GDR Groupement de recherche
GDR-GDM n°2043
Géométrie Différentielle et Mécanique

AFM ASSOCIATION FRANÇAISE DE MÉCANIQUE

Intégrateurs géométriques et dynamique non-régulière

Anthony Gravouil
LaMCoS, INSA Lyon

cnrs

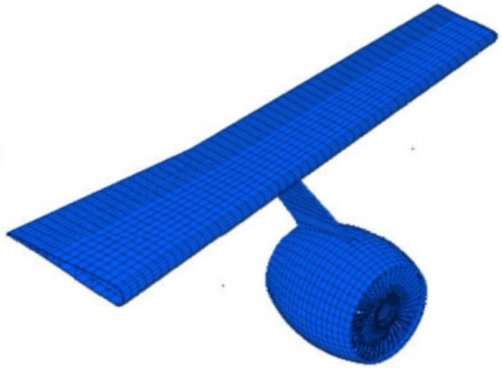
LaMCoS
Laboratoire de Mécanique des Contacts et des Structures
UMR 5259

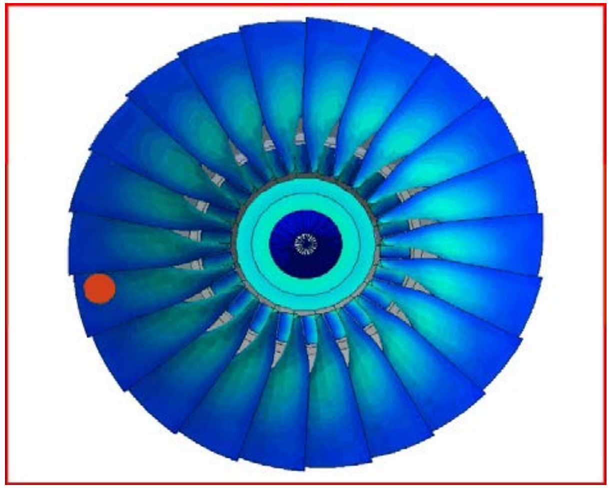
INSA INSTITUT NATIONAL DES SCIENCES APPLIQUÉES LYON

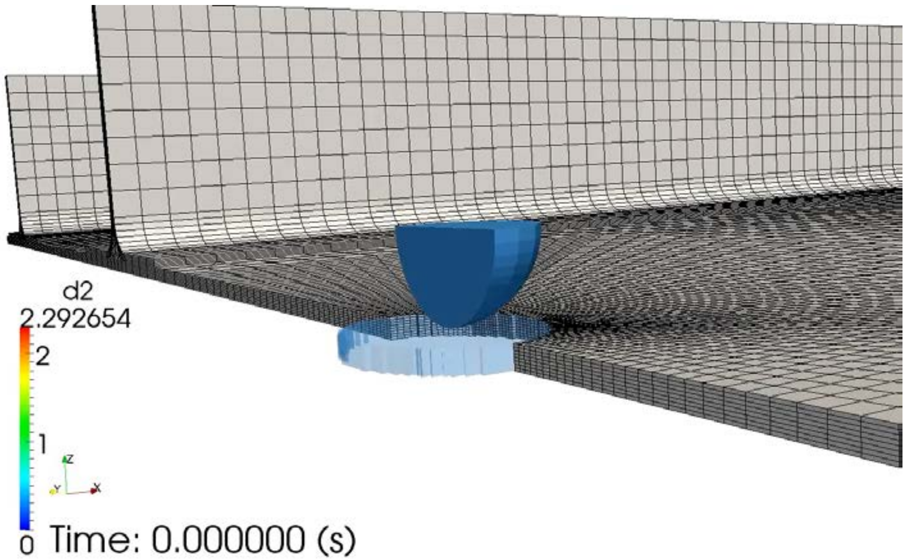
1

Introduction aux intégrateurs temporels conventionnels en dynamique des structures

Introduction aux intégrateurs temporels conventionnels



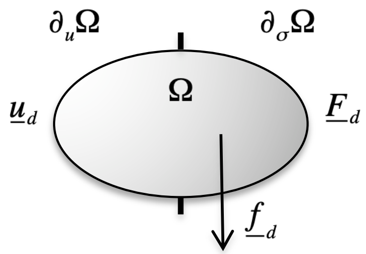




Définition d'un problème de référence en mécanique des milieux continus

- *Un problème de référence en mécanique des solides:*

➡ *On considère un solide élastique linéaire dans l'hypothèse des petites perturbations:*



où $\underline{F}_d$ *sont les efforts imposés sur le bord* $\partial_\sigma\Omega$ *(Neumann) ,*
$\underline{f}_d$ *sont les efforts donnés dans le volume* Ω *,*
$\underline{u}_d$ *sont les déplacements imposés sur le bord* $\partial_u\Omega$ *(Dirichlet)*

- *Formulation forte du problème de référence (I):*
  - *Equations d'équilibre en volume et sur le bord*
  - *Loi de comportement (élastique linéaire)*
  - *Mesure de déformation*
  - *Conditions aux limites*
  - *Conditions initiales*

$$\left\{\begin{array}{ll} \underline{div}(\sigma)+\underline{f}_d=\rho\frac{d^2\mathbf{u}}{dt^2} & \\ \sigma\underline{n}=\underline{R} & sur\ \partial_u\Omega \\ \sigma\underline{n}=\underline{F}_d & sur\ \partial_\sigma\Omega \end{array}\right.$$

$$\sigma=\mathbf{K}\varepsilon(\underline{u})$$

$$\varepsilon(\underline{u})=\tfrac{1}{2}\left(\mathbf{Grad}\underline{u}+\mathbf{Grad}\underline{u}^T\right)$$

$$\underline{u}\big|_{\partial_u\Omega}=\underline{u}_d$$

$$\underline{u}(0)=\underline{u}_0 \qquad \tfrac{du}{dt}(0)=\underline{v}_0$$

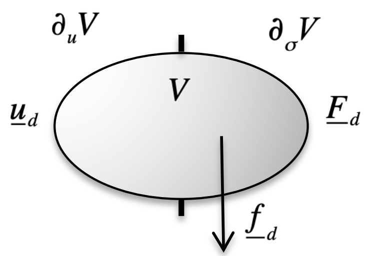
- *Le but est de trouver la solution exacte* $(\underline{u}_{ex},\sigma_{ex})$ *du problème (I) pour chaque élément de volume de* Ω *, à chaque instant de l'intervalle de temps* $[0,T]$ *.*

- *On considère le Principe des Puissances virtuelles dans le cas de la dynamique linéaire:*

$$\sigma = \boldsymbol{K}\varepsilon(\underline{u})$$

$$-\int_V Tr\left(\sigma\varepsilon\left(\underline{u}^*\right)\right)dv + \int_{\partial_\sigma V} \underline{F}_d.\underline{u}^*\, ds + \int_V \underline{f}_d.\underline{u}^*\, dv = \int_V \rho \frac{d^2\underline{u}}{dt^2}.\underline{u}^*\, dv$$

$$\forall \underline{u}^* \ \text{avec}\ \ \underline{u}^*\big|_{\partial_u V} = 0 \qquad \text{et} \quad \underline{u}(0) = \underline{u}_0 \quad \tfrac{d\underline{u}}{dt}(0) = \underline{v}_0$$



- *On introduit l'approximation éléments finis dans le Principe des Puissances Virtuelles avec un comportement élastique linéaire:*

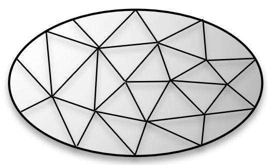
$$\underline{u}(\underline{x},t) = \sum_1^n u_i(t)\underline{\varphi}_i(\underline{x})$$

$$\underline{u}^*(\underline{x}) = \sum_1^n u_i^*\underline{\varphi}_i(\underline{x})$$

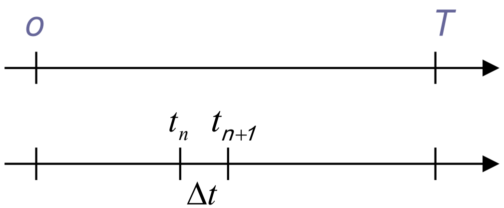
$$P_{int}^* = -\sum_{i=1}^n \left[\int_\Omega Tr\left[\sigma\varepsilon\left(\underline{\varphi}_i\right)\right]dv\right]u_i^* = -\mathbf{F}_{int}^T\mathbf{U}^* \qquad P_{int}^* = -\mathbf{U}^T\mathbf{K}\mathbf{U}^*$$

$$P_{ext}^* = \sum_{i=1}^n \left[\int_\Omega \underline{f}_d.\underline{\varphi}_i\, d\Omega + \int_{\partial_2\Omega} \underline{F}_d.\underline{\varphi}_i\, dS\right]u_i^* = \mathbf{F}_{ext}^T\mathbf{U}^*$$

$$P_{acc}^* = \sum_{j=1}^n\sum_{i=1}^n \ddot{u}_i \left[\int_V \rho\underline{\varphi}_i.\underline{\varphi}_j\, dv\right]u_j^* = \ddot{\mathbf{U}}^T\mathbf{M}\mathbf{U}^*$$



- *On obtient:* $\left(-\mathbf{KU} + \mathbf{F}_{ext} - \mathbf{M}\ddot{\mathbf{U}}\right)^T \mathbf{U}^* = 0 \quad \forall \mathbf{U}^* \qquad \forall t \in [0,T] \qquad \boxed{\mathbf{M}\ddot{\mathbf{U}}(t) + \mathbf{KU}(t) = \mathbf{F}_{ext}(t)}$



Temps continu

Temps discrétisé

Discrétisation espace-temps: $\boxed{\mathbf{M}\ddot{\mathbf{U}}_{n+1} + \mathbf{KU}_{n+1} = \mathbf{F}_{ext_{n+1}}}$

- *Considérons de nouveau la dynamique transitoire linéaire semi-discrétisée en espace par la M.E.F.:*

$$\boxed{\begin{array}{ll} \mathbf{M}\ddot{\mathbf{U}}(t)+\mathbf{K}\mathbf{U}(t)=\mathbf{F}_{ext}(t) & \forall t\in[0,T] \\ \dot{\mathbf{U}}(t)=\frac{d\mathbf{U}(t)}{dt},\ddot{\mathbf{U}}(t)=\frac{d^2\mathbf{U}(t)}{dt^2} & \mathbf{U}(0)=\mathbf{U}_0 \quad \dot{\mathbf{U}}(0)=\mathbf{V}_0 \end{array}}$$

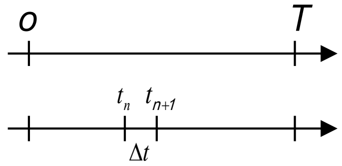
- *Développement en série de Taylor au second ordre de la variable temporelle* ***[Newmark 1959]***

$$\begin{cases} \mathbf{U}_{n+1}=\mathbf{U}_n^p+\beta\Delta t^2\ddot{\mathbf{U}}_{n+1} \\ \dot{\mathbf{U}}_{n+1}=\dot{\mathbf{U}}_n^p+\gamma\Delta t\ddot{\mathbf{U}}_{n+1} \end{cases} \qquad \begin{cases} \mathbf{U}_n^p=\mathbf{U}_n+\Delta t\dot{\mathbf{U}}_n+\Delta t^2\left(\frac{1}{2}-\beta\right)\ddot{\mathbf{U}}_n \\ \dot{\mathbf{U}}_n^p=\dot{\mathbf{U}}_n+\Delta t(1-\gamma)\ddot{\mathbf{U}}_n \end{cases}$$

➡

$$\tilde{\mathbf{M}}\equiv\mathbf{M}+\beta\Delta t^2\mathbf{K}$$
$$\mathbf{X}\equiv\ddot{\mathbf{U}}_{n+1}$$
$$\mathbf{B}\equiv\mathbf{F}_{ext_{n+1}}-\mathbf{K}\mathbf{U}_n^p$$

$$\boxed{\tilde{\mathbf{M}}\mathbf{X}=\mathbf{B}}$$



➡ *Le choix d'un schéma numérique en temps est essentiellement lié au contenu fréquentiel du chargement et à la réponse de la structure*

- *Schéma explicite de la différence centrée (Abaqus explicite) (hautes fréquences, chocs, impacts)*

$$\boxed{\gamma=1/2 \quad \beta=0 \qquad \mathbf{M}_{lump}\ddot{\mathbf{U}}_{n+1}=\left(\mathbf{F}_{ext_{n+1}}-\mathbf{K}\mathbf{U}_n^p\right) \qquad \Delta t\leq\frac{2}{\omega_{\max}}}$$

- *Schéma implicite de l'accélération moyenne (Abaqus standard) (basses fréquences, vibrations associées aux grandes longueurs d'onde de la structure)*

$$\boxed{\gamma=1/2 \quad \beta=1/4 \qquad \left(\tfrac{4}{\Delta t^2}\mathbf{M}+\mathbf{K}\right)\mathbf{U}_{n+1}=\left(\mathbf{F}_{ext_{n+1}}+\tfrac{4}{\Delta t^2}\mathbf{M}\mathbf{U}_n^p\right)}$$

- *Etude de la A-stabilité (matrice d'amplification):*

$$\mathbf{X}(t_{n+1}) - \mathbf{A}\mathbf{X}(t_n) - \mathbf{L}_n = \Delta t \boldsymbol{\tau}(t_n) \qquad \boxed{\mathbf{X}_{n+1} - \mathbf{A}\mathbf{X}_n - \mathbf{L}_n = 0} \qquad \mathbf{X}_{n+1} = \begin{pmatrix} \dot{\mathbf{U}}_{n+1} \\ \mathbf{U}_{n+1} \end{pmatrix} \qquad e(\mathrm{t}_{n+1}) = \mathbf{X}(\mathrm{t}_{n+1}) - \mathbf{X}_{n+1}$$

- *Théorème de Lax:*

Si:	$\lvert\mathbf{A}\rvert \le 1$			*(stabilité)*
	$\lvert\boldsymbol{\tau}(t)\rvert \le \mathrm{c}\Delta t^k$	$\forall t \in [0,T]$	$\mathrm{k} > 0$	*(consistance)*
Alors	$e(t_n) \to 0$	*lorsque*	$\Delta t \to 0$	*(convergence)*

- *Propriétés de stabilité du schéma de Newmark:*

$$\mathbf{A} = \mathbf{A}_1^{-1}\mathbf{A}_2 \qquad \textit{avec} \qquad \mathbf{A}_1 = \begin{pmatrix} \mathbf{M} & \gamma\Delta t\mathbf{K} \\ 0 & \mathbf{M} + \beta\Delta t^2\mathbf{K} \end{pmatrix} \qquad \mathbf{A}_2 = \begin{pmatrix} \mathbf{M} & -(\gamma-1)\Delta t\mathbf{K} \\ \Delta t\mathbf{M} & \mathbf{M} - (\frac{1}{2} - \beta)\Delta t^2\mathbf{K} \end{pmatrix}$$

➡ *Analyse modale de la matrice d'amplification (valeurs propres inférieures ou égales à 1)*

$$\mathbf{A}_1 = \begin{pmatrix} 1 & \gamma\Delta t\omega^2 \\ 0 & 1 + \beta\Delta t^2\omega^2 \end{pmatrix} \qquad \mathbf{A}_2 = \begin{pmatrix} 1 & -(\gamma-1)\Delta t\omega^2 \\ \Delta t & 1 - (\frac{1}{2} - \beta)\Delta t^2\omega^2 \end{pmatrix}$$

$\frac{1}{2} \le \gamma \le 2\beta$		*méthode inconditionnellement stable*	
$\frac{1}{2} \le \gamma$	*et* $\quad 2\beta \le \gamma$	*méthode conditionnellement stable*	$\Delta t \le \dfrac{1}{\omega_{\max}\sqrt{\frac{\gamma}{2} - \beta}}$

➡ *Les schémas de la différence centrée et de l'accélération moyenne sont d'ordre 2*

➡ *Ces résultats se généralisent en présence d'amortissement numérique*

Un formalisme général d'intégrateurs temporels avec multiplicateurs de Lagrange

- *Cas particulier des intégrateurs temporels CH-α introduits par Chung et Hulbert en 1993 :*

$$\boxed{\begin{aligned}&(1-\alpha_g)\mathbf{M}\dot{\mathbf{v}}_{n+1}+\alpha_g\mathbf{M}\dot{\mathbf{v}}_n+(1-\alpha_f)\mathbf{K}\mathbf{u}_{n+1}+\alpha_f\mathbf{K}\mathbf{u}_n\\&=(1-\alpha_f)\mathbf{f}_{n+1}+(1-\alpha_f)\mathbf{f}_n+\mathbf{L}^t\boldsymbol{\Lambda}_{n+1-\alpha_f}\end{aligned}} \qquad \begin{cases}\xi_g=1-\alpha_g\\ \xi_f=1-\alpha_f\end{cases}$$

Scheme	α_g	α_f	γ	β
HHT-α	0	$-\alpha_{HHT}=\frac{1-\rho_\infty}{1+\rho_\infty}$	$\frac{1}{2}-\alpha_{HHT}$	$\frac{1}{4}(1-\alpha_{HHT})^2$
WBZ-α	$\alpha_{WBZ}=\frac{\rho_\infty-1}{1+\rho_\infty}$	0	$\frac{1}{2}-\alpha_{WBZ}$	$\frac{1}{4}(1-\alpha_{WBZ})^2$
CH-α	$\frac{2\rho_\infty-1}{1+\rho_\infty}$	$\frac{\rho_\infty}{1+\rho_\infty}$	$\frac{3}{2}-2\alpha_f$	$(1-\alpha_f)^2$

➡ *En pratique, les quatre paramètres sont définis de manière à garantir une stabilité inconditionnelle, une précision du second ordre, la maîtrise des oscillations parasites à haute fréquence et la minimisation de la dissipation numérique aux basses fréquences. Les deux paramètres α peuvent être définis comme des fonctions du rayon spectral jusqu'à la limite des hautes fréquences, afin de contrôler le taux de dissipation numérique dans cet intervalle.*

- *Cas particulier de Newmark:* $\xi_g=\xi=\gamma \qquad \xi_f=\xi=\gamma$

➡ *Aucun intégrateur temporel de Newmark ne permet de contrôler les oscillations parasites à haute fréquence tout en garantissant une précision du second ordre.*

➡ *Ce formalisme inclut également les intégrateurs de Simo, Krenk et Verlet comme cas particuliers avec multiplicateurs de Lagrange.*

Eléments finis en temps – Méthode des Résidus pondérés

- On considère ici une formulation faible en temps sur l'intervalle $\left[-\Delta t, \Delta t\right]$ avec discrétisation E.F. en espace.
- On définit les fonctions de forme en temps de la façon suivante:

$$\boxed{\mathbf{U}(t) = \sum_{k=1}^{n} \mathbf{U}_k \phi_k(t)}$$

[Zienkiewicz et al. 1984]

- On effectue un changement de variable qui permet de considérer l'intervalle de temps $\left[-1,1\right]$: $\xi = \frac{t}{\Delta t} \quad -1 < \xi < 1$
- On propose ici de choisir des fonctions de forme quadratiques en temps:

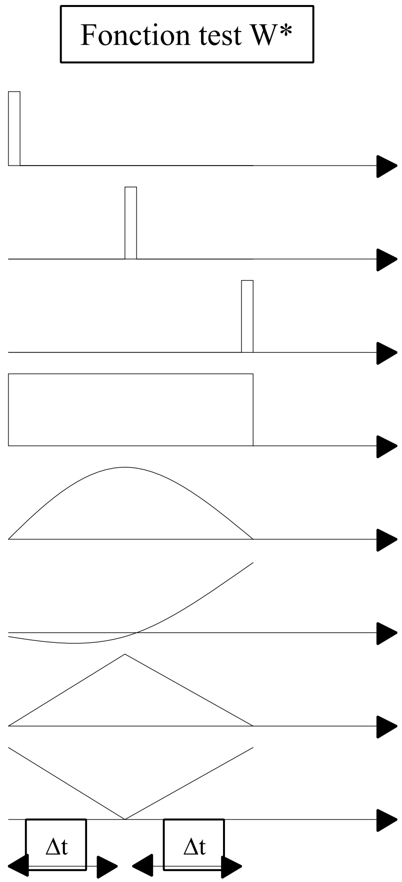
$$\phi_{k+1} = \xi(1+\xi)/2$$
$$\phi_k = (1-\xi)(1+\xi)$$
$$\phi_{k-1} = -\xi(1-\xi)/2$$

- On obtient:

$$\boxed{\int_{-1}^{1} \mathbf{W}^* \begin{pmatrix} \mathbf{M}\left(\mathbf{U}_{k-1}\ddot{\phi}_{k-1} + \mathbf{U}_k\ddot{\phi}_k + \mathbf{U}_{k+1}\ddot{\phi}_{k+1}\right) \\ +\mathbf{K}\left(\mathbf{U}_{k-1}\phi_{k-1} + \mathbf{U}_k\phi_k + \mathbf{U}_{k+1}\phi_{k+1}\right) - \mathbf{F}_{ext} \end{pmatrix} d\xi = 0 \quad \forall \mathbf{W}^*}$$

