ between $\mathbf{W} = 0$ and $\mathbf{rot}\,\mathbf{rot}^t\,\boldsymbol{\varepsilon} = 0$
- Natural formulation of Airy and Beltrami stress potentials in this framework

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