- Définitions:

$$\begin{cases} \gamma = \int_{-1}^{1} \mathbf{W}^* \left(\xi + \frac{1}{2}\right) d\xi \Big/ \int_{-1}^{1} \mathbf{W}^* d\xi \\ \beta = \int_{-1}^{1} \mathbf{W}^* \frac{1}{2}\xi(1+\xi) d\xi \Big/ \int_{-1}^{1} \mathbf{W}^* d\xi \end{cases}$$



γ	β
$-\frac{1}{2}$	0
$\frac{1}{2}$	0
$\frac{3}{2}$	1
$\frac{1}{2}$	$\frac{1}{6}$
$\frac{1}{2}$	$\frac{1}{10}$
$\frac{3}{2}$	$\frac{4}{5}$
$\frac{1}{2}$	$\frac{1}{12}$
$\frac{1}{2}$	$\frac{1}{4}$

- Dans la méthode des résidus pondérés, les fonctions tests $\mathbf{W}^*$ peuvent être choisies différemment des fonctions de forme ϕ On fait le lien ici entre le choix des fonctions tests et le schéma de Newmark:

$$\boxed{\left(\mathbf{M} + \beta\Delta t^2\mathbf{K}\right)\mathbf{U}_{k+1} + \left(-2\mathbf{M} + \left(\frac{1}{2} - 2\beta + \gamma\right)\Delta t^2\mathbf{K}\right)\mathbf{U}_k + \left(\mathbf{M} + \left(\frac{1}{2} + \beta - \gamma\right)\Delta t^2\mathbf{K}\right)\mathbf{U}_{k-1} - \left(\beta\mathbf{F}_{ext_{k+1}} + \left(\frac{1}{2} - 2\beta + \gamma\right)\mathbf{F}_{ext_k} + \left(\frac{1}{2} + \beta - \gamma\right)\mathbf{F}_{ext_{k-1}}\right)\Delta t^2 = 0}$$

⇒ Autre écriture du schéma de Newmark sous la forme d'un schéma à 3 pas

Cas du schéma numérique Galerkin discontinu en temps

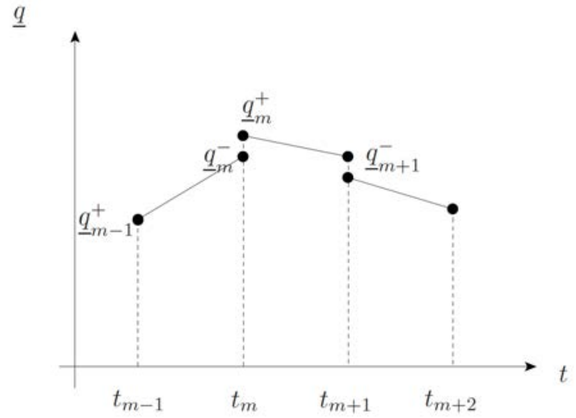
- Écriture sous forme d'un système du premier ordre

$$q_m^- = \lim_{\varepsilon \to 0^-} q(t_m + \varepsilon)$$

$$q_m^+ = \lim_{\varepsilon \to 0^+} q(t_m + \varepsilon)$$

$$\boxed{\begin{array}{l} M\,\dot{v} = f - K\,u \\ \dot{u} - v = 0 \end{array}}$$

$$\left\{ \begin{array}{rcl} v_m^- & = & v^- \\ u_m^- & = & u^- \\ u_m^+ & = & u1 \\ u_{m+1}^- & = & u2 \\ v_m^+ & = & v1 \\ v_{m+1}^- & = & v2 \end{array} \right.$$



- Écriture globale en temps (déplacements et vitesses)

$$\boxed{\begin{array}{l} \displaystyle\int_{I_m} w_1 (M\,\dot{v} + K\,u - f)dt + \int_{I_m} w_2\,K(\dot{u} - v)dt \qquad [t_m, t_{m+1}] \\ +\quad w_{2m}\,K[u_m] + w_{1m}\,M[v_m] = 0 \qquad \forall w_1, \forall w_2 \end{array}}$$

- Écriture discrétisée en espace et en temps

$$\boxed{\begin{bmatrix} \frac{1}{2}\mathrm{M} + \frac{1}{36}\Delta t^2 \mathrm{K} & \mathrm{M}^* \\ \mathrm{M}^* & -\frac{1}{2}\mathrm{M} + \frac{7}{36}\Delta t^2 \mathrm{K} \end{bmatrix} \cdot \begin{pmatrix} v2_m \\ v1_m \end{pmatrix} = \begin{pmatrix} F2_m^* \\ F1_m^* \end{pmatrix}}$$

$$\boxed{\begin{array}{l} u1_m = u_{m-1}^- + \frac{1}{6}\Delta t v1_m - \frac{1}{6}\Delta t v2_m \\ u2_m = u_{m-1}^- + \frac{1}{2}\Delta t v1_m + \frac{1}{2}\Delta t v2_m \end{array}}$$

$$\mathrm{M}^* = \frac{1}{2}\mathrm{M} + \frac{5}{36}\Delta t^2 \mathrm{K}$$

$$F1_m = \Delta t \left(\frac{1}{3} f_{m-1} + \frac{1}{6} f_m \right)$$

$$F2_m = \Delta t \left(\frac{1}{6} f_{m-1} + \frac{1}{3} f_m \right)$$

$$F1_m^* = -\frac{1}{2}\Delta t \mathrm{K}.u_{m-1}^- + F2_m$$

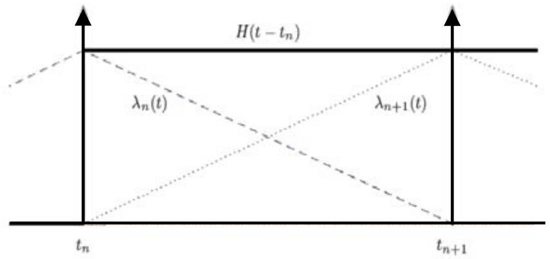
$$F2_m^* = \mathrm{M}.v_{m-1}^- - \frac{1}{2}\Delta t \mathrm{K}.u_{m-1}^- + F1_m$$

[Li X.D. et Wiberg N.-E 1998]

Éléments finis étendus en temps

- $v(t) = v^{\mathrm{c}}(t) + v^{\mathrm{e}}(t) \qquad v^{\mathrm{c}}(t) = v_n^{\mathrm{c}}\lambda_n(t) + v_{n+1}^{\mathrm{c}}\lambda_{n+1}(t)$

  $u(t) = u^{\mathrm{c}}(t) + u^{\mathrm{e}}(t) \qquad v^{\mathrm{e}}(t) = v_{n+1}^{\mathrm{e}}\lambda_n(t)H(t-t_n)$



- $$\boxed{\int_{t_n^+}^{t_{n+1}^-} W(t)[m\dot{v}(t) + ku(t)]\,\mathrm{d}t + \underbrace{W(t_n^+)mv_{n+1}^{\mathrm{e}}}_{\textit{Continuité faible}} = \int_{t_n^+}^{t_{n+1}^-} W(t)f(t)\,\mathrm{d}t}$$

- **Formulation Eléments Finis Etendus du problème en temps**

  Connaissant u_n^c, u_n^e, v_n^c et v_n^e, trouver $u_{n+1}^c, u_{n+1}^e, v_{n+1}^c$ et v_{n+1}^e tels que :

$$
\begin{aligned}
u_{n+1}^c &= u_n^c + \frac{\Delta t}{2}\left(v_{n+1}^c + v_n^c\right)\\
u_{n+1}^e &= u_n^e + \frac{\Delta t}{2} v_{n+1}^e
\end{aligned}
$$

$$
\begin{aligned}
\left[m + \beta_1 \Delta t^2 k\right] v_{n+1}^c + \left[(\delta_1 - 1)m + (\gamma_1 - \beta_1)\Delta t^2 k\right] v_{n+1}^e &= -\Delta t k u_n^c - \Delta t k u_n^e\\
&+ \left[m + (\beta_1 - \gamma_1)\,\Delta t^2 k\right] v_n^c + \Delta t\left[(1-\gamma_1)\,f_n + \gamma_1 f_{n+1}\right]\\
\left[m + \beta_2 \Delta t^2 k\right] v_{n+1}^c + \left[(\delta_2 - 1)m + (\gamma_2 - \beta_2)\Delta t^2 k\right] v_{n+1}^e &= -\Delta t k u_n^c - \Delta t k u_n^e\\
&+ \left[m + (\beta_2 - \gamma_2)\,\Delta t^2 k\right] v_n^c + \Delta t\left[(1-\gamma_2)\,f_n + \gamma_2 f_{n+1}\right]
\end{aligned}
$$

- $\beta_1 = \frac{1}{12}, \gamma_1 = \frac{1}{3}, \delta_1 = 2$

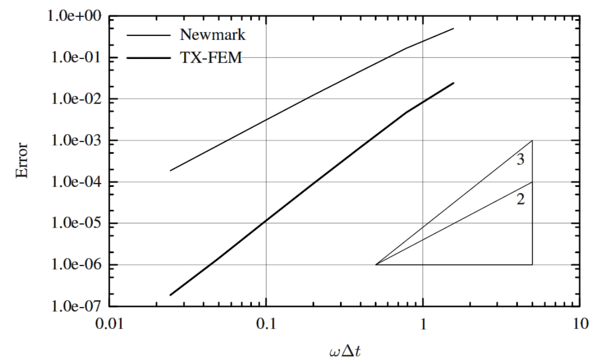
  $\beta_2 = \frac{1}{4}, \gamma_2 = \frac{2}{3}, \delta_2 = 0$

  T-XFEM $\Longrightarrow$ *Galerkin Discontinu en temps*

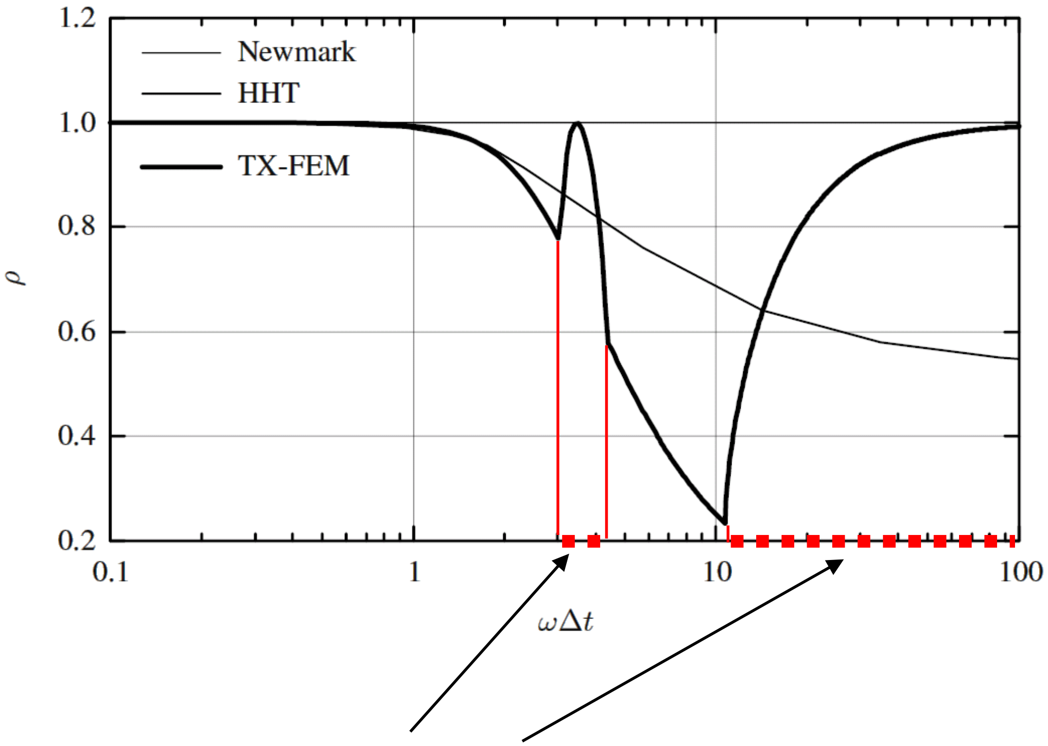
[Réthoré, Gravouil, Combescure, 2005]

Éléments finis étendus en temps

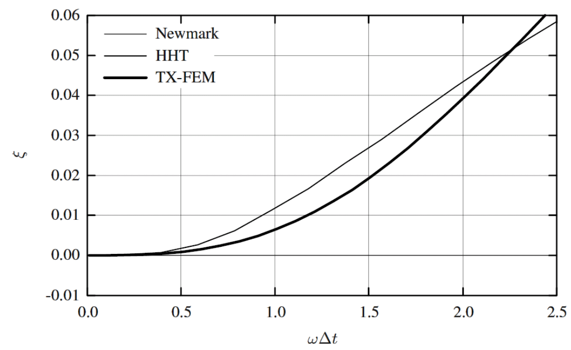
Norme L2



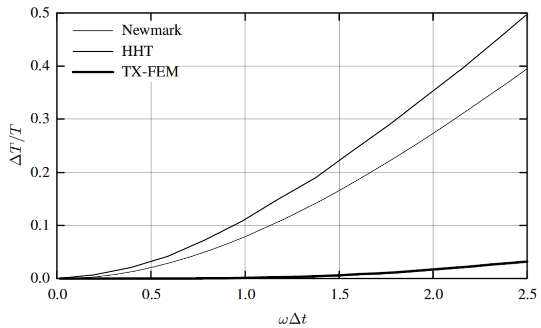
Rayon spectral



Amortissement numérique



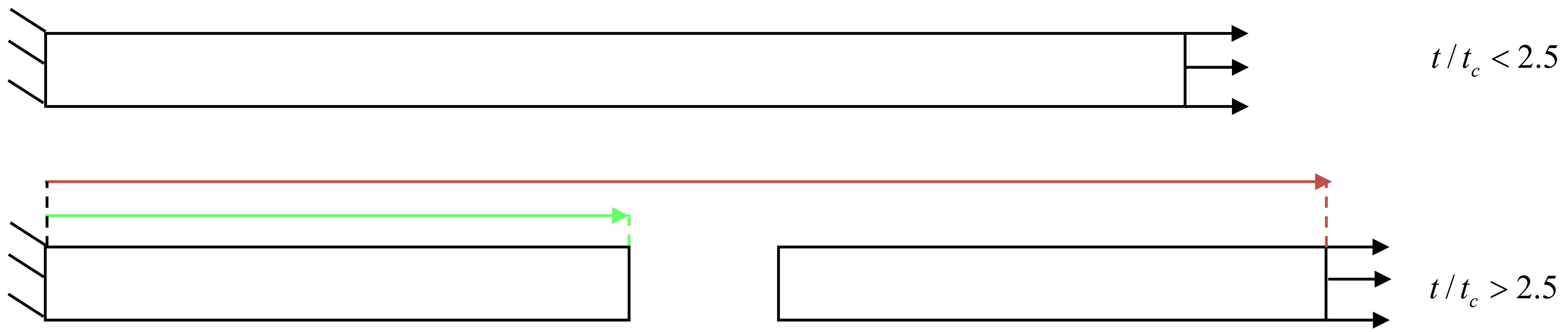
Erreur en phase

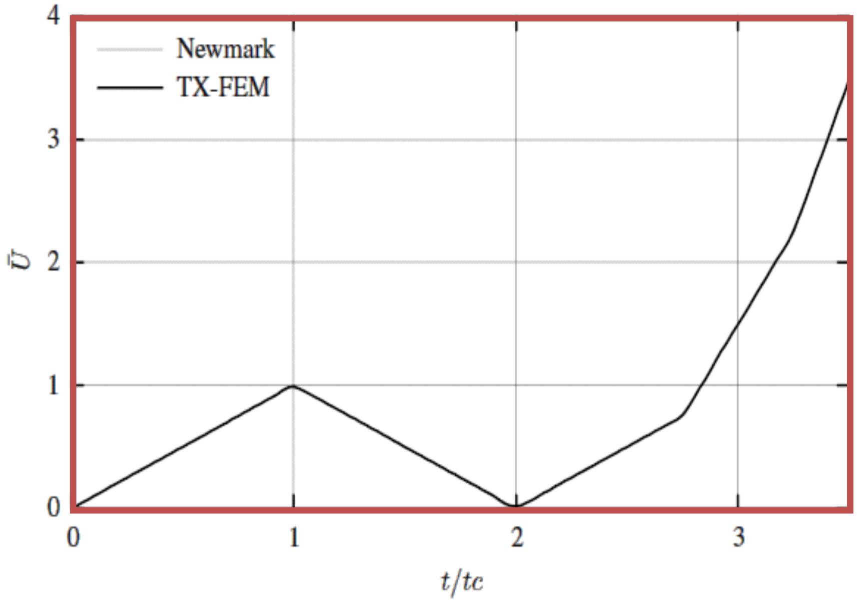


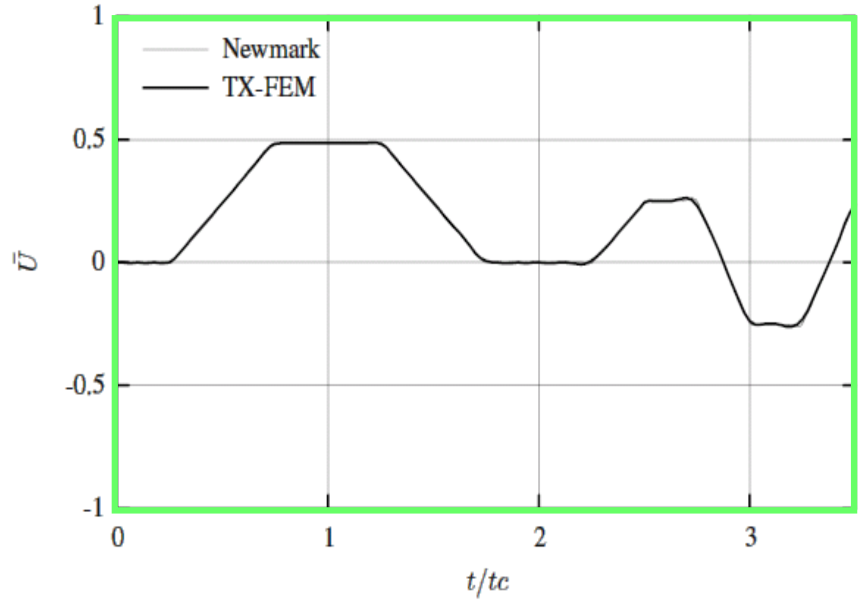
Réponse apériodique

$\Rightarrow$ Pas d'oscillations parasites

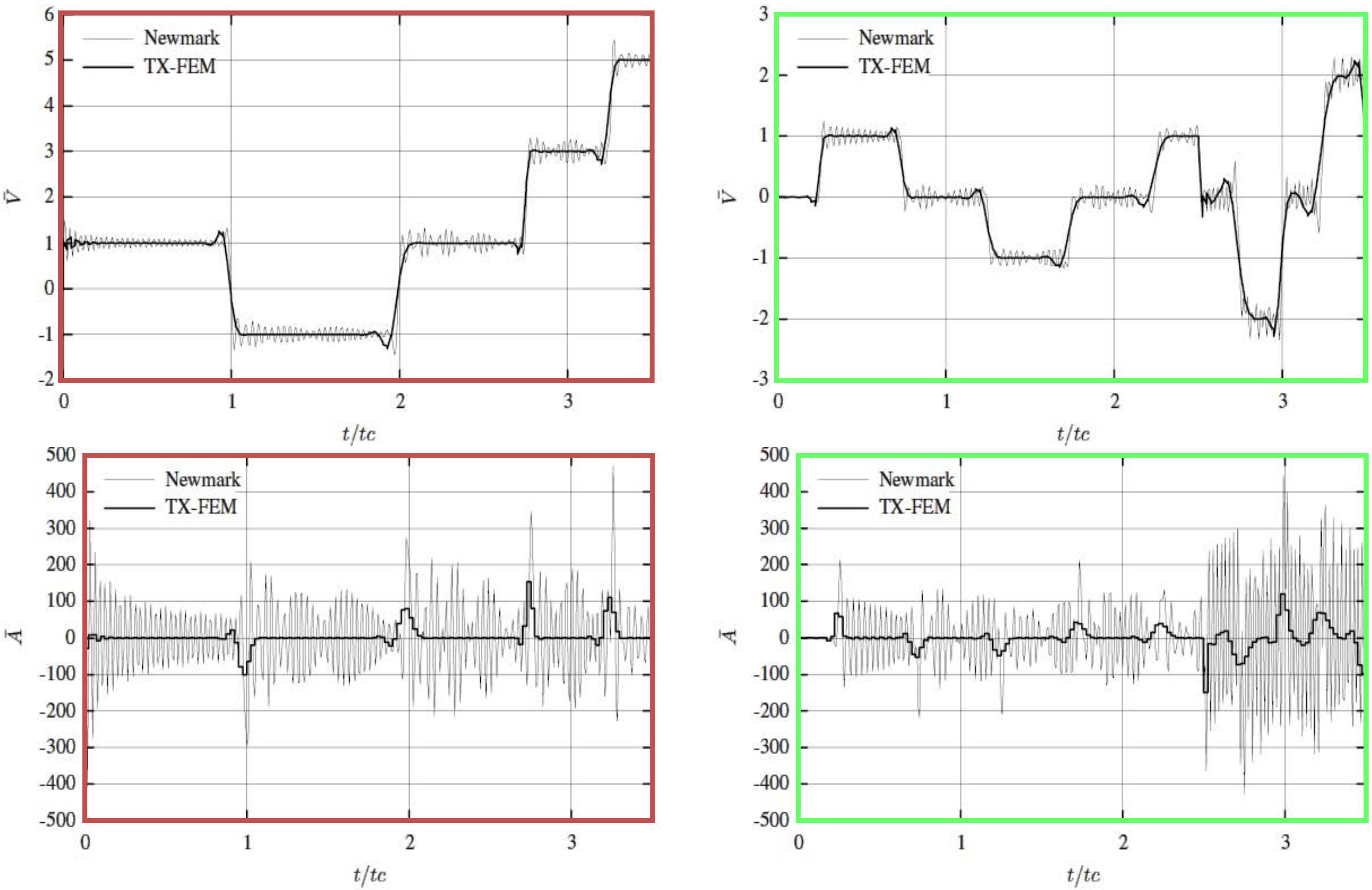
Poutre en tension avec rupture



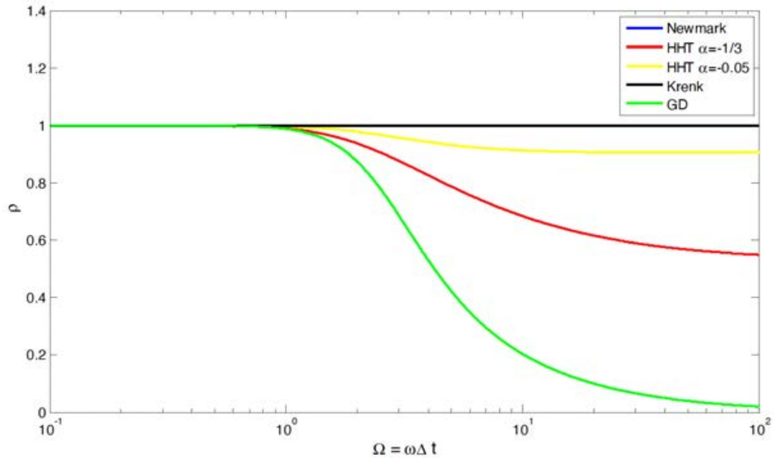




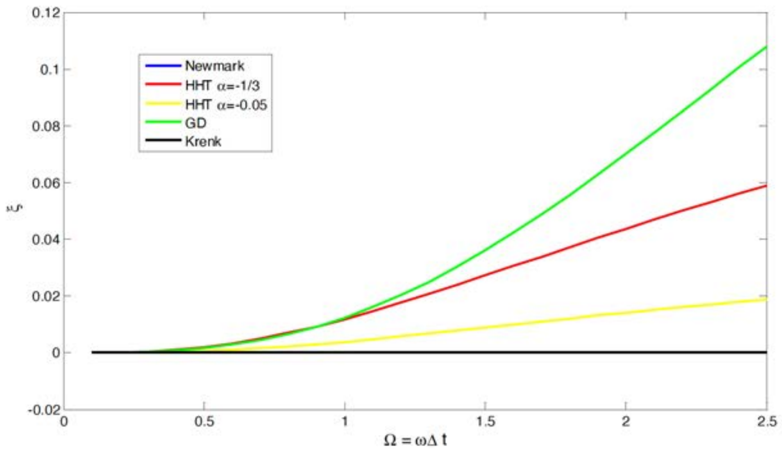
Poutre en tension avec rupture



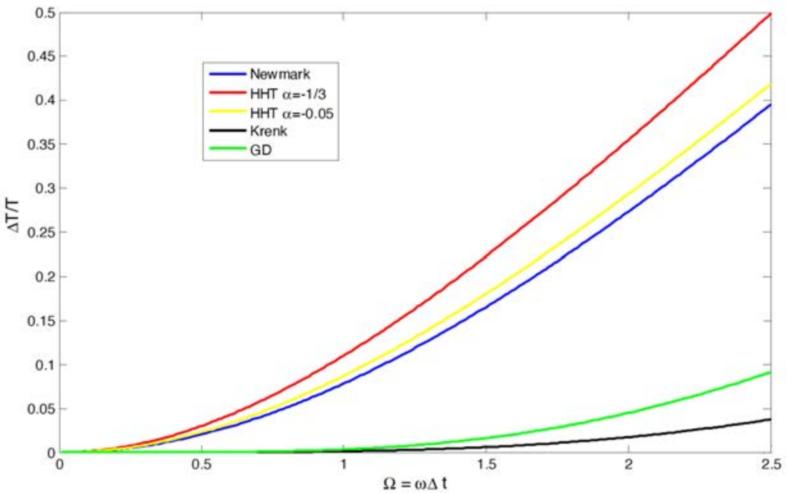
Comparaisons des différentes méthodes d'intégration temporelle



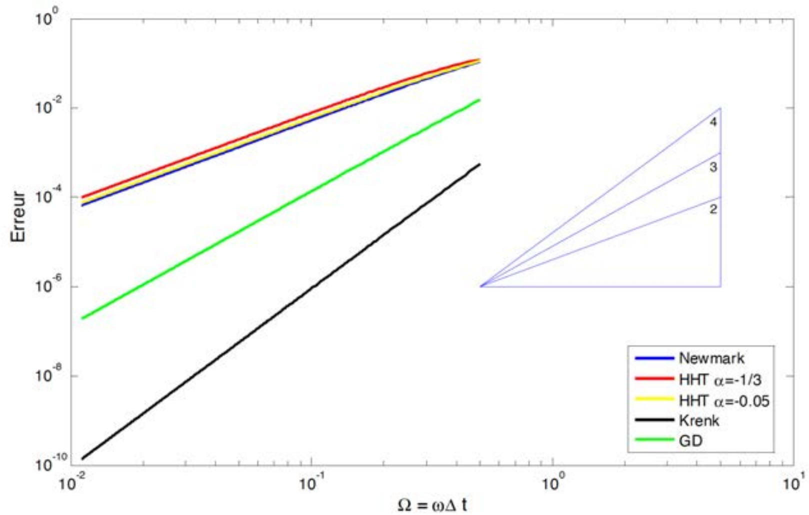
Rayon spectral des différents méthodes étudiées en fonction de Ω=ωΔt



Allongement de la période des différents schémas étudiés en fonction de Ω=ωΔt



Amortissement numérique des différents schémas étudiés en fonction de Ω=ωΔt

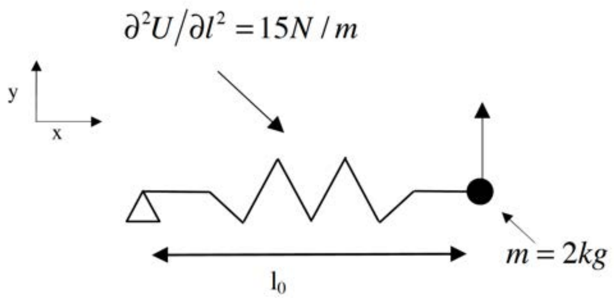


ordre de convergence des différents schémas étudiés

[Krenk 2006]

Cas du système masse ressort tournant

- Système conservatif non-linéaire à deux degrés de libertés



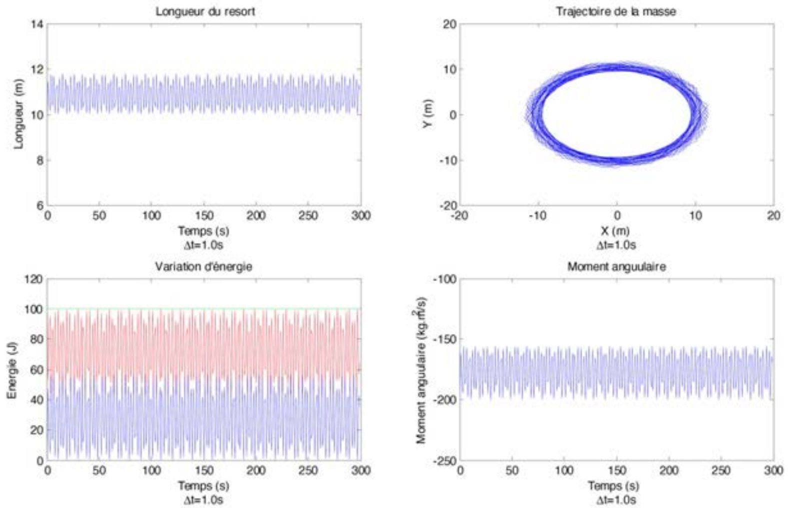
- Energie potentielle

$$U = \frac{EA_0}{2l_0}(l - l_0)^2$$

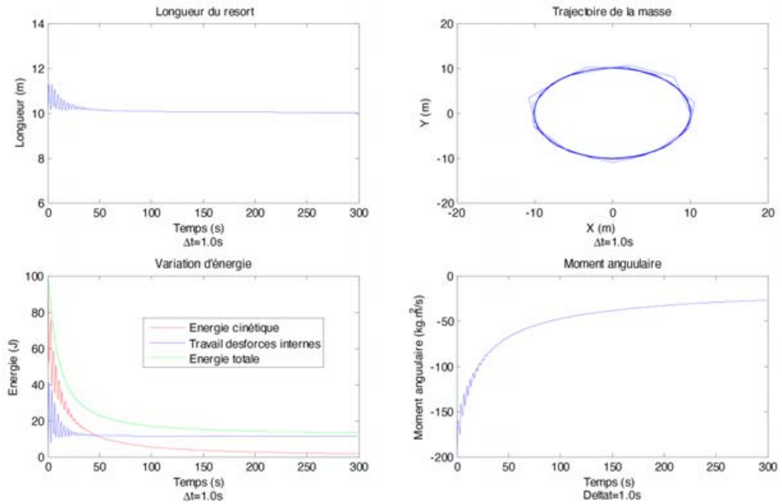
- Écriture matricielle de l'équation du mouvement

$$\begin{bmatrix} \frac{\rho_0 A_0 l_0}{2} & 0 \\ 0 & \frac{\rho_0 A_0 l_0}{2} \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix} + \begin{Bmatrix} U''\left(1 - \frac{l_0}{\sqrt{x^2+y^2}}\right)x \\ U''\left(1 - \frac{l_0}{\sqrt{x^2+y^2}}\right)y \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

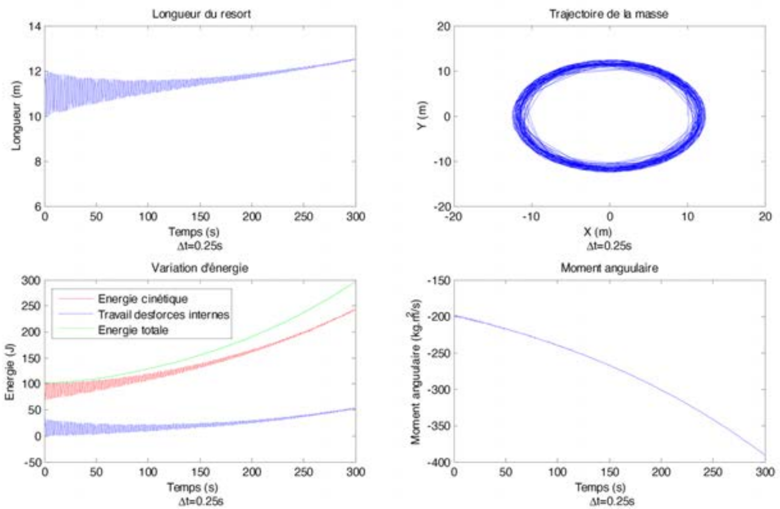
Comparaisons des différentes méthodes d'intégration temporelle



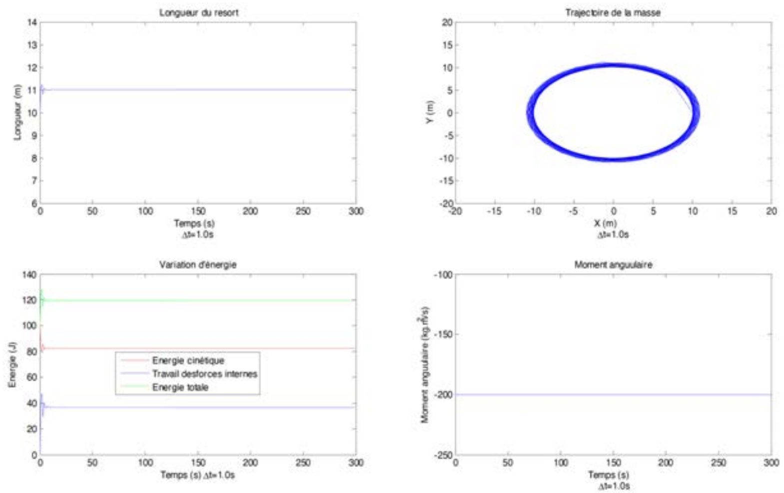
Intégration temporelle de Newmark avec Δt=1.0s



Intégration temporelle HHT-α (α=-0.05) avec Δt=1.0s



Intégration temporelle Galerkin Discontinu avec Δt=0.25s



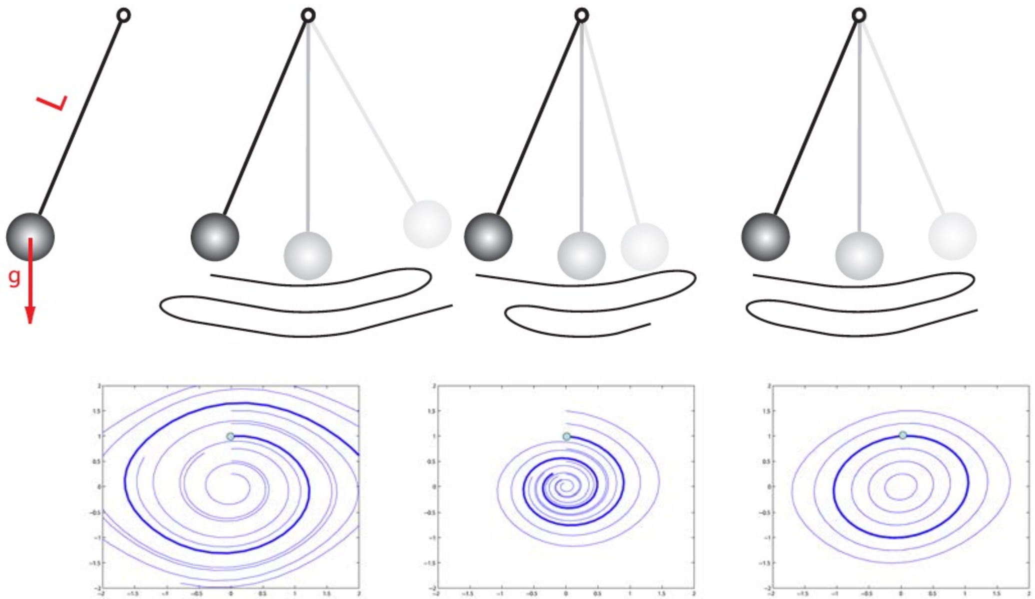
Intégration temporelle de Krenk avec Δt=1.0s

2

Introduction aux intégrateurs variationnels

Introduction

- Accurate numerical simulations of dynamical systems are essential in virtually all areas of physics and mechanics.
- However, most traditional numerical methods do not take into account the underlying geometric structure of the physical system, leading to simulation results that may suggest non-physical behaviors.
- The field of geometric numerical integration (GNI) concerns numerical methods that respect the fundamental physics of a problem by preserving the geometric properties of the associated differential equations.



Introduction

- *Since the emergence of computational methods, fundamental properties such as accuracy, stability, convergence, and computational efficiency are considered crucial in deciding the usefulness of a numerical algorithm.*

  Today, aspects related to the preservation of the underlying structure are considered essential, in addition to these fundamental properties.

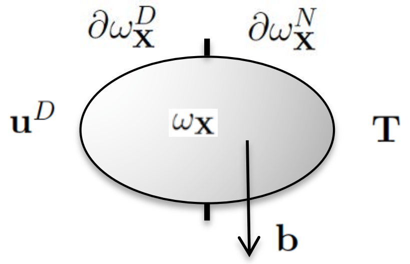
  One of the key ideas of approximate structure preservation is to treat the numerical method as a discrete dynamical system that approximates the flow of the continuous differential equation instead of focusing on the numerical approximation of a single trajectory .

  Such an approach allows a better understanding of the invariants and the qualitative properties of the numerical method.

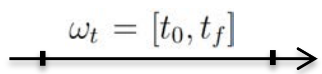
- *MAIN GOAL: develop an explicit variational integrator for non-smooth elastodynamics satisfying exactly ALL conservation equations in the discrete sense.*

Variational formalism for elastodynamics

- *Deformed configuration defined by the position vector:* $\mathbf{x} \equiv \varphi(t, \mathbf{X}) \quad \forall (t, \mathbf{X}) \in \omega_t \times \omega_{\mathbf{X}}$



- *State vector:* $\mathbf{m} \equiv (t, \mathbf{X}, \varphi)$



- *Action of the field theory of elastodynamics (non-deformed configuration, first gradient theory):*

$$\boxed{\mathscr{A}[\mathbf{m}] = \int_{\omega_t} \int_{\omega_{\mathbf{X}}} \mathscr{L}(t, \mathbf{X}, \varphi, d_t\varphi, d_{\mathbf{x}}\varphi)\, \mathrm{d}\mathbf{X}\mathrm{d}t}$$

$d_t\varphi = \frac{d\varphi}{dt} = \mathbf{V}$ *velocity*

$d_{\mathbf{x}}\varphi = \frac{d\varphi}{d\mathbf{X}} = \mathbf{F}$ *Transformation gradient tensor*

- *Corresponding Lagrangian:*

$$\mathcal{L} = T(d_t\varphi) - V(\varphi, d_{\mathbf{x}}\varphi)$$

$$T(d_t\varphi) = \tfrac{1}{2}\rho d_t\varphi . d_t\varphi \quad \text{and} \quad V(\varphi, d_{\mathbf{x}}\varphi) = V_{int}(d_{\mathbf{x}}\varphi) - V_{ext}(\varphi)$$

$$V_{int} = \psi(d_{\mathbf{x}}\varphi) \quad \text{and} \quad V_{ext}(\varphi) = \mathbf{b}.\varphi$$

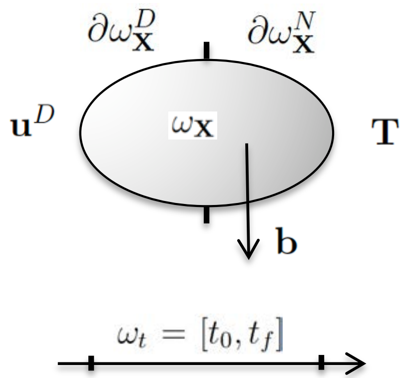
Variational formalism for elastodynamics

- *Hamilton's principle:*

$$\boxed{\begin{aligned}&\delta\mathcal{A} + \int_{\omega_t}\int_{\partial\omega_{\mathbf{X}}^N} \mathbf{T}.\delta\varphi\,\mathrm{d}\mathbf{S}\mathrm{d}t = 0\\ &\forall\delta\mathbf{m}\in\mathcal{M}_0(\delta\mathbf{m}\in H^1(\omega_t\times\omega_{\mathbf{X}});\delta t|_{\partial\omega_t}=0;\delta\mathbf{X}|_{\partial\omega_{\mathbf{X}}}=0;\ldots\\ &\ldots\delta\varphi|_{\partial\omega_t}=0;\delta\varphi|_{\partial\omega_{\mathbf{X}}^D}=0)\end{aligned}}$$

$$\partial\omega_{\mathbf{X}} = \partial\omega_{\mathbf{X}}^D \cup \partial\omega_{\mathbf{X}}^N \quad and \quad \partial\omega_{\mathbf{X}}^D \cap \partial\omega_{\mathbf{X}}^N = \emptyset$$

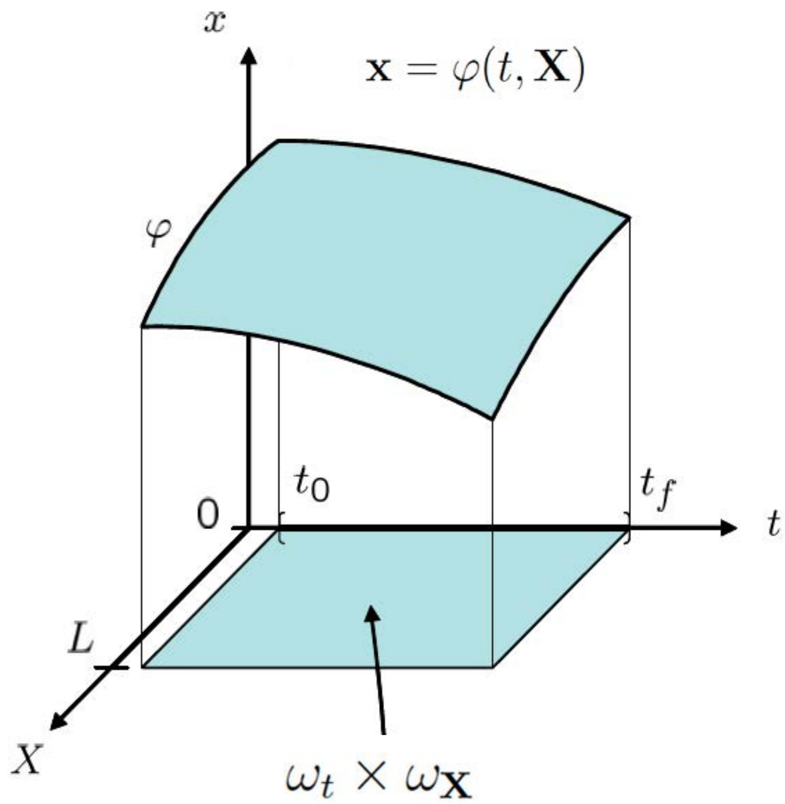
$$\varphi|_{\partial\omega_{\mathbf{X}}^D} = \mathbf{u}^D$$



- *Corresponding infinitesimal generators:*

$$\delta\mathcal{A} = \frac{\partial\mathcal{A}}{\partial\mathbf{m}}.\delta\mathbf{m} = \delta_t\mathcal{A} + \delta_{\mathbf{X}}\mathcal{A} + \delta_\varphi\mathcal{A} = D_t\mathcal{A}.\delta t + D_{\mathbf{X}}\mathcal{A}.\delta\mathbf{X} + D_\varphi\mathcal{A}.\delta\varphi$$

Variational formalism for elastodynamics



[Zielonka 2006, Kolev et al. 2023]

Variational formalism for elastodynamics

- *Vertical variation:* $\boxed{\delta \mathbf{m} \equiv (\delta t = 0, \delta \mathbf{X} = 0, \delta \varphi)}$

- *Determination of the vertical infinitesimal generator* $D_\varphi \mathcal{A}$ *:*

$$
\begin{aligned}
\delta_\varphi \mathcal{A} &= \int_{\omega_t} \int_{\omega_\mathbf{X}} \left(\frac{\partial \mathcal{L}}{\partial \varphi} . \delta \varphi + \frac{\partial \mathcal{L}}{\partial d_t \varphi} . \delta d_t \varphi + \frac{\partial \mathcal{L}}{\partial d_\mathbf{x} \varphi} : \delta d_\mathbf{x} \varphi \right) \mathrm{d}\mathbf{X}\mathrm{d}t \\
&= \int_{\omega_t} \int_{\omega_\mathbf{X}} \left(\frac{\partial \mathcal{L}}{\partial \varphi} . \delta \varphi \right) \mathrm{d}\mathbf{X}\mathrm{d}t \\
&+ \int_{\omega_t} \int_{\omega_\mathbf{X}} d_t \left(\frac{\partial \mathcal{L}}{\partial d_t \varphi} . \delta \varphi \right) \mathrm{d}\mathbf{X}\mathrm{d}t - \int_{\omega_t} \int_{\omega_\mathbf{X}} d_t \left(\frac{\partial \mathcal{L}}{\partial d_t \varphi} \right) . \delta \varphi \, \mathrm{d}\mathbf{X}\mathrm{d}t \\
&+ \int_{\omega_t} \int_{\omega_\mathbf{X}} d_\mathbf{x} . \left(\frac{\partial \mathcal{L}}{\partial d_\mathbf{x} \varphi} . \delta \varphi \right) \mathrm{d}\mathbf{X}\mathrm{d}t - \int_{\omega_t} \int_{\omega_\mathbf{X}} d_\mathbf{x} . \left(\frac{\partial \mathcal{L}}{\partial d_\mathbf{x} \varphi} \right) . \delta \varphi \, \mathrm{d}\mathbf{X}\mathrm{d}t \\
&= \int_{\omega_t} \int_{\omega_\mathbf{X}} \left(\frac{\partial \mathcal{L}}{\partial \varphi} . \delta \varphi \right) \mathrm{d}\mathbf{X}\mathrm{d}t \\
&+ \left[\int_{\omega_\mathbf{X}} \left(\frac{\partial \mathcal{L}}{\partial d_t \varphi} . \delta \varphi \right) \mathrm{d}\mathbf{X} \right]_{t_0}^{t_f} - \int_{\omega_t} \int_{\omega_\mathbf{X}} d_t \left(\frac{\partial \mathcal{L}}{\partial d_t \varphi} \right) . \delta \varphi \, \mathrm{d}\mathbf{X}\mathrm{d}t \\
&+ \int_{\omega_t} \int_{\partial \omega_\mathbf{X}} \left(\frac{\partial \mathcal{L}}{\partial d_\mathbf{x} \varphi} . \mathbf{N} \right) . \delta \varphi \, \mathrm{d}\mathbf{S}\mathrm{d}t - \int_{\omega_t} \int_{\omega_\mathbf{X}} d_\mathbf{x} . \left(\frac{\partial \mathcal{L}}{\partial d_\mathbf{x} \varphi} \right) . \delta \varphi \, \mathrm{d}\mathbf{X}\mathrm{d}t \\
&= D_\varphi \mathcal{A} . \delta \varphi
\end{aligned}
$$

Variational formalism for elastodynamics

- *Vertical variation:* $\delta \mathbf{m} \equiv (\delta t = 0, \delta \mathbf{X} = 0, \delta \varphi)$

- *Infinitesimal generator:*

$$\delta_\varphi \mathcal{A} = D_\varphi \mathcal{A}.\delta\varphi$$
$$= \int_{\omega_t}\int_{\omega_{\mathbf{X}}} \left(\frac{\partial \mathcal{L}}{\partial \varphi} - d_t\left(\frac{\partial \mathcal{L}}{\partial d_t \varphi}\right) - d_{\mathbf{x}}.\left(\frac{\partial \mathcal{L}}{\partial d_{\mathbf{x}} \varphi}\right)\right).\delta\varphi \,\mathrm{d}\mathbf{X}\mathrm{d}t$$
$$+ \int_{\omega_t}\int_{\partial\omega_{\mathbf{X}}^N} \left(\frac{\partial \mathcal{L}}{\partial d_{\mathbf{x}} \varphi}.\mathbf{N}\right).\delta\varphi \,\mathrm{d}\mathbf{S}\mathrm{d}t$$

- *Hamilton's principle:*

$$\boxed{D_\varphi \mathcal{A}.\delta\varphi + \int_{\omega_t}\int_{\partial\omega_{\mathbf{X}}^N} \mathbf{T}.\delta\varphi \,\mathrm{d}\mathbf{S}\mathrm{d}t = 0 \quad \forall \delta\varphi \in \mathcal{M}_0}$$

- *Euler-Lagrange equations for elastodynamics:*

$$\boxed{\begin{array}{l} \dfrac{\partial \mathcal{L}}{\partial \varphi} - d_t\left(\dfrac{\partial \mathcal{L}}{\partial d_t \varphi}\right) - d_{\mathbf{x}}.\left(\dfrac{\partial \mathcal{L}}{\partial d_{\mathbf{x}} \varphi}\right) = 0 \quad in\ \omega_{\mathbf{X}} \\ \dfrac{\partial \mathcal{L}}{\partial d_{\mathbf{x}} \varphi}.\mathbf{N} + \mathbf{T} = 0 \quad on\ \partial\omega_{\mathbf{X}}^N \end{array}}$$

Variational formalism for elastodynamics

- Expressions of the first derivatives of the Lagrangian:

$$\frac{\partial \mathcal{L}}{\partial \varphi} = \mathbf{b}$$

$$\frac{\partial \mathcal{L}}{\partial d_t \varphi} = \rho d_t \varphi = \rho \dot{\varphi} = \mathbf{p}$$ Momentum

$$\frac{\partial \mathcal{L}}{\partial d_{\mathbf{x}} \varphi} = -\frac{\partial \psi}{\partial d_{\mathbf{x}} \varphi} = -\mathbf{\Pi}$$ Non-symmetric stress tensor of Piola–Kirchhoff 1

- Field equations of elastodynamics:

$$\boxed{\begin{array}{l} \mathbf{b} - d_t \mathbf{p} + d_{\mathbf{x}}.\mathbf{\Pi} = 0 \quad in\ \omega_t \times \omega_{\mathbf{X}} \\ -\mathbf{\Pi}.\mathbf{N} + \mathbf{T} = 0 \quad in\ \omega_t \times \partial \omega_{\mathbf{X}}^N \end{array}}$$

Variational formalism for elastodynamics

- *Horizontal variation:* $\boxed{\delta \mathbf{m} \equiv (\delta t, \delta \mathbf{X}, \delta\varphi = 0)}$

- *Determination of horizontal infinitesimal generators* $D_t\mathcal{A}$ *and* $D_{\mathbf{X}}\mathcal{A}$ *:*

$$\begin{aligned}
&\delta_t\mathcal{A} + \delta_{\mathbf{X}}\mathcal{A} = \\
&\int_{\omega_t}\int_{\omega_{\mathbf{X}}}\left(\frac{\partial\mathcal{L}}{\partial t}\delta t + \frac{\partial\mathcal{L}}{\partial\varphi}.d_t\varphi\delta t + \frac{\partial\mathcal{L}}{\partial d_t\varphi}.d_{tt}\varphi\delta t + \frac{\partial\mathcal{L}}{\partial d_{\mathbf{x}}\varphi}.d_{t\mathbf{x}}\varphi\delta t\right)\,\mathrm{d}\mathbf{X}\mathrm{d}t \\
&+\int_{\omega_t}\int_{\omega_{\mathbf{X}}}\left(\frac{\partial\mathcal{L}}{\partial\mathbf{X}}.\delta\mathbf{X} + \frac{\partial\mathcal{L}}{\partial\varphi}.d_{\mathbf{x}}\varphi.\delta\mathbf{X} + \frac{\partial\mathcal{L}}{\partial d_t\varphi}.d_{\mathbf{x}t}\varphi.\delta\mathbf{X} + \frac{\partial\mathcal{L}}{\partial d_{\mathbf{x}}\varphi} : d_{\mathbf{x}\mathbf{x}}\varphi.\delta\mathbf{X}\right)\,\mathrm{d}\mathbf{X}\mathrm{d}t \\
&+\int_{\omega_t}\int_{\omega_{\mathbf{X}}}\mathcal{L}\,\delta\left(d\mathbf{X}\mathrm{d}t\right) = \delta\mathcal{A}^1 + \delta\mathcal{A}^2 + \delta\mathcal{A}^3
\end{aligned}$$

with:

$$\begin{cases}
\delta\mathcal{A}^1 = \int_{\omega_t}\int_{\omega_{\mathbf{X}}}\left(\frac{\partial\mathcal{L}}{\partial t} + d_t\left(\frac{\partial\mathcal{L}}{\partial d_t\varphi}.d_t\varphi\right) + d_{\mathbf{x}}.\left(\frac{\partial\mathcal{L}}{\partial d_{\mathbf{x}}\varphi}.d_t\varphi\right)\right)\delta t \\
\quad + \left(\frac{\partial\mathcal{L}}{\partial\varphi} - d_t(\frac{\partial\mathcal{L}}{\partial d_t\varphi}) - d_{\mathbf{x}}.(\frac{\partial\mathcal{L}}{\partial d_{\mathbf{x}}\varphi})\right).d_t\varphi\delta t\,\mathrm{d}\mathbf{X}\mathrm{d}t \\
\delta\mathcal{A}^2 = \int_{\omega_t}\int_{\omega_{\mathbf{X}}}\left(\frac{\partial\mathcal{L}}{\partial\mathbf{X}} + d_{\mathbf{x}}.(\frac{\partial\mathcal{L}}{\partial d_{\mathbf{x}}\varphi}.d_{\mathbf{x}}\varphi) + d_t(\frac{\partial\mathcal{L}}{\partial d_t\varphi}.d_{\mathbf{x}}\varphi)\right).\delta\mathbf{X} \\
+\left(\frac{\partial\mathcal{L}}{\partial\varphi} - d_t(\frac{\partial\mathcal{L}}{\partial d_t\varphi}) - d_{\mathbf{x}}.(\frac{\partial\mathcal{L}}{\partial d_{\mathbf{x}}\varphi})\right).d_{\mathbf{x}}\varphi.\delta\mathbf{X}\,\mathrm{d}\mathbf{X}\mathrm{d}t \\
\delta\mathcal{A}^3 = -\int_{\omega_t}\int_{\omega_{\mathbf{X}}}((d_{\mathbf{X}}\mathcal{L}).\delta\mathbf{X} + (d_t\mathcal{L})\delta t)\mathrm{d}\mathbf{X}\mathrm{d}t
\end{cases}$$

Variational formalism for elastodynamics

- *Note on the third term:*

$$\delta\mathcal{A}^3 = \int_{\omega_\tau}\int_{\omega_\Xi}\mathcal{L}\,\delta\left(d_\Xi\mathbf{X}d_\tau t\right)\mathrm{d}\Xi\mathrm{d}\tau$$

$$\forall(\delta t(\tau),\delta\mathbf{X}(\Xi))\in\square_0$$

$$\square_0\equiv((\delta t(\tau),\delta\mathbf{X}(\Xi))\in H^1(\omega_\tau\times\omega_\Xi);\delta t(\tau)|_{\partial\omega_\tau}=0;\delta\mathbf{X}(\Xi)|_{\partial\omega_\Xi}=0)$$

- *Parametric configuration:* $\omega_\tau=[0,1]$ and $\omega_\Xi=[0,1]^3$

$$\delta\mathcal{A}^3=\int_{\omega_\tau}\int_{\omega_\Xi}\mathcal{L}\left(\delta(d_\Xi\mathbf{X})d_\tau t+d_\Xi\mathbf{X}\delta(d_\tau t)\right)\mathrm{d}\Xi\mathrm{d}\tau$$

$$=\int_{\omega_\tau}\int_{\omega_\Xi}\mathcal{L}(d_\Xi\delta\mathbf{X})d_\tau t\mathrm{d}\Xi\mathrm{d}\tau+\int_{\omega_\tau}\int_{\omega_\Xi}\mathcal{L}(d_\tau\delta t)d_\Xi\mathbf{X}\mathrm{d}\Xi\mathrm{d}\tau$$

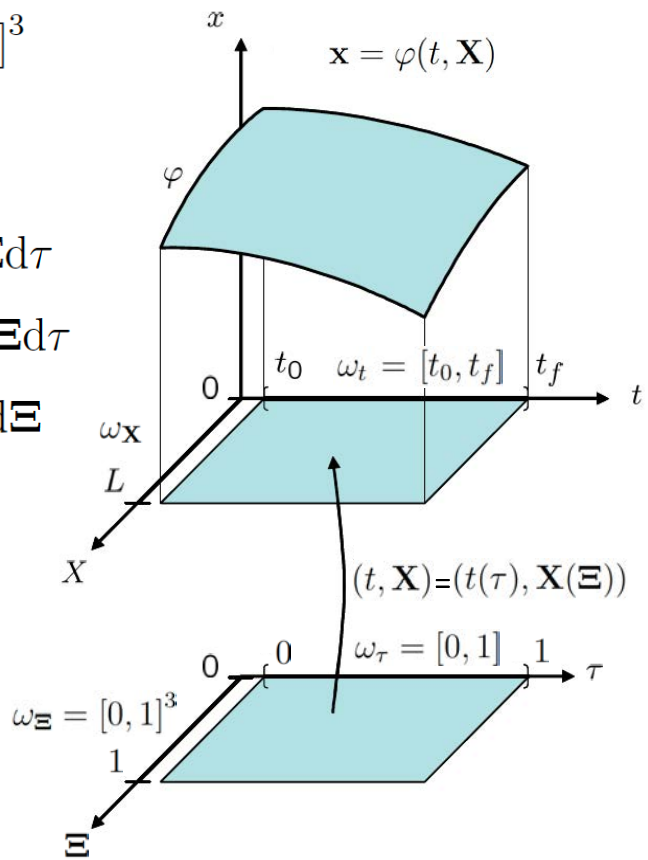
$$=\int_{\omega_\tau}\int_{\omega_\Xi}d_\Xi.(\mathcal{L}\delta\mathbf{X})d_\tau t\mathrm{d}\Xi\mathrm{d}\tau-\int_{\omega_\tau}\int_{\omega_\Xi}(d_\Xi\mathcal{L}).\delta\mathbf{X}d_\tau t\mathrm{d}\Xi\mathrm{d}\tau$$

$$+\int_{\omega_\Xi}\int_{\omega_\tau}d_\tau(\mathcal{L}\delta t)d_\Xi\mathbf{X}\mathrm{d}\tau\mathrm{d}\Xi-\int_{\omega_\Xi}\int_{\omega_\tau}(d_\tau\mathcal{L})\delta t d_\Xi\mathbf{X}\mathrm{d}\tau\mathrm{d}\Xi$$

with:

$$\begin{cases}\mathbf{x}=\varphi(t,\mathbf{X})=\varphi(t(\tau),\mathbf{X}(\Xi))\\ \mathrm{d}\mathbf{X}\mathrm{d}t=(d_\Xi\mathbf{X}d_\tau t)\mathrm{d}\Xi\mathrm{d}\tau\end{cases}$$

$$\boxed{\delta\mathcal{A}^3=-\int_{\omega_t}\int_{\omega_\mathbf{X}}((d_\mathbf{X}\mathcal{L}).\delta\mathbf{X}+(d_t\mathcal{L})\delta t)\mathrm{d}\mathbf{X}\mathrm{d}t}$$



Variational formalism for elastodynamics

- *Hamilton's principle:*

$$
\begin{aligned}
&\delta_t \mathcal{A} + \delta_{\mathbf{x}} \mathcal{A} = \\
&\int_{\omega_t} \int_{\omega_{\mathbf{X}}} \left(\frac{\partial \mathcal{L}}{\partial t} + d_t \left(\frac{\partial \mathcal{L}}{\partial d_t \varphi} . d_t \varphi \right) + d_{\mathbf{x}} . \left(\frac{\partial \mathcal{L}}{\partial d_{\mathbf{x}} \varphi} . d_t \varphi \right) - d_t \mathcal{L} \right) \delta t \mathrm{d}\mathbf{X} \mathrm{d}t \\
&+ \int_{\omega_t} \int_{\omega_{\mathbf{X}}} \left(\frac{\partial \mathcal{L}}{\partial \mathbf{X}} + d_{\mathbf{x}} . \left(\frac{\partial \mathcal{L}}{\partial d_{\mathbf{x}} \varphi} . d^s_{\mathbf{x}} \varphi \right) + d_t \left(\frac{\partial \mathcal{L}}{\partial d_t \varphi} . d_{\mathbf{x}} \varphi \right) - d_{\mathbf{X}} \mathcal{L} \right) . \delta \mathbf{X} \mathrm{d}\mathbf{X} \mathrm{d}t \\
&= D_t \mathcal{A} . \delta t + D_{\mathbf{X}} \mathcal{A} . \delta \mathbf{X} = 0
\end{aligned}
$$

- *Conservation of mechanical energy (horizontal infinitesimal time generator):*

$$
\boxed{\frac{\partial \mathcal{L}}{\partial t} + d_t \left(\frac{\partial \mathcal{L}}{\partial d_t \varphi} . d_t \varphi - \mathcal{L} \right) + d_{\mathbf{x}} . \left(\frac{\partial \mathcal{L}}{\partial d_{\mathbf{x}} \varphi} . d_t \varphi \right) = 0}
$$

- *Conservation of configurational forces (horizontal infinitesimal space generator):*

$$
\boxed{\frac{\partial \mathcal{L}}{\partial \mathbf{X}} + d_{\mathbf{x}} . \left(\frac{\partial \mathcal{L}}{\partial d_{\mathbf{x}} \varphi} . d_{\mathbf{x}} \varphi - \mathcal{L} \mathbf{I} \right) + d_t \left(\frac{\partial \mathcal{L}}{\partial d_t \varphi} . d_{\mathbf{x}} \varphi \right) = 0}
$$

Variational formalism for elastodynamics

- *Space-time matrix formulation:*

$$\boxed{\left(\frac{\partial \mathcal{L}}{\partial t}, \frac{\partial \mathcal{L}}{\partial \mathbf{X}}\right) + (d_t, d_{\mathbf{x}}).\begin{bmatrix} \frac{\partial \mathcal{L}}{\partial d_t\varphi}.d_t\varphi - \mathcal{L} & \frac{\partial \mathcal{L}}{\partial d_t\varphi}.d_{\mathbf{x}}\varphi \\ \frac{\partial \mathcal{L}}{\partial d_{\mathbf{x}}\varphi}.d_t\varphi & \frac{\partial \mathcal{L}}{\partial d_{\mathbf{x}}\varphi}.d_{\mathbf{x}}\varphi - \mathcal{L}\mathbf{I} \end{bmatrix} = (0,0)}$$

- *Energy momentum tensor (space-time Eshelby tensor) for elastodynamics:*

$$\left(\frac{\partial \mathcal{L}}{\partial t}, \frac{\partial \mathcal{L}}{\partial \mathbf{X}}\right) + (d_t, d_{\mathbf{x}}).\mathbf{C}_{t\mathbf{x}} = (0,0)$$

- *Case where the Lagrangian does not explicitly depend on time and space:* $\mathcal{L}\,(\varphi, d_t\varphi, d_{\mathbf{x}}\varphi)$

$$\boxed{\begin{array}{l} d_t(T+V) = d_{\mathbf{x}}.(\mathbf{\Pi}.\mathbf{V}) \\ d_{\mathbf{x}}.(\mathbf{C}) - \mathbf{F}^T.\mathbf{b} + d_t(\mathbf{F}^T.\mathbf{p}) = 0 \\ \mathbf{C} = (-T+\psi)\mathbf{I} - \mathbf{\Pi}^T.\mathbf{F} \end{array}}$$

(mechanical energy)

(configurational forces)

(Eshelby dynamic tensor)

- *Integral form of mechanical energy conservation:*

$$\boxed{\left[\int_{\omega_{\mathbf{X}}} (T+V)\,\mathrm{d}\mathbf{X}\right]_{t_0}^{t_f} = \int_{\omega_t}\int_{\partial\omega_{\mathbf{X}}^D} \mathbf{\Pi}.\mathbf{N}.\mathbf{V}\,\mathrm{d}\mathbf{S}\mathrm{d}t + \int_{\omega_t}\int_{\partial\omega_{\mathbf{X}}^N} \mathbf{T}.\mathbf{V}\,\mathrm{d}\mathbf{S}\mathrm{d}t}$$

[Desmorat et al. 2026]

Variational formalism for elastodynamics

- Vertical generator for any subdomain: $\omega'_{\mathbf{X}} \subset \omega_{\mathbf{X}}$

$$\delta_\varphi \mathcal{A}' = D_\varphi \mathcal{A}'.\delta\varphi = \left[\int_{\omega'_{\mathbf{X}}} \left(\frac{\partial \mathcal{L}}{\partial d_t \varphi}\right).\delta\varphi \, \mathrm{d}\mathbf{X}\right]_{t_0}^{t_f} + \int_{\omega_t} \int_{\partial \omega'_{\mathbf{X}}} \left(\frac{\partial \mathcal{L}}{\partial d_{\mathbf{x}} \varphi}.\mathbf{N}\right).\delta\varphi \, \mathrm{d}\mathbf{S}\mathrm{d}t$$
$$- \int_{\omega_t} \int_{\omega'_{\mathbf{X}}} \frac{\partial \mathcal{L}}{\partial \varphi}.\delta\varphi \, \mathrm{d}\mathbf{X}\mathrm{d}t$$

- Noether's theorem:

$$\boxed{\frac{d\mathcal{A}'}{d\epsilon}(\varphi^\epsilon)|_{\epsilon=0} = D_\varphi \mathcal{A}'.\frac{d\varphi^\epsilon}{d\epsilon}|_{\epsilon=0} = 0 \quad \forall \delta\varphi^\epsilon \in \mathcal{M} \equiv (\delta\varphi^\epsilon \in H^1(\omega_t \times \omega'_{\mathbf{X}}))}$$

- For any vector $\mathbf{v}$ of R^3 : $\boxed{\varphi^\epsilon = \varphi + \epsilon \mathbf{v}}$

$$\frac{d\mathcal{A}'}{d\epsilon}(\varphi^\epsilon)|_{\epsilon=0} = D_\varphi \mathcal{A}'.\mathbf{v}$$
$$= \left[\int_{\omega'_{\mathbf{X}}} \left(\frac{\partial \mathcal{L}}{\partial d_t \varphi}\right).\mathbf{v} \, \mathrm{d}\mathbf{X}\right]_{t_0}^{t_f} + \int_{\omega_t} \int_{\partial \omega'_{\mathbf{X}}} \left(\frac{\partial \mathcal{L}}{\partial d_{\mathbf{x}} \varphi}.\mathbf{N}\right).\mathbf{v} \, \mathrm{d}\mathbf{S}\mathrm{d}t - \int_{\omega_t} \int_{\omega'_{\mathbf{X}}} \frac{\partial \mathcal{L}}{\partial \varphi}.\mathbf{v} \, \mathrm{d}\mathbf{X}\mathrm{d}t$$
$$= \int_{\omega_t} \int_{\omega'_{\mathbf{X}}} (d_t(\frac{\partial \mathcal{L}}{\partial d_t \varphi}.\mathbf{v}) + d_{\mathbf{x}}.(\frac{\partial \mathcal{L}}{\partial d_{\mathbf{x}} \varphi}.\mathbf{v}) - \frac{\partial \mathcal{L}}{\partial \varphi}.\mathbf{v}) \, \mathrm{d}\mathbf{X}\mathrm{d}t$$
$$= \int_{\omega_t} \int_{\omega'_{\mathbf{X}}} (d_t(\frac{\partial \mathcal{L}}{\partial d_t \varphi}) + d_{\mathbf{x}}.(\frac{\partial \mathcal{L}}{\partial d_{\mathbf{x}} \varphi}) - \frac{\partial \mathcal{L}}{\partial \varphi}).\mathbf{v} \, \mathrm{d}\mathbf{X}\mathrm{d}t = 0 \quad \forall \mathbf{v} \in R^3$$

➡ We again obtain the balance of linear momentum

Variational formalism for elastodynamics

- *For any antisymmetric tensor* $\mathbf{\Omega}$ *:* $\boxed{\varphi^\epsilon = exp(\mathbf{\Omega}\epsilon)\varphi}$

$$
\begin{aligned}
&\frac{d\mathcal{A}'}{d\epsilon}(\varphi^\epsilon)|_{\epsilon=0} = D_\varphi \mathcal{A}'.\mathbf{\Omega}\varphi \\
&= \int_{\omega_t}\int_{\omega'_{\mathbf{X}}} (d_t(\frac{\partial\mathcal{L}}{\partial d_t\varphi}.\mathbf{\Omega}\varphi) + d_{\mathbf{x}}.(\frac{\partial\mathcal{L}}{\partial d_{\mathbf{x}}\varphi}.\mathbf{\Omega}\varphi) - \frac{\partial\mathcal{L}}{\partial\varphi}.\mathbf{\Omega}\varphi)\,\mathrm{d}\mathbf{X}\mathrm{d}t \\
&= \int_{\omega_t}\int_{\omega'_{\mathbf{X}}} (d_t(\frac{\partial\mathcal{L}}{\partial d_t\varphi}) + d_{\mathbf{x}}.(\frac{\partial\mathcal{L}}{\partial d_{\mathbf{x}}\varphi}) - \frac{\partial\mathcal{L}}{\partial\varphi}).\mathbf{\Omega}\varphi\,\mathrm{d}\mathbf{X}\mathrm{d}t \\
&+ \int_{\omega_t}\int_{\omega'_{\mathbf{X}}} \frac{\partial\mathcal{L}}{\partial d_{\mathbf{x}}\varphi} : (\mathbf{\Omega} d_{\mathbf{x}}\varphi)\,\mathrm{d}\mathbf{X}\mathrm{d}t \\
&= 0 \quad \forall\mathbf{\Omega}
\end{aligned}
$$

It comes:

$$
\begin{aligned}
&\frac{\partial\mathcal{L}}{\partial d_{\mathbf{x}}\varphi} : (\mathbf{\Omega} d_{\mathbf{x}}\varphi) = 0 \ \forall\mathbf{\Omega} \iff (\mathbf{\Pi}.\mathbf{F}^T) : \mathbf{\Omega} = 0 \ \forall\mathbf{\Omega} \\
&\iff \mathbf{\Pi}.\mathbf{F}^T = (\mathbf{\Pi}.\mathbf{F}^T)^T
\end{aligned}
$$

$$
\begin{cases} J = det(\mathbf{F}) \\ \sigma = J^{-1}\mathbf{\Pi}.\mathbf{F}^T \end{cases}
$$

$\boxed{\sigma = \sigma^T}$ *balance of angular momentum*

Noether's theorem can also be applied to horizontal generators (e.g. rotational invariance causes Eshelby's dynamic tensor symmetry)

$$\boxed{\mathbf{C} = (-T+\psi)\mathbf{I} - \mathbf{\Pi}^T.\mathbf{F}}$$

[Lew et al 2004]

Multi-symplectic formalism for elastodynamics

- Partial Legendre–Fenchel transforms (see Eshelby space-time tensor):

$$\boxed{\begin{aligned} T^*\left(\frac{\partial T}{\partial d_t\varphi}\right) &= \sup_{d_t\varphi}\left(\frac{\partial T}{\partial d_t\varphi}.d_t\varphi - T(d_t\varphi)\right) \\ \psi^*\left(\frac{\partial \psi}{\partial d_{\mathbf{x}}\varphi}\right) &= \sup_{d_{\mathbf{x}}\varphi}\left(\frac{\partial \psi}{\partial d_{\mathbf{x}}\varphi} : d_{\mathbf{x}}\varphi - \psi(d_{\mathbf{x}}\varphi)\right) \end{aligned}}$$

$$\begin{aligned} T^*(\mathbf{p}) &= \sup_{d_t\varphi}(\mathbf{p}.d_t\varphi - T(d_t\varphi)) \\ \psi^*(\mathbf{\Pi}) &= \sup_{d_{\mathbf{x}}\varphi}(\mathbf{\Pi} : d_{\mathbf{x}}\varphi - \psi(d_{\mathbf{x}}\varphi)) \end{aligned}$$

- Action with two partial dualizations on momentum and stress fields:

$$\boxed{\begin{aligned} &\mathscr{A}^*[\mathbf{m}^*] = \int_{\omega_t}\int_{\omega_{\mathbf{X}}} \left(\mathbf{p}.d_t\varphi - \Pi : d_{\mathbf{x}}\varphi - \mathscr{L}^*(t,\mathbf{X},\varphi,\mathbf{p},\Pi)\right) \mathrm{d}\mathbf{X}\mathrm{d}t \\ &\mathscr{L}^*(t,\mathbf{X},\varphi,\mathbf{p},\Pi) = T^*(\mathbf{p}) - \psi^*(\Pi) - V_{\text{ext}}(\varphi) \ \text{ and } \ T^*(\mathbf{p}) = \tfrac{1}{2}\rho^{-1}\mathbf{p}.\mathbf{p} \end{aligned}}$$

- Corresponding state vector: $\mathbf{m}^* \equiv (t, \mathbf{X}, \varphi, \mathbf{p}, \mathbf{\Pi})$

- Hamilton's principle:

$$\boxed{\begin{aligned} &\delta\mathcal{A}^* + \int_{\omega_t}\int_{\partial\omega_{\mathbf{X}}^N} \mathbf{T}.\delta\varphi \,\mathrm{d}\mathbf{S}\mathrm{d}t = 0 \quad \forall \delta\mathbf{m}^* \in \mathcal{M}_0^* \\ &\mathcal{M}_0^* \equiv (\delta\mathbf{m}^* \in H^1(\omega_t \times \omega_{\mathbf{X}}); \delta t|_{\partial\omega_t} = 0; \delta\mathbf{X}|_{\partial\omega_{\mathbf{X}}} = 0; \ldots \\ &\ldots\delta(\varphi, \mathbf{p}, \mathbf{\Pi})|_{\partial\omega_t} = 0; \delta\varphi|_{\partial\omega_{\mathbf{X}}^D} = 0) \end{aligned}}$$

Multi-symplectic formalism for elastodynamics

- Notations of the 5 corresponding infinitesimal generators (2 horizontal and 3 vertical):

$$\boxed{\begin{aligned}&\delta\mathcal{A}^* = \frac{\partial\mathcal{A}^*}{\partial\mathbf{m}^*}.\delta\mathbf{m}^* = \delta_t\mathcal{A}^* + \delta_{\mathbf{x}}\mathcal{A}^* + \delta_\varphi\mathcal{A}^* + \delta_{\mathbf{p}}\mathcal{A}^* + \delta_{\mathbf{\Pi}}\mathcal{A}^*\\ &= D_t\mathcal{A}^*.\delta t + D_{\mathbf{X}}\mathcal{A}^*.\delta\mathbf{X} + D_\varphi\mathcal{A}^*.\delta\varphi + D_{\mathbf{p}}\mathcal{A}^*.\delta\mathbf{p} + D_{\mathbf{\Pi}}\mathcal{A}^*.\delta\mathbf{\Pi}\end{aligned}}$$

- Calculation of the 3 vertical infinitesimal generators:

$$\begin{aligned}
&\delta_\varphi\mathcal{A}^* + \delta_{\mathbf{p}}\mathcal{A}^* + \delta_{\mathbf{\Pi}}\mathcal{A}^* =\\
&\int_{\omega_t}\int_{\omega_{\mathbf{X}}}\left(\delta\mathbf{p}.d_t\varphi + \mathbf{p}.\delta d_t\varphi - \delta\mathbf{\Pi} : d_{\mathbf{x}}\varphi - \mathbf{\Pi} : \delta d_{\mathbf{x}}\varphi\right)\mathrm{d}\mathbf{X}\mathrm{d}t\\
&-\int_{\omega_t}\int_{\omega_{\mathbf{X}}}\left(\frac{\partial\mathcal{L}^*}{\partial\varphi}.\delta\varphi + \frac{\partial\mathcal{L}^*}{\partial\mathbf{p}}.\delta\mathbf{p} + \frac{\partial\mathcal{L}^*}{\partial\mathbf{\Pi}}.\delta\mathbf{\Pi}\right)\mathrm{d}\mathbf{X}\mathrm{d}t\\
&\iff \int_{\omega_t}\int_{\omega_{\mathbf{X}}}\left(\delta\mathbf{p}.d_t\varphi - d_t\mathbf{p}.\delta\varphi - \delta\mathbf{\Pi} : d_{\mathbf{x}}\varphi + d_{\mathbf{x}}.\mathbf{\Pi}.\delta\varphi\right)\mathrm{d}\mathbf{X}\mathrm{d}t\\
&+\int_{\omega_t}\int_{\partial\omega_{\mathbf{X}}^N}(-\mathbf{\Pi}.\mathbf{N}).\delta\varphi\,\mathrm{d}\mathbf{S}\mathrm{d}t\\
&-\int_{\omega_t}\int_{\omega_{\mathbf{X}}}\left(\frac{\partial\mathcal{L}^*}{\partial\varphi}.\delta\varphi + \frac{\partial\mathcal{L}^*}{\partial\mathbf{p}}.\delta\mathbf{p} + \frac{\partial\mathcal{L}^*}{\partial\mathbf{\Pi}} : \delta\mathbf{\Pi}\right)\mathrm{d}\mathbf{X}\mathrm{d}t\\
&\iff \int_{\omega_t}\int_{\omega_{\mathbf{X}}}\left(-\frac{\partial\mathcal{L}^*}{\partial\varphi} - d_t\mathbf{p} + d_{\mathbf{x}}.\mathbf{\Pi}\right).\delta\varphi\,\mathrm{d}\mathbf{X}\mathrm{d}t\\
&+\int_{\omega_t}\int_{\partial\omega_{\mathbf{X}}^N}(-\mathbf{\Pi}.\mathbf{N}).\delta\varphi\,\mathrm{d}\mathbf{S}\mathrm{d}t\\
&+\int_{\omega_t}\int_{\omega_{\mathbf{X}}}\left(-\frac{\partial\mathcal{L}^*}{\partial\mathbf{p}} + d_t\varphi\right).\delta\mathbf{p}\,\mathrm{d}\mathbf{X}\mathrm{d}t\\
&+\int_{\omega_t}\int_{\omega_{\mathbf{X}}}\left(-\frac{\partial\mathcal{L}^*}{\partial\mathbf{\Pi}} - d_{\mathbf{x}}\varphi\right) : \delta\mathbf{\Pi}\,\mathrm{d}\mathbf{X}\mathrm{d}t
\end{aligned}$$

Multi-symplectic formalism for elastodynamics

- Hamilton's principle:

$$D_{\varphi}\mathcal{A}^*.\delta\varphi + D_{\mathbf{p}}\mathcal{A}^*.\delta\mathbf{p} + D_{\mathbf{\Pi}}\mathcal{A}^*.\delta\mathbf{\Pi} + \int_{\omega_t}\int_{\partial\omega_{\mathbf{X}}^N} \mathbf{T}.\delta\varphi\, \mathrm{d}\mathbf{S}\mathrm{d}t = 0$$
$$\forall(\delta\varphi, \delta\mathbf{p}, \delta\mathbf{\Pi}) \in \mathcal{M}_0^*$$

- Euler-Lagrange equations of 3-field mixed formulation of elastodynamics: $\mathbf{s} \equiv (\varphi, \mathbf{p}, \mathbf{\Pi})$

$$\boxed{\begin{array}{lll} -\frac{\partial\mathcal{L}^*}{\partial\varphi} - d_t\mathbf{p} + d_{\mathbf{x}}.\mathbf{\Pi} & = 0 & in\ \omega_t \times \omega_{\mathbf{X}} \\ -\frac{\partial\mathcal{L}^*}{\partial\mathbf{p}} + d_t\varphi & = 0 & in\ \omega_t \times \omega_{\mathbf{X}} \\ -\frac{\partial\mathcal{L}^*}{\partial\mathbf{\Pi}} - d_{\mathbf{x}}\varphi & = 0 & in\ \omega_t \times \omega_{\mathbf{X}} \\ -\mathbf{\Pi}.\mathbf{N} + \mathbf{T} & = 0 & on\ \omega_t \times \partial\omega_{\mathbf{X}}^N \end{array}}$$

- Multi-symplectic formalism: $\mathbf{W}.d_t\mathbf{s} + \mathbf{K}.d_{\mathbf{x}}\mathbf{s} = \partial_{\mathbf{s}}\mathcal{L}^*$

$$\boxed{\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} d_t \begin{bmatrix} \varphi \\ \mathbf{p} \\ \mathbf{\Pi} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} d_{\mathbf{x}} \begin{bmatrix} \varphi \\ \mathbf{p} \\ \mathbf{\Pi} \end{bmatrix} = \begin{bmatrix} \frac{\partial\mathcal{L}^*}{\partial\varphi} \\ \frac{\partial\mathcal{L}^*}{\partial\mathbf{p}} \\ \frac{\partial\mathcal{L}^*}{\partial\mathbf{\Pi}} \end{bmatrix}}$$

[Razafindralandy et al 2019]

Multi-symplectic formalism for elastodynamics

- Expressions for the first derivatives of the Lagrangian:

$$\frac{\partial \mathcal{L}^*}{\partial \varphi} = -\mathbf{b}$$
$$\frac{\partial \mathcal{L}^*}{\partial \mathbf{p}} = \frac{\partial T^*}{\partial \mathbf{p}} = \rho^{-1}\mathbf{p}$$
$$\frac{\partial \mathcal{L}^*}{\partial \mathbf{\Pi}} = -\frac{\partial \psi^*}{\partial \mathbf{\Pi}} = -\mathbf{F}$$

- Euler-Lagrange equations of 3-field elastodynamics:

$$\boxed{\begin{array}{lll} \mathbf{b} - d_t\mathbf{p} + d_{\mathbf{x}}.\mathbf{\Pi} & = 0 & \mathit{in}\ \omega_t \times \omega_{\mathbf{X}} \\ -\rho^{-1}\mathbf{p} + d_t\varphi & = 0 & \mathit{in}\ \omega_t \times \omega_{\mathbf{X}} \\ \mathbf{F} - d_{\mathbf{x}}\varphi & = 0 & \mathit{in}\ \omega_t \times \omega_{\mathbf{X}} \\ -\mathbf{\Pi}.\mathbf{N} + \mathbf{T} & = 0 & \mathit{in}\ \omega_t \times \partial\omega_{\mathbf{X}}^N \end{array}}$$

- Note: we find a simple symplectic form when the Lagrangian does not depend on $d_{\mathbf{x}}\varphi$ and $\mathbf{\Pi}$:

$$\mathbf{s} \equiv (\varphi, \mathbf{p}) \qquad \begin{cases} \mathcal{A}^*(t, \mathbf{X}, \varphi, \mathbf{p}) = \int_{\omega_t} \int_{\omega_{\mathbf{X}}} (\mathbf{p}.d_t\varphi - \mathcal{L}^*(t, \mathbf{X}, \varphi, \mathbf{p}))\, \mathrm{d}\mathbf{X}\mathrm{d}t \\ \mathcal{L}^* = \mathcal{H} = T^*(\mathbf{p}) + V(\varphi) \end{cases}$$

$$\boxed{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} d_t \begin{bmatrix} \varphi \\ \mathbf{p} \end{bmatrix} = \begin{bmatrix} \frac{\partial \mathcal{L}^*}{\partial \varphi} \\ \frac{\partial \mathcal{L}^*}{\partial \mathbf{p}} \end{bmatrix}} \qquad \mathbf{J}^{-1}.d_t\mathbf{s} = \partial_{\mathbf{s}}\mathcal{L}^*$$

Exercice: quels sont les schémas symplectiques de la famille de Newmark? (GDR GDM, Quiberon 2025)

- $$\boxed{\begin{aligned} \dot{\boldsymbol{q}}_{n+1} &= \dot{\boldsymbol{q}}_n + (1-\gamma)h\ddot{\boldsymbol{q}}_n + \gamma h\ddot{\boldsymbol{q}}_{n+1} \\ \boldsymbol{q}_{n+1} &= \boldsymbol{q}_n + h\dot{\boldsymbol{q}}_n + h^2\left(\frac{1}{2}-\beta\right)\ddot{\boldsymbol{q}}_n + h^2\beta\ddot{\boldsymbol{q}}_{n+1} \end{aligned}}$$

$$\begin{aligned} \boldsymbol{M}\ddot{\boldsymbol{q}}_n &= -\boldsymbol{C}\dot{\boldsymbol{q}}_n - \boldsymbol{K}\boldsymbol{q}_n + \boldsymbol{p}_n \\ \boldsymbol{M}\ddot{\boldsymbol{q}}_{n+1} &= -\boldsymbol{C}\dot{\boldsymbol{q}}_{n+1} - \boldsymbol{K}\boldsymbol{q}_{n+1} + \boldsymbol{p}_{n+1} \end{aligned}$$

- *Matrice d'amplification (slide 7):*

$$\boxed{X_{n+1} = AX_n \qquad X_{n+1} = \begin{pmatrix} q_{n+1} \\ \dot{q}_{n+1} \end{pmatrix}}$$

$$\begin{aligned} D &= \det(\boldsymbol{H}_1) = 1 + 2\varepsilon\gamma\omega h + \beta\omega^2 h^2 \\ t_{11} &= \frac{1}{D}\left[1 + 2(\gamma-1)\varepsilon\omega h + (\beta-\gamma)\,\omega^2h^2 + (\gamma-2\beta)\varepsilon\omega^3h^3\right] \\ t_{22} &= \frac{1}{D}\left[1 + 2\gamma\varepsilon\omega h + \left(\beta-\frac{1}{2}\right)\omega^2h^2 + (2\beta-\gamma)\varepsilon^2\omega^3h^3\right] \\ t_{12} &= \frac{1}{D}\left[-\omega^2h + \left(\frac{\gamma}{2}-\beta\right)\omega^4h^3\right] \\ t_{21} &= \frac{1}{D}\left[h + (2\gamma-1)\varepsilon\omega h^2 + 2(2\beta-\gamma)\varepsilon^2\omega^2h^3\right] \end{aligned}$$

- *Structure symplectique imposée à la matrice d'amplification:*

$$\boxed{A^TJA = J \qquad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}$$

$$Q = A^TJA - J,$$

$$Q = \begin{pmatrix} 0 & \det(A) \\ -\det(A) & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad a = \det(A) - 1$$

- *Cas linéaire:* $\gamma = 1/2$
- *Cas non-linéaire:* $\gamma = 1/2 \quad \beta = 0$

$$a \simeq \frac{\left(\gamma-\frac{1}{2}\right)\omega_n^2h^2}{1+\beta\omega_n^2h^2}$$

$$a \simeq \frac{\left(\gamma-\frac{1}{2}\right)\omega_n^2h^2 + \beta(\omega_{n+1}^2-\omega_n^2)h^2}{1+\beta\omega_{n+1}^2h^2}$$

[Simo 1991] [Géradin et Rixen 1993]

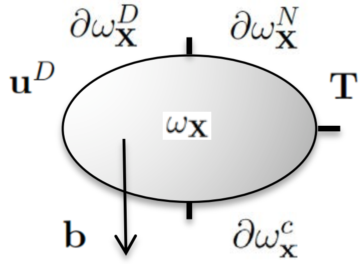
3

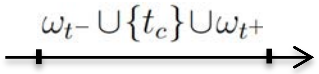
Application à l'élastodynamique non-régulière

Multi-symplectic variational formalism for non-smooth elastodynamics

- *Definition of the potentially contact edge and gap:*

$$C \equiv (\mathbf{X} \in \partial\omega^c_{\mathbf{X}}; g_N(\mathbf{X},t) \geq 0)$$
$$g_N = (\varphi - \mathbf{X}_0).(-\mathbf{N})$$
$$\partial\omega_{\mathbf{X}} = \partial\omega^D_{\mathbf{X}} \cup \partial\omega^N_{\mathbf{X}} \cup \partial\omega^c_{\mathbf{X}}$$





$$\omega_{t^+} \equiv [t_c^+, t_f]$$
$$\omega_{t^-} \equiv [t_0, t_c^-]$$

- *7-field action with Lagrange multiplier for non-smooth elastodynamics:*

$$\boxed{\begin{aligned}&\widetilde{\mathcal{A}}(t, t_c, \mathbf{X}, \varphi, \mathbf{p}, \mathbf{\Pi}, \lambda)\\ &= \mathcal{A}^{*-}(t, \mathbf{X}, \varphi, \mathbf{p}, \mathbf{\Pi}, \lambda) + \mathcal{A}^{*+}(t, \mathbf{X}, \varphi, \mathbf{p}, \mathbf{\Pi}, \lambda) + \int_{\partial\omega^c_{\mathbf{X}}} \lambda(t_c).g_N(t_c)\, d\mathbf{S}\end{aligned}}$$

with:

$$\begin{cases} \mathcal{A}^{*\pm}(t, \mathbf{X}, \varphi, \mathbf{p}, \mathbf{\Pi}) \\ = \int_{\omega_{t^\pm}} \int_{\omega_{\mathbf{X}}} (\mathbf{p}.d_t\varphi - \mathbf{\Pi} : d_{\mathbf{X}}\varphi - \mathcal{L}^*(t, \mathbf{X}, \varphi, \mathbf{p}, \mathbf{\Pi}))\, d\mathbf{X}dt \\ + \int_{\omega_{t^\pm}} \int_{\partial\omega^c_{\mathbf{X}}} \lambda(t).\dot{g}_N(t)\, d\mathbf{S}dt \\ \mathcal{L}^* = T^*(\mathbf{p}) - \psi^*(\mathbf{\Pi}) - V_{ext}(\varphi) \quad T^*(\mathbf{p}) = \frac{1}{2}\rho^{-1}\mathbf{p}.\mathbf{p} \end{cases}$$

- *Karush-Kuhn-Tucker complementary conditions over the entire time domain:*

$$\boxed{\lambda(\mathbf{X},t).g_N(\mathbf{X},t) = 0 \quad \lambda(\mathbf{X},t) \geq 0 \quad g_N(\mathbf{X},t) \geq 0 \ \forall \mathbf{X} \in C\ \forall t \in [t_0, t_f]}$$

[Lee 1983, Fetecau 2003]

Multi-symplectic variational formalism for non-smooth elastodynamics

- *Definition of the state vector:*

$$\tilde{\mathbf{m}} \equiv (t, t_c, \mathbf{X}, \varphi, \mathbf{p}, \mathbf{\Pi}, \lambda)$$

- *Hamilton's principle for non-smooth elastodynamics:*

$$\boxed{\begin{array}{l}\delta\tilde{\mathcal{A}} + \int_{\omega_t}\int_{\partial\omega_{\mathbf{X}}^N} \mathbf{T}.\delta\varphi\,\mathrm{d}\mathbf{S}\mathrm{d}t = 0 \quad \forall\delta\tilde{\mathbf{m}} \in \tilde{\mathcal{M}}_0 \\ \tilde{\mathcal{M}}_0 \equiv (\delta(t, t_c, \mathbf{X}, \varphi, \mathbf{p}, \mathbf{\Pi}) \in H^1(\omega_t \times \omega_{\mathbf{X}}); \delta t|_{[t_0,t_f]} = 0; \delta\mathbf{X}|_{\partial\omega_{\mathbf{X}}} = 0; .. \\ ...; \delta(\varphi, \mathbf{p}, \mathbf{\Pi})|_{[t_0,t_f]} = 0; \delta\varphi|_{\partial\omega_{\mathbf{X}}^D} = 0; \delta\lambda \in H^{\frac{1}{2}}(\omega_t \times C); \delta\lambda(t_c) > 0)\end{array}}$$

- *Definition of the 7 infinitesimal generators (3 horizontal, 4 vertical):*

$$\delta\tilde{\mathcal{A}} = \frac{\partial\tilde{\mathcal{A}}}{\partial\tilde{\mathbf{m}}}.\delta\tilde{\mathbf{m}} = \delta_t\tilde{\mathcal{A}} + \delta_{\mathbf{x}}\tilde{\mathcal{A}} + \delta_\varphi\tilde{\mathcal{A}} + \delta_{\mathbf{p}}\tilde{\mathcal{A}} + \delta_{\mathbf{\Pi}}\tilde{\mathcal{A}} + \delta_{t_c}\tilde{\mathcal{A}} + \delta_\lambda\tilde{\mathcal{A}}$$
$$= D_t\tilde{\mathcal{A}}.\delta t + D_{\mathbf{X}}\tilde{\mathcal{A}}.\delta\mathbf{X} + D_\varphi\tilde{\mathcal{A}}.\delta\varphi + D_{\mathbf{p}}\tilde{\mathcal{A}}.\delta\mathbf{p} + D_{\mathbf{\Pi}}\tilde{\mathcal{A}}.\delta\mathbf{\Pi}$$
$$+ D_{t_c}\tilde{\mathcal{A}}.\delta t_c + D_\lambda\tilde{\mathcal{A}}.\delta\lambda$$

Multi-symplectic variational formalism for non-smooth elastodynamics

- Definition of a specific state vector (without space):

$$\boxed{\delta\widetilde{\mathbf{m}} \equiv (\delta t, \delta t_c, \delta\mathbf{X} = 0, \delta\varphi, \delta\mathbf{p}, \delta\mathbf{\Pi}, \delta\lambda)}$$

- Calculation of the 6 corresponding infinitesimal generators (2 horizontal, 4 vertical):

$$
\begin{aligned}
&\delta_t\widetilde{\mathcal{A}} + \delta_\varphi\widetilde{\mathcal{A}} + \delta_{\mathbf{p}}\widetilde{\mathcal{A}} + \delta_{\mathbf{\Pi}}\widetilde{\mathcal{A}} + \delta_{t_c}\widetilde{\mathcal{A}} + \delta_\lambda\widetilde{\mathcal{A}} = \\
&\int_{\omega_{t-}\cup\omega_{t+}}\int_{\omega_{\mathbf{X}}} (\delta\mathbf{p}.d_t\varphi + \mathbf{p}.\delta d_t\varphi - \delta\mathbf{\Pi} : d_{\mathbf{x}}\varphi - \mathbf{\Pi} : \delta d_{\mathbf{x}}\varphi)\,\mathrm{d}\mathbf{X}\mathrm{d}t \\
&- \int_{\omega_{t-}\cup\omega_{t+}}\int_{\omega_{\mathbf{X}}} \left(\tfrac{\partial\mathcal{L}^*}{\partial\varphi}.\delta\varphi + \tfrac{\partial\mathcal{L}^*}{\partial\mathbf{p}}.\delta\mathbf{p} + \tfrac{\partial\mathcal{L}^*}{\partial\mathbf{\Pi}} : \delta\mathbf{\Pi}\right)\mathrm{d}\mathbf{X}\mathrm{d}t \\
&+ \int_{\omega_{t-}\cup\omega_{t+}}\int_{\partial\omega_{\mathbf{x}}^c}(\delta\lambda.\dot{g}_N + \lambda.\delta\dot{g}_N)\,\mathrm{d}\mathbf{S}\mathrm{d}t \\
&+ \int_{\omega_{t-}\cup\omega_{t+}}\int_{\omega_{\mathbf{X}}} \left(\tfrac{\partial(\mathbf{p}.d_t\varphi)}{\partial d_t\varphi}.d_{tt}\varphi.\delta t + \tfrac{\partial(\mathbf{p}.d_t\varphi)}{\partial\mathbf{p}}.d_t\mathbf{p}\delta t\right)\mathrm{d}\mathbf{X}\mathrm{d}t \\
&- \int_{\omega_{t-}\cup\omega_{t+}}\int_{\omega_{\mathbf{X}}} \left(\tfrac{\partial(\mathbf{\Pi}:d_{\mathbf{x}}\varphi)}{\partial d_{\mathbf{x}}\varphi}.d_{t\mathbf{x}}\varphi.\delta t + \tfrac{\partial(\mathbf{\Pi}:d_{\mathbf{x}}\varphi)}{\partial\mathbf{\Pi}} : d_t\mathbf{\Pi}\delta t\right)\mathrm{d}\mathbf{X}\mathrm{d}t \\
&- \int_{\omega_{t-}\cup\omega_{t+}}\int_{\omega_{\mathbf{X}}} \left(\tfrac{\partial\mathcal{L}^*}{\partial t}.\delta t + \tfrac{\partial\mathcal{L}^*}{\partial\varphi}.d_t\varphi\delta t + \tfrac{\partial\mathcal{L}^*}{\partial\mathbf{p}}.d_t\mathbf{p}\delta t + \tfrac{\partial\mathcal{L}^*}{\partial\mathbf{\Pi}} : d_t\mathbf{\Pi}\delta t\right)\mathrm{d}\mathbf{X}\mathrm{d}t \\
&+ \int_{\omega_{t-}\cup\omega_{t+}}\int_{\partial\omega_{\mathbf{x}}^c}\left(\tfrac{\partial(\lambda.\dot{g}_N)}{\partial\lambda}.d_t\lambda.\delta t + \tfrac{\partial(\lambda.\dot{g}_N)}{\partial\dot{g}_N}.d_t g_N\delta t\right)\mathrm{d}\mathbf{S}\mathrm{d}t \\
&+ \int_{\omega_{t-}\cup\omega_{t+}}\left(\int_{\omega_{\mathbf{X}}}(\mathbf{p}.d_t\varphi - \mathbf{\Pi} : d_{\mathbf{x}}\varphi - \mathcal{L}^*)\,\mathrm{d}\mathbf{X} + \int_{\partial\omega_{\mathbf{x}}^c}\lambda.\dot{g}_N\,\mathrm{d}\mathbf{S}\right)\delta(\mathrm{d}t) \\
&+ \int_{\partial\omega_{\mathbf{x}}^c}\delta\lambda(t_c).g_N(t_c)\,\mathrm{d}\mathbf{S} + \int_{\partial\omega_{\mathbf{x}}^c}\lambda(t_c).\delta g_N(t_c)\,\mathrm{d}\mathbf{S}
\end{aligned}
$$

Multi-symplectic variational formalism for non-smooth elastodynamics

- *Hamilton's principle:*

$$D_t\widetilde{\mathcal{A}}.\delta t + D_\varphi\widetilde{\mathcal{A}}.\delta\varphi + D_{\mathbf{p}}\widetilde{\mathcal{A}}.\delta\mathbf{p} + D_{\mathbf{\Pi}}\widetilde{\mathcal{A}}.\delta\mathbf{\Pi}$$
$$+D_{t_c}\widetilde{\mathcal{A}}.\delta t_c + D_\lambda\widetilde{\mathcal{A}}.\delta\lambda + \int_{\omega_t}\int_{\partial\omega_{\mathbf{X}}^N}\mathbf{T}.\delta\varphi\,\mathrm{d}\mathbf{S}\mathrm{d}t = 0$$
$$\forall(\delta t, \delta t_c, \delta\varphi, \delta\mathbf{p}, \delta\mathbf{\Pi}, \delta\lambda) \in \widetilde{\mathcal{M}}_0$$

- *Euler–Lagrange equations of non-smooth elastodynamics:*

$$\begin{aligned}
&-\frac{\partial\mathcal{L}^*}{\partial\varphi} - d_t\mathbf{p} + d_{\mathbf{X}}.\mathbf{\Pi} &&= 0 \quad \text{in } (\omega_{t^-}\cup\omega_{t^+})\times\omega_{\mathbf{X}}\\
&-\frac{\partial\mathcal{L}^*}{\partial\mathbf{p}} + d_t\varphi &&= 0 \quad \text{if } (\omega_{t^-}\cup\omega_{t^+})\times\omega_{\mathbf{X}}\\
&-\frac{\partial\mathcal{L}^*}{\partial\mathbf{\Pi}} - d_{\mathbf{X}}\varphi &&= 0 \quad \text{in } (\omega_{t^-}\cup\omega_{t^+})\times\omega_{\mathbf{X}}\\
&\mathbf{\Pi}.\mathbf{N} &&= \dot{\lambda}\mathbf{N} \quad \text{on } (\omega_{t^-}\cup\omega_{t^+})\times\partial\omega_{\mathbf{X}}^c\\
&\mathbf{\Pi}.\mathbf{N} &&= \mathbf{T} \quad \text{on } (\omega_{t^-}\cup\omega_{t^+})\times\partial\omega_{\mathbf{X}}^N\\
&\frac{\partial\mathcal{L}^*}{\partial t} + d_t(\mathbf{p}.d_t\varphi - \mathcal{L}^*) - d_{\mathbf{X}}(\mathbf{\Pi}.d_t\varphi) &&= 0 \quad \text{in } (\omega_{t^-}\cup\omega_{t^+})\times\omega_{\mathbf{X}}\\
&[\mathbf{p}]_{t_c^-}^{t_c^+} &&= 0 \quad \text{in } \omega_{\mathbf{X}}\\
&[\mathbf{p}.d_t\varphi - \mathcal{L}^*]_{t_c^-}^{t_c^+} &&= 0 \quad \text{in } \omega_{\mathbf{X}}\\
&\lambda(t_c).\dot{g}_N(t_c) &&= 0 \quad \text{on } \partial\omega_{\mathbf{X}}^c\\
&g_N(t_c) &&= 0 \quad \text{on } \partial\omega_{\mathbf{X}}^c\\
&+\text{KKT conditions}
\end{aligned}$$

Multi-symplectic variational formalism for non-smooth elastodynamics

- The KKT conditions combined with the two previous contact conditions are written:

$$\begin{cases} \lambda(\mathbf{X},t).g_N(\mathbf{X},t) = 0 \quad \lambda(\mathbf{X},t) \geq 0 \quad g_N(\mathbf{X},t) \geq 0 \ \forall \mathbf{X} \in C \ \forall t \in [t_0, t_f] \\ \lambda(t_c).\dot{g}_N(t_c) = 0 \quad \text{on } \partial\omega^c_{\mathbf{x}} \\ g_N(t_c) = 0 \quad \text{on } \partial\omega^c_{\mathbf{x}} \end{cases}$$

- These conditions are equivalent to the Hertz-Signorini-Moreau (HSM) conditions:

$$\boxed{\begin{cases} g_N(t) \geq 0 \\ \lambda(t) \geq 0 \\ \lambda(t).g_N(t) = 0 \\ \lambda(t_c).\dot{g}_N(t_c) = 0 \\ g_N(t_c) = 0 \end{cases} \iff \begin{cases} \text{if } \ g_N(t) > 0 \ \text{ then } \ \lambda(t) = 0 \\ \text{if } \ g_N(t) = 0 \ \text{ then } \begin{cases} \dot{g}_N(t) \geq 0 \\ \lambda(t) \geq 0 \\ \lambda(t).\dot{g}_N(t) = 0 \end{cases} \end{cases}}$$

[Moreau 1999]

Multi-symplectic variational formalism for non-smooth elastodynamics

- *From the first derivatives of the Lagrangian, the Euler-Lagrange equations of non-smooth elastodynamics are obtained as part of a mixed formulation, to which we associate the HSM conditions:*

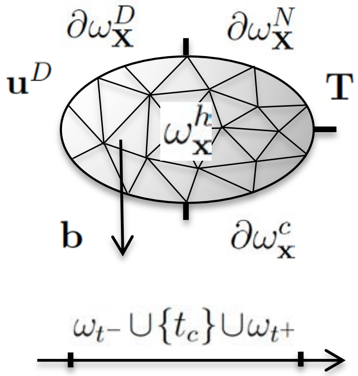
$$
\boxed{
\begin{array}{lll}
\mathbf{b} - d_t\mathbf{p} + d_{\mathbf{x}}.\mathbf{\Pi} & = 0 & \text{in } (\omega_{t^-} \cup \omega_{t^+}) \times \omega_{\mathbf{X}} \\
-\rho^{-1}\mathbf{p} + d_t\varphi & = 0 & \text{if } (\omega_{t^-} \cup \omega_{t^+}) \times \omega_{\mathbf{X}} \\
\mathbf{F} - d_{\mathbf{x}}\varphi & = 0 & \text{in } (\omega_{t^-} \cup \omega_{t^+}) \times \omega_{\mathbf{X}} \\
\mathbf{\Pi}.\mathbf{N} & = \dot{\lambda}.\mathbf{N} & \text{on } (\omega_{t^-} \cup \omega_{t^+}) \times \partial\omega_{\mathbf{X}}^{c} \\
\mathbf{\Pi}.\mathbf{N} & = \mathbf{T} & \text{on } (\omega_{t^-} \cup \omega_{t^+}) \times \partial\omega_{\mathbf{X}}^{N} \\
d_t(T^*(\mathbf{p}) + \psi^*(\mathbf{\Pi})) - d_{\mathbf{x}}(\mathbf{\Pi}.d_t\varphi) & = 0 & \text{in } (\omega_{t^-} \cup \omega_{t^+}) \times \omega_{\mathbf{X}} \\
[\mathbf{p}]_{t_c^-}^{t_c^+} & = 0 & \text{in } \omega_{\mathbf{X}} \\
[T^*(\mathbf{p})]_{t_c^-}^{t_c^+} & = 0 & \text{in } \omega_{\mathbf{X}} \\
+\text{HSM conditions} & &
\end{array}
}
$$

- *Similarly, the expression of the equation for the conservation of configurational forces (mixed formulation) can be calculated from the horizontal space generator $D_{\mathbf{X}}\mathcal{A}$:*

$$\frac{\partial \mathcal{L}^*}{\partial \mathbf{X}} + d_{\mathbf{x}}.(-\mathbf{\Pi}.d_{\mathbf{x}}\varphi - \mathcal{L}^*\mathbf{I}) + d_t(\mathbf{p}.d_{\mathbf{x}}\varphi) = 0$$

Space semi-discretized variational formulation for non-smooth elastodynamics

- *Mesh of geometry:*



Node coordinates: $\mathbf{X}_j$

- *Spatial discretization by the finite element method:*

$$\boxed{\begin{aligned}
\varphi(t,\mathbf{X}) &\simeq \varphi^h(t,\mathbf{X}) = \sum_{i=1}^{n} u_i(t).\phi_i^u(\mathbf{X})\\
\mathbf{p}(t,\mathbf{X}) &\simeq \mathbf{p}^h(t,\mathbf{X}) = \sum_{i=1}^{n} p_i(t).\phi_i^p(\mathbf{X})\\
\mathbf{\Pi}(t,\mathbf{X}) &\simeq \mathbf{\Pi}^h(t,\mathbf{X}) = \sum_{i=1}^{n} s_i(t).\phi_i^s(\mathbf{X})\\
\lambda(t,\mathbf{X}) &\simeq \lambda^h(t,\mathbf{X}) = \sum_{i=1}^{p} \lambda_i(t).\phi_i^\lambda(\mathbf{X})
\end{aligned}}$$

- *Partition of unity and compact support:* $\sum_{i=1}^{n}\phi_i^u(\mathbf{X}) = 1 \qquad \phi_i^u(\mathbf{X}_j) = \delta_{ij}$

Space semi-discretized variational formulation for non-smooth elastodynamics

- Spatial semi-discretized multi-field action for non-smooth elastodynamics:

$$\boxed{\begin{aligned}&\tilde{\mathscr{A}}^h[\tilde{\mathbf{m}}^h] = \int_{\omega_{t^-}\cup\omega_{t^+}} \left(\mathbf{P}^T.\dot{\mathbf{U}} - \mathbf{F}_{\text{int}}^T.\mathbf{U} - \mathscr{L}^{h^*} + \Lambda^T.\dot{\mathbf{g}}_N\right) \mathrm{d}t + \Lambda(t_c)^T.\mathbf{g}_N(t_c)\\ &+ \text{KKT conditions}\\ &\text{with} \quad \mathscr{L}^{h^*} = T^{h^*}(\mathbf{P}) - \Psi^{h^*}(\mathbf{F}_{\text{int}}) - {\mathbf{F}_{\text{ext}}^b}^T.\mathbf{U}\end{aligned}}$$

- State vector: $\tilde{\mathbf{m}}^h \equiv (t, t_c, \mathbf{U}, \mathbf{P}, \mathbf{F}_{\text{int}}, \Lambda)$

- Spatial discretization of the different terms:
  - $$\begin{aligned}\int_{\omega_{\mathbf{X}}} \mathbf{p}.d_t\varphi\,\mathrm{d}\mathbf{X} &\simeq \int_{\omega_{\mathbf{X}}^h} \left(\sum_{i=1}^n p_i(t).\phi_i^p(\mathbf{X})\right).d_t\left(\sum_{j=1}^n u_j(t).\phi_j^u(\mathbf{X})\right)\mathrm{d}\mathbf{X}\\ &= \sum_{i=1}^n\sum_{i=1}^n p_i(t)\left(\int_{\omega_{\mathbf{X}}^h}\phi_i^p.\phi_j^u\,\mathrm{d}\mathbf{X}\right)d_t u_j(t) = \mathbf{P}^T.\dot{\mathbf{U}}\end{aligned}$$
  - $$\int_{\omega_{\mathbf{X}}} \Pi : d_{\mathbf{X}}\varphi\,\mathrm{d}\mathbf{X} \simeq \sum_{j=1}^n f_{\text{int}_j} u_j(t) = \mathbf{F}_{\text{int}}^T.\mathbf{U}$$
  - $$\int_{\omega_{\mathbf{X}}} \mathbf{b}.\varphi\,\mathrm{d}\mathbf{X} \simeq \sum_{i=1}^n \left(\int_{\omega_{\mathbf{X}}^h}\mathbf{b}.\phi_i^u(\mathbf{X})\,\mathrm{d}\mathbf{X}\right)u_i(t) = {\mathbf{F}_{\text{ext}}^b}^T.\mathbf{U}$$
  - $$\int_{\partial\omega_{\mathbf{X}}^c} \lambda(t_c).g_N(t_c)\,\mathrm{d}\mathbf{S} \simeq \Lambda(t_c)^T.\mathbf{g}_N(t_c)$$
  - $$\begin{aligned}\int_{\partial\omega_{\mathbf{X}}^c} \dot{\lambda}.\delta g_N\,\mathrm{d}\mathbf{S} &= \int_{\partial\omega_{\mathbf{X}}^c} \dot{\lambda}.d_\varphi g_N\delta\varphi\,\mathrm{d}\mathbf{S}\\ &\simeq \sum_{i=1}^p\sum_{j=1}^n \dot{\lambda}_i.\left(\int_{\partial\omega_{\mathbf{X}}^c}\phi_i^\lambda.\phi_j^u.(-\mathbf{N})\,\mathrm{d}\mathbf{S}\right)\delta U_j = \dot{\Lambda}^T.(-\mathbf{L}).\delta\mathbf{U} = \dot{\Lambda}^T.\delta\mathbf{g}_N\end{aligned}$$

Space semi-discretized variational formulation for non-smooth elastodynamics

- Hamilton's principle semi-discretized in space for non-smooth elastodynamics:

$$\boxed{\delta\tilde{\mathscr{A}}^h + \int_{\omega_t} \mathbf{F}^{s\;T}_{\text{ext}}.\delta\mathbf{U}\,\mathrm{d}t = 0 \qquad \forall\delta\tilde{\mathbf{m}}^h \in \tilde{\mathscr{M}}^h_0}$$

with: $$\int_{\partial\omega^N_{\mathbf{X}}} \mathbf{T}.\delta\varphi\,\mathrm{d}\mathbf{S} \simeq \sum_{i=1}^{n} \Big(\int_{\partial\omega^N_{\mathbf{X}}} \mathbf{T}.\phi_i\,\mathrm{d}\mathbf{S}\Big).\delta U_i = \mathbf{F}^{s\;T}_{\text{ext}}.\delta\mathbf{U}$$

- Definition of the 6 space-separated infinitesimal generators (2 horizontal and 4 vertical):

$$\begin{aligned}\delta\tilde{\mathscr{A}}^h &= \frac{\partial\tilde{\mathscr{A}}^h}{\partial\tilde{\mathbf{m}}^h}.\delta\tilde{\mathbf{m}}^h = \delta_t\tilde{\mathscr{A}}^h + \delta_{t_c}\tilde{\mathscr{A}}^h + \delta_{\mathbf{U}}\tilde{\mathscr{A}}^h + \delta_{\mathbf{P}}\tilde{\mathscr{A}}^h + \delta_{\mathbf{F}_{\text{int}}}\tilde{\mathscr{A}}^h + \delta_{\Lambda}\tilde{\mathscr{A}}^h \\ &= D_t\tilde{\mathscr{A}}^h.\delta t + D_{t_c}\tilde{\mathscr{A}}^h.\delta t_c + D_{\mathbf{U}}\tilde{\mathscr{A}}^h.\delta\mathbf{U} + D_{\mathbf{P}}\tilde{\mathscr{A}}^h.\delta\mathbf{P} + D_{\mathbf{F}_{\text{int}}}\tilde{\mathscr{A}}^h.\delta\mathbf{F}_{\text{int}} \\ &+ D_{\Lambda}\tilde{\mathscr{A}}.\delta\Lambda\end{aligned}$$

Space semi-discretized variational formulation for non-smooth elastodynamics

- *Euler–Lagrange equations semi-discretized in space for non-smooth elastodynamics:*

$$
\begin{aligned}
\frac{\partial \mathscr{L}^{h*}}{\partial \mathbf{U}} + \dot{\mathbf{P}} + \mathbf{F}_{\text{int}} = \mathbf{F}^{s}_{\text{ext}} + \mathbf{L}^T \dot{\Lambda} \qquad & \text{in } (\omega_{t^-} \cup \omega_{t^+}) \times \omega^h_{\mathbf{x}} \\
-\frac{\partial \mathscr{L}^{*}}{\partial \mathbf{P}} + \dot{\mathbf{U}} = 0 \qquad & \text{in } (\omega_{t^-} \cup \omega_{t^+}) \times \omega^h_{\mathbf{x}} \\
-\frac{\partial \mathscr{L}^{*}}{\partial \mathbf{F}_{\text{int}}} - \mathbf{U} = 0 \qquad & \text{in } (\omega_{t^-} \cup \omega_{t^+}) \times \omega^h_{\mathbf{x}} \\
-\frac{\partial \mathscr{L}^{h*}}{\partial t} + d_t(\mathscr{L}^{h*} + \mathbf{F}^T_{\text{int}}\mathbf{U}) = (\mathbf{L}^T\dot{\Lambda})^T.\dot{\mathbf{U}} + \mathbf{F}_{\text{ext}}{}^T.\dot{\mathbf{U}} \qquad & \text{in } (\omega_{t^-} \cup \omega_{t^+}) \times \omega^h_{\mathbf{x}} \\
[\mathbf{P}]^{t_c^+}_{t_c^-} = 0 \qquad & \text{in } \omega^h_{\mathbf{x}} \\
[\mathbf{P}^T.\dot{\mathbf{U}} - \mathscr{L}^{h*}]^{t_c^+}_{t_c^-} = 0 \qquad & \text{in } \omega^h_{\mathbf{x}} \\
\Lambda(t_c)^T.\dot{\mathbf{g}}_N(t_c) = 0 \qquad & \text{on } \partial\omega^c_{\mathbf{X}} \\
\mathbf{g}_N(t_c) = 0 \qquad & \text{on } \partial\omega^c_{\mathbf{X}} \\
\text{and KKT conditions} \qquad &
\end{aligned}
$$

Space semi-discretized variational formulation for non-smooth elastodynamics

- *Expressions of the first derivatives of the space-semi-discretized Lagrangian:*

$$\frac{\partial \mathscr{L}^{h^*}}{\partial \mathbf{U}} = -\mathbf{F}^b_{\text{ext}}$$
$$\frac{\partial \mathscr{L}^{h^*}}{\partial \mathbf{P}} = \frac{\partial T^*}{\partial \mathbf{P}} = \mathbf{M}^{-1}\mathbf{P}$$
$$\frac{\partial \mathscr{L}^{h^*}}{\partial \mathbf{F}_{\text{int}}} = -\frac{\partial \psi^*}{\partial \mathbf{F}_{\text{int}}} = -\mathbf{K}^{-1}(\mathbf{U})\mathbf{F}_{\text{int}}$$

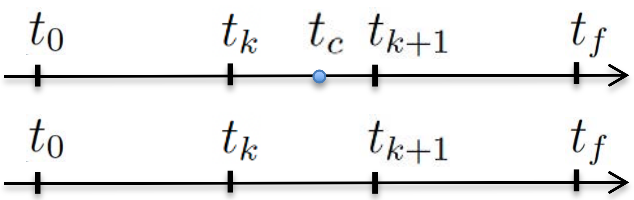
- *Euler–Lagrange equations semi-discretized in space for non-smooth elastodynamics:*

$$\boxed{\begin{array}{ll}
\dot{\mathbf{P}} + \mathbf{F}_{\text{int}} = \mathbf{F}_{\text{ext}} + \mathbf{L}^T\dot{\Lambda} & \text{in } (\omega_{t^-} \cup \omega_{t^+}) \times \omega^h_{\mathbf{x}} \\
-\mathbf{M}^{-1}\mathbf{P} + \dot{\mathbf{U}} = 0 & \text{in } (\omega_{t^-} \cup \omega_{t^+}) \times \omega^h_{\mathbf{x}} \\
\mathbf{K}^{-1}(\mathbf{U})\mathbf{F}_{\text{int}} - \mathbf{U} = 0 & \text{in } (\omega_{t^-} \cup \omega_{t^+}) \times \omega^h_{\mathbf{x}} \\
d_t(T^{h^*}(\mathbf{P}) + \psi^*(\mathbf{F}_{\text{int}})) = (\mathbf{F}_{\text{ext}} + \mathbf{L}^T\dot{\Lambda})^T.\dot{\mathbf{U}} & \text{in } (\omega_{t^-} \cup \omega_{t^+}) \times \omega^h_{\mathbf{x}} \\
[\mathbf{P}]^{t_c^+}_{t_c^-} = 0 & \text{in } \times \omega^h_{\mathbf{x}} \\
[\frac{1}{2}\mathbf{P}^T.\mathbf{M}^{-1}.\mathbf{P}]^{t_c^+}_{t_c^-} = 0 & \text{in } \times \omega^h_{\mathbf{x}} \\
\text{and HSM conditions} &
\end{array}}$$

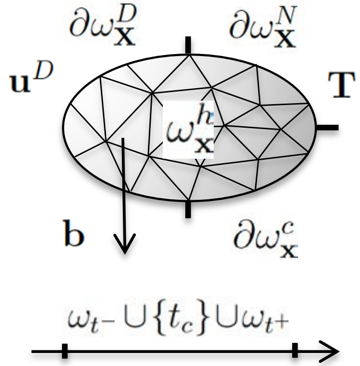
with: $\mathbf{F}_{\text{ext}} = \mathbf{F}^b_{\text{ext}} + \mathbf{F}^s_{\text{ext}}$

Variational formulation discretized in space and time for non-smooth elastodynamics

- Two families of time integrators in non-smooth elastodynamics:
  - Event driven
  - Time stepping



$$\tau = t_{k+1} - t_k$$



➡ *Unlike the continuous-time formulation, and following Moreau's ideas, the impact event has not to be detected. Indeed, the impact is here taken in the weak sense. In other words, many impacts can occur during a time step. However, it is not necessary to detect them precisely. Here, a main difference must be made between "event-driven" integrators (detection of the event of an impact) and "time-stepping" time integrators (no detection of the event of an impact in the strong sense). Here, the "time-stepping" approach will be favored in order to possibly allow an infinity of impact events in a finite time (Zeno's paradox). Therefore, the time approximation no longer contains the instant of impact as a variable of the action.*

[Acary 2015]

Variational formulation discretized in space and time for non-smooth elastodynamics

- Multi-field action discretized in space and time for non-smooth elastodynamics (central difference method, Verlet time integrator):

$$\tilde{\mathscr{A}}^h \simeq \sum_{k=0}^{N-1} \tilde{\mathscr{A}}^{h\tau}(t_{k+1}, \mathbf{U}_{k+1}, \mathbf{P}_{k+\frac{1}{2}}, \mathbf{F}_{\mathrm{int}\,k+1}, \Lambda_{k+\frac{1}{2}})$$

$$\boxed{\begin{aligned}&\tilde{\mathscr{A}}^{h\tau}(t_{k+1}, \mathbf{U}_{k+1}, \mathbf{P}_{k+\frac{1}{2}}, \mathbf{F}_{\mathrm{int}\,k+1}, \Lambda_{k+\frac{1}{2}})\\ &= \tau \mathbf{P}^T_{k+\frac{1}{2}}.\dot{\mathbf{U}}_{k+\frac{1}{2}} - \frac{\tau}{2}(\mathbf{F}_{\mathrm{int}}{}^T_k.\mathbf{U}_k + \mathbf{F}_{\mathrm{int}}{}^T_{k+1}.\mathbf{U}_{k+1})\\ &-\frac{\tau}{2}(\mathscr{L}^{h^*}(\mathbf{U}_k, \mathbf{P}_{k+\frac{1}{2}}, \mathbf{F}_{\mathrm{int}\,k}) + \mathscr{L}^{h^*}(\mathbf{U}_{k+1}, \mathbf{P}_{k+\frac{1}{2}}, \mathbf{F}_{\mathrm{int}\,k+1}))\\ &+\tau \Lambda^T_{k+\frac{1}{2}}.\dot{\mathbf{g}}_{N_{k+\frac{1}{2}}}\end{aligned}}$$

with: $\mathscr{L}^{h^*}(\mathbf{U}_k, \mathbf{P}_{k+\frac{1}{2}}, \mathbf{F}_{\mathrm{int}\,k}) = T^{h^*}(\mathbf{P}_{k+\frac{1}{2}}) - \Psi^{h^*}(\mathbf{F}_{\mathrm{int}\,k}) - \mathbf{F}^{b\ T}_{\mathrm{ext}\,k}.\mathbf{U}_k$

$$\begin{cases} \text{if } \; g^i_{N\,k+1} > 0 \; \text{ then } \; \Lambda^i_{k+\frac{3}{2}} = 0 \\ \text{if } \; g^i_{N\,k+1} \leq 0 \; \text{ then } \; \begin{cases} \dot{g}^i_{N_{k+\frac{3}{2}}} \geq 0 \\ \Lambda^i_{k+\frac{3}{2}} \geq 0 \\ \Lambda^T_{k+\frac{3}{2}}.\dot{g}^i_{N_{k+\frac{3}{2}}} = 0 \end{cases} \end{cases} \qquad i \in \{1, \ldots p\}$$

[de Vogelaere 1956, Verlet 1967, Wendlanlt 1997, Marsden 2001, Cirak 2005, Desmoures 2017, Fekak 2017]

Variational formulation discretized in space and time for non-smooth elastodynamics

- State vector:

$$\boxed{\tilde{\mathbf{m}}^{h\tau} \equiv (t_{k+1}, \mathbf{U}_{k+1}, \mathbf{P}_{k+\frac{1}{2}}, \mathbf{F}_{int\,k+1}, \mathbf{\Lambda}_{k+\frac{1}{2}})}$$

- Discrete space-time Hamilton's principle:

$$\boxed{\delta\tilde{\mathscr{A}}^{h\tau} + \sum_{k=0}^{N-1} \tau \mathbf{F}^{s}_{\mathrm{ext}\,k+1}{}^{T}.\delta\mathbf{U}_{k+1} = 0 \qquad \forall \delta\tilde{\mathbf{m}}^{h\tau} \in \tilde{\mathscr{M}}_0^{h\tau}}$$

- Discrete space-time infinitesimal generators (1 horizontal and 4 vertical):

$$\begin{aligned}
\delta\tilde{\mathscr{A}}^{h\tau} &= \frac{\partial\tilde{\mathscr{A}}^{h\tau}}{\partial\tilde{\mathbf{m}}^{h\tau}}.\delta\tilde{\mathbf{m}}^{h\tau} \\
&= \delta_t\tilde{\mathscr{A}}^{h\tau} + \delta_{\mathbf{U}}\tilde{\mathscr{A}}^{h\tau} + \delta_{\mathbf{P}}\tilde{\mathscr{A}}^{h\tau} + \delta_{\mathbf{F}_{\mathrm{int}}}\tilde{\mathscr{A}}^{h\tau} + \delta_{\Lambda}\tilde{\mathscr{A}}^{h\tau} \\
&= D_t\tilde{\mathscr{A}}^{h\tau}.\delta t_{k+1} + D_{\mathbf{U}}\tilde{\mathscr{A}}^{h\tau}.\delta\mathbf{U}_{k+1} + D_{\mathbf{P}}\tilde{\mathscr{A}}^{h\tau}.\delta\mathbf{P}_{k+\frac{1}{2}} \\
&+ D_{\mathbf{F}_{\mathrm{int}}}\tilde{\mathscr{A}}^{h\tau}.\delta\mathbf{F}_{\mathrm{int}\,k+1} + D_{\Lambda}\tilde{\mathscr{A}}^{h\tau}.\delta\Lambda_{k+\frac{1}{2}}
\end{aligned}$$

- Calculation of the 5 corresponding infinitesimal generators (1 horizontal and 4 vertical) from Hamilton's principle:

$$\begin{aligned}
&\delta\sum_{k=0}^{N-1}[\tau\mathbf{P}^{T}_{k+\frac{1}{2}}.\frac{(\mathbf{U}_{k+1}-\mathbf{U}_k)}{\tau} - \frac{\tau}{2}[\mathbf{F}_{\mathrm{int}}{}^{T}_{k}.\mathbf{U}_k + \mathbf{F}_{\mathrm{int}}{}^{T}_{k+1}.\mathbf{U}_{k+1}] \\
&-\frac{\tau}{2}[\mathscr{L}^{h^*}(\mathbf{U}_k, \mathbf{P}_{k+\frac{1}{2}}, \mathbf{F}_{\mathrm{int}\,k}) + \mathscr{L}^{h^*}(\mathbf{U}_{k+1}, \mathbf{P}_{k+\frac{1}{2}}, \mathbf{F}_{\mathrm{int}\,k+1})] \\
&+\tau\Lambda^{T}_{k+\frac{1}{2}}.\dot{\mathbf{g}}_{N_{k+\frac{1}{2}}}] + \tau\mathbf{F}^{s}_{\mathrm{ext}\,k+1}{}^{T}.\delta\mathbf{U}_{k+1} = 0 \qquad \forall\delta\tilde{\mathbf{m}}^{h\tau} \in \tilde{\mathscr{M}}_0^{h\tau}
\end{aligned}$$

and HSM conditions

Variational formulation discretized in space and time for non-smooth elastodynamics

- Calculation of the 5 corresponding infinitesimal generators (1 horizontal and 4 vertical) from Hamilton's principle:

$$
\begin{aligned}
&\sum_{k=0}^{N-1}\Big[\delta \mathbf{P}^T_{k+\frac{1}{2}}.(\mathbf{U}_{k+1}-\mathbf{U}_k)+\mathbf{P}^T_{k+\frac{1}{2}}.\delta\mathbf{U}_{k+1}-\mathbf{P}^T_{k+\frac{1}{2}}.\delta\mathbf{U}_k\\
&-\tfrac{\delta\tau}{2}[\mathbf{F}_{\text{int}}{}^T_k.\mathbf{U}_k+\mathbf{F}_{\text{int}}{}^T_{k+1}.\mathbf{U}_{k+1}]\\
&-\tfrac{\tau}{2}[\delta\mathbf{F}_{\text{int}}{}^T_k.\mathbf{U}_k+\mathbf{F}_{\text{int}}{}^T_k.\delta\mathbf{U}_k+\delta\mathbf{F}_{\text{int}}{}^T_{k+1}.\mathbf{U}_{k+1}+\mathbf{F}_{\text{int}}{}^T_{k+1}.\delta\mathbf{U}_{k+1}]\\
&-\tfrac{\delta\tau}{2}[\mathscr{L}^{h^*}(\mathbf{U}_k,\mathbf{P}_{k+\frac{1}{2}},\mathbf{F}_{\text{int}\,k})+\mathscr{L}^{h^*}(\mathbf{U}_{k+1},\mathbf{P}_{k+\frac{1}{2}},\mathbf{F}_{\text{int}\,k+1})]\\
&-\tfrac{\tau}{2}[\tfrac{\partial\mathscr{L}^{h^*}}{\partial\mathbf{U}}(\mathbf{U}_k,\mathbf{P}_{k+\frac{1}{2}},\mathbf{F}_{\text{int}\,k})\delta\mathbf{U}_k+\tfrac{\partial\mathscr{L}^{h^*}}{\partial\mathbf{P}}(\mathbf{U}_k,\mathbf{P}_{k+\frac{1}{2}},\mathbf{F}_{\text{int}\,k})\delta\mathbf{P}_{k+\frac{1}{2}}+\dots\\
&\dots+\tfrac{\partial\mathscr{L}^{h^*}}{\partial\mathbf{F}_{\text{int}}}(\mathbf{U}_k,\mathbf{P}_{k+\frac{1}{2}},\mathbf{F}_{\text{int}\,k})\delta\mathbf{F}_{\text{int}\,k}]\\
&-\tfrac{\tau}{2}[\tfrac{\partial\mathscr{L}^{h^*}}{\partial\mathbf{U}}(\mathbf{U}_{k+1},\mathbf{P}_{k+\frac{1}{2}},\mathbf{F}_{\text{int}\,k+1})\delta\mathbf{U}_{k+1}+\dots\\
&\dots+\tfrac{\partial\mathscr{L}^{h^*}}{\partial\mathbf{P}}(\mathbf{U}_{k+1},\mathbf{P}_{k+\frac{1}{2}},\mathbf{F}_{\text{int}\,k+1})\delta\mathbf{P}_{k+\frac{1}{2}}+\dots\\
&\dots+\tfrac{\partial\mathscr{L}^{h^*}}{\partial\mathbf{F}_{\text{int}}}(\mathbf{U}_{k+1},\mathbf{P}_{k+\frac{1}{2}},\mathbf{F}_{\text{int}\,k+1})\delta\mathbf{F}_{\text{int}\,k+1}]\\
&+\delta\tau\Lambda^T_{k+\frac{1}{2}}.\dot{\mathbf{g}}_{N_{k+\frac{1}{2}}}+\tau\delta\Lambda^T_{k+\frac{1}{2}}.\dot{\mathbf{g}}_{N_{k+\frac{1}{2}}}+\tau\Lambda^T_{k+\frac{1}{2}}.\delta\dot{\mathbf{g}}_{N_{k+\frac{3}{2}}}\\
&+\tau\mathbf{F}^s_{\text{ext}\,k+1}{}^T.\delta\mathbf{U}_{k+1}\Big]=0\qquad\forall\delta\tilde{\mathbf{m}}^{h\tau}\in\tilde{\mathscr{M}}_0^{h\tau}\\
&\text{and HSM conditions}
\end{aligned}
$$

Variational formulation discretized in space and time for non-smooth elastodynamics

- Vertical generator associated with linear momentum:

$$\delta \mathbf{P}_{k+\frac{1}{2}} \qquad D_{\mathbf{P}} \tilde{\mathscr{A}}^{h\tau} = (\mathbf{U}_{k+1} - \mathbf{U}_k) - \frac{\tau}{2} \frac{\partial \mathscr{L}^{h*}}{\partial \mathbf{P}} (\mathbf{U}_k, \mathbf{P}_{k+\frac{1}{2}}, \mathbf{F}_{\text{int}\,k}) + \ldots$$
$$\ldots - \frac{\tau}{2} \frac{\partial \mathscr{L}^{h*}}{\partial \mathbf{P}} (\mathbf{U}_{k+1}, \mathbf{P}_{k+\frac{1}{2}}, \mathbf{F}_{\text{int}\,k+1}) = 0$$

$$\boxed{\mathbf{U}_{k+1} = \mathbf{U}_k + \tau \mathbf{M}^{-1} \mathbf{P}_{k+\frac{1}{2}}}$$

- Vertical generator associated with displacement (space-time discretized equation of motion):

$$\delta \mathbf{U}_{k+1} \qquad D_{\mathbf{U}} \tilde{\mathscr{A}}^{h\tau} = -\mathbf{P}_{k+\frac{3}{2}} + \mathbf{P}_{k+\frac{1}{2}} - \tau \mathbf{F}_{\text{int}\,k+1}$$
$$- \frac{\tau}{2} \left[\frac{\partial \mathscr{L}^{h*}}{\partial \mathbf{U}} (\mathbf{U}_{k+1}, \mathbf{P}_{k+\frac{1}{2}}, \mathbf{F}_{\text{int}\,k+1}) - \frac{\tau}{2} \frac{\partial \mathscr{L}^{h*}}{\partial \mathbf{U}} (\mathbf{U}_{k+1}, \mathbf{P}_{k+\frac{3}{2}}, \mathbf{F}_{\text{int}\,k+1}) \right]$$
$$+ \mathbf{L}^T_{k+1} \Lambda_{k+\frac{3}{2}} - \mathbf{L}^T_k \Lambda_{k+\frac{1}{2}} + \tau \mathbf{F}^s_{\text{ext}\,k+1} = 0$$

with: $$\delta \dot{\mathbf{g}}_{N_{k+\frac{1}{2}}} = -\mathbf{L}_k \delta \dot{\mathbf{U}}_{k+\frac{1}{2}}$$

$$\boxed{\mathbf{P}_{k+\frac{3}{2}} = \mathbf{P}_{k+\frac{1}{2}} + \tau (\mathbf{F}_{\text{ext}\,k+1}^{b} + \mathbf{F}^s_{\text{ext}\,k+1} - \mathbf{F}_{\text{int}\,k+1}) + \mathbf{L}^T_{k+1} \Lambda_{k+\frac{3}{2}} - \mathbf{L}_k \Lambda_{k+\frac{1}{2}}}$$

Variational formulation discretized in space and time for non-smooth elastodynamics

- Vertical generator associated with internal forces:

$\delta \mathbf{F}_{int\,k+1}$

$$D_{\mathbf{F}_{\text{int}}} \tilde{\mathscr{A}}^{h\tau} = \mathbf{U}_{k+1} + \frac{1}{2}.[\frac{\partial \mathscr{L}^{h^*}}{\partial \mathbf{F}_{\text{int}}}(\mathbf{U}_{k+1}, \mathbf{P}_{k+\frac{1}{2}}, \mathbf{F}_{\text{int}\,k+1}) + \frac{\partial \mathscr{L}^{h^*}}{\partial \mathbf{F}_{\text{int}}}(\mathbf{U}_{k+1}, \mathbf{P}_{k+\frac{3}{2}}, \mathbf{F}_{\text{int}\,k+1})] = 0$$

$$\boxed{\mathbf{U}_{k+1} - \mathbf{K}^{-1}(\mathbf{U}_{k+1})\mathbf{F}_{\text{int}\,k+1} = 0}$$

- Horizontal generator associated with time (space-time discretized energy balance):

δt_{k+1}

$$\boxed{\begin{aligned} &[T^{h^*}(\mathbf{P}_{k+\frac{3}{2}}) + \frac{1}{2}(\Psi^{h^*}(\mathbf{F}_{\text{int}\,k+1}) + \Psi^{h^*}(\mathbf{F}_{\text{int}\,k+2})] \\ &-[T^{h^*}(\mathbf{P}_{k+\frac{1}{2}}) + \frac{1}{2}(\Psi^{h^*}(\mathbf{F}_{\text{int}\,k}) + \Psi^{h^*}(\mathbf{F}_{\text{int}\,k+1})] \\ &= \frac{1}{2}[(\mathbf{F}^T_{\text{ext}\,k+1}.\mathbf{U}_{k+1} + \mathbf{F}^T_{\text{ext}\,k+2}.\mathbf{U}_{k+2}) - (\mathbf{F}^T_{\text{ext}\,k}.\mathbf{U}_k + \mathbf{F}^T_{\text{ext}\,k+1}.\mathbf{U}_{k+1})] \\ &-(\Lambda^T_{k+\frac{3}{2}}.\dot{\mathbf{g}}_{N_{k+\frac{3}{2}}} - \Lambda^T_{k+\frac{1}{2}}.\dot{\mathbf{g}}_{N_{k+\frac{1}{2}}}) \end{aligned}}$$

$\Rightarrow$ Non-standard discrete energy balance

[Hughes 1989, Di stasio 2021]

Variational formulation discretized in space and time for non-smooth elastodynamics

- *Discretized Euler–Lagrange equations for non-smooth elastodynamics:*

$$
\begin{aligned}
\mathbf{U}_{k+1} &= \mathbf{U}_k + \tau \mathbf{M}^{-1}\mathbf{P}_{k+\frac{1}{2}} \\
\mathbf{F}_{\text{int}\,k+1} &= \mathbf{K}(\mathbf{U}_{k+1})\mathbf{U}_{k+1} \\
\mathbf{P}_{k+\frac{3}{2}} &= \mathbf{P}_{k+\frac{1}{2}} + \tau(\mathbf{F}_{\text{ext}\,k+1} - \mathbf{F}_{\text{int}\,k+1}) + \mathbf{L}^T_{k+1}\Lambda_{k+\frac{3}{2}} - \mathbf{L}_k\Lambda_{k+\frac{1}{2}} \\
& \text{and HSM conditions}
\end{aligned}
$$

with:

$$
\begin{aligned}
\mathbf{g}_{N\,k+1} &= -(\mathbf{L}_{k+1}\mathbf{U}_{k+1} - \mathbf{X}_0.\mathbf{N}) \\
\dot{\mathbf{g}}_{N_{\text{free}}} &= -\mathbf{L}_{k+1}\mathbf{M}^{-1}\mathbf{P}_{\text{free}} \\
\dot{\mathbf{g}}_{N_{k+\frac{3}{2}}} &= -\mathbf{L}_{k+1}\mathbf{M}^{-1}\mathbf{P}_{k+\frac{3}{2}}
\end{aligned}
$$

$$\mathbf{P}_{\text{free}} = \mathbf{P}_{k+\frac{1}{2}} + \tau(\mathbf{F}_{\text{ext}\,k+1} - \mathbf{F}_{\text{int}\,k+1}) - \mathbf{L}^T_k\Lambda_{k+\frac{1}{2}}$$

- *Solving / LCP type problem ($\mathbf{M}$ diagonal):*

$$
\begin{cases}
\mathbf{H}\Lambda_{k+\frac{3}{2}} = -(\dot{\mathbf{g}}_{N_{k+\frac{3}{2}}} - \dot{\mathbf{g}}_{N_{\text{free}}}) \\
\text{if} \quad g^i_{N\,k+1} > 0 \quad \text{then} \quad \Lambda^i_{k+\frac{3}{2}} = 0 \\
\text{if} \quad g^i_{N\,k+1} \leq 0 \quad \text{then} \quad
\begin{cases}
\dot{g}^i_{N_{k+\frac{3}{2}}} \geq 0 \\
\Lambda^i_{k+\frac{3}{2}} \geq 0 \\
\Lambda^T_{k+\frac{3}{2}} \cdot \dot{g}^i_{N_{k+\frac{3}{2}}} = 0
\end{cases}
\end{cases}
\qquad \mathbf{H} = \mathbf{L}_{k+1}\mathbf{M}^{-1}\mathbf{L}^T_{k+1} \qquad i \in \{1,\ldots p\}
$$

Steklov-Poincaré operator (Delassus)

Variational formulation discretized in space and time for non-smooth elastodynamics

- *Explicit algorithm of the LCP problem (explicit CD-Lagrange variational integrator):*

---

1 - compute $\mathbf{U}_{k+1} = \mathbf{U}_k + \tau \mathbf{M}^{-1} \mathbf{P}_{k+\frac{1}{2}}$

2 - compute $\mathbf{F}_{\text{int}\,k+1}(\mathbf{U}_{k+1})$

3 - compute $\dot{\mathbf{g}}_{N_\text{free}} = -\mathbf{L}_{k+1} \mathbf{M}^{-1} \mathbf{P}_\text{free}$ and $\mathbf{P}_\text{free} = \mathbf{P}_{k+\frac{1}{2}} + \tau(\mathbf{F}_{\text{ext}\,k+1} - \mathbf{F}_{\text{int}\,k+1}) - \mathbf{L}_k^T \Lambda_{k+\frac{1}{2}}$

4 - compute $\Lambda^i_{k+\frac{3}{2}} = \begin{cases} 0 & \text{if} \quad g^i_{N\,k+1} > 0 \\ \text{otherwise} & \max(0, H^{-1}_{ii}\, \dot{g}_{N^i_\text{free}}) \end{cases}$

5 - compute $\mathbf{P}_{k+\frac{3}{2}} = \mathbf{P}_\text{free} + \mathbf{L}^T_{k+1} \Lambda_{k+\frac{3}{2}}$

6 - compute $\dot{\mathbf{g}}_{N_{k+\frac{3}{2}}} = -\mathbf{L}_{k+1} \mathbf{M}^{-1} \mathbf{P}_{k+\frac{3}{2}}$ (only needed for post-processing energy balance)

[Fekak 2017, Di stasio 2021, Larousse 2023]

Variational formulation discretized in space and time for non-smooth elastodynamics

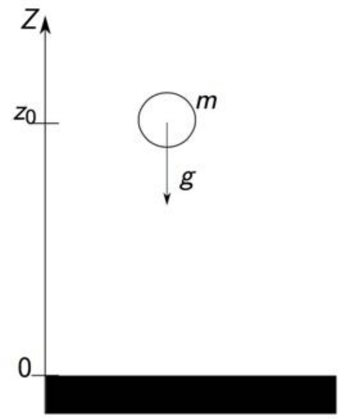
- *Multi-symplectic form discretized in space and time for non-smooth elastodynamics:*

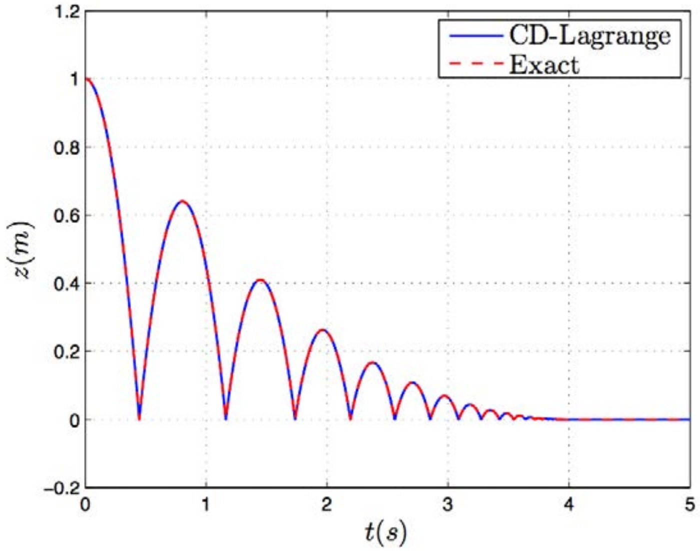
$$
\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\begin{bmatrix} \frac{1}{\tau}(\mathbf{U}_{k+1} - \mathbf{U}_k) \\ \frac{1}{\tau}(\mathbf{P}_{k+\frac{3}{2}} - \mathbf{P}_{k+\frac{1}{2}}) \\ d_t \mathbf{F}_{\text{int}} \end{bmatrix}
+
\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}
\begin{bmatrix} \mathbf{U}_{k+1} \\ d_{\mathbf{x}} \mathbf{p} \\ \mathbf{F}_{\text{int}\,k+1} \end{bmatrix}
=
\begin{bmatrix} \mathbf{F}_{\text{ext}\,k+1} + \mathbf{F}^{s}_{\text{ext}\,k+1} \\ \mathbf{M}^{-1}\mathbf{P}_{k+\frac{1}{2}} \\ -\mathbf{K}^{-1}(\mathbf{U}_{k+1})\mathbf{F}_{\text{int}\,k+1} \end{bmatrix}
+
\begin{bmatrix} \frac{1}{\tau}(\mathbf{L}^T_{k+\frac{3}{2}} \Lambda_{k+1} - \mathbf{L}^T_k \Lambda_{k+\frac{1}{2}}) \\ 0 \\ 0 \end{bmatrix}
$$

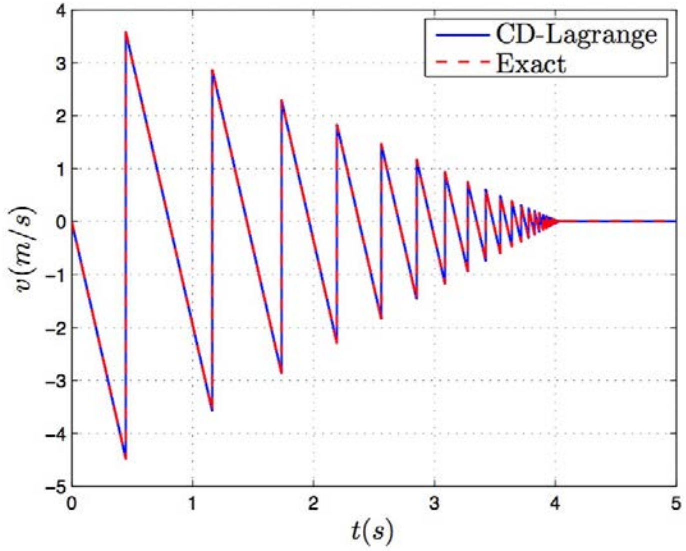
The proposed explicit variational time integrator preserves the multi-symplectic discretized form of non-smooth elastodynamics. As a result, ***the linear and angular momentum equations are verified exactly****, even with Lagrange multipliers. Nevertheless, it can be shown that* ***the discretized energy balance is not guaranteed exactly****. However, it can be shown that the corresponding error decays exponentially, which characterizes the high robustness of variational time integrators over long times. To ensure an exact discretized energy balance, a non-uniform time step will be considered.*

The bouncing ball example: rigid / rigid analytical solution

- First rebound: $$t_0 = \sqrt{\frac{2h}{g}}$$
- Nth rebound: $$t_n = t_0\left(2\frac{1-e^n}{1-e} - 1\right)$$
- Downtime of the mass: $$T = t_0\frac{1+e}{1-e}$$

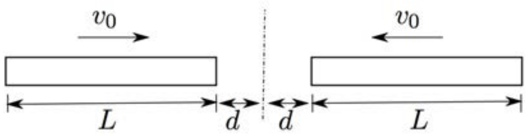




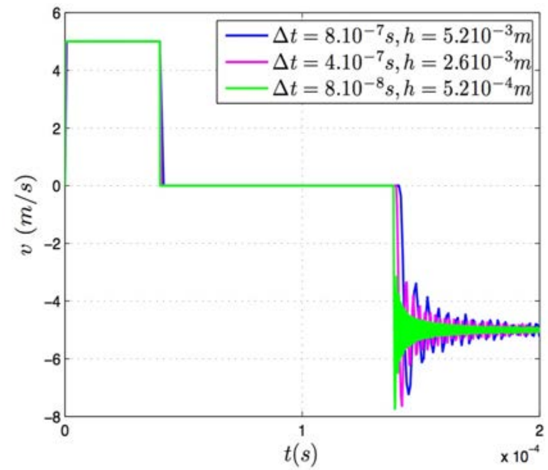


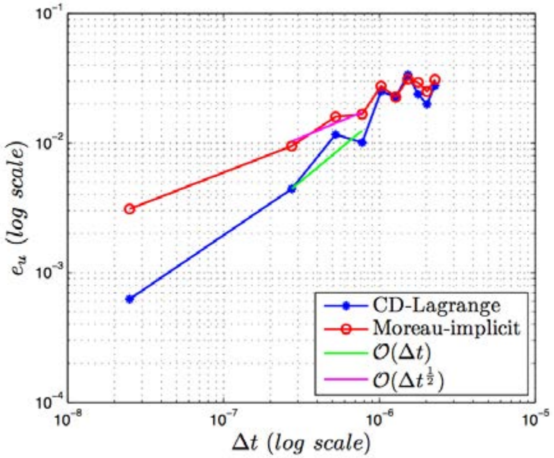
The mass does an infinity of rebounds before it stops in a finite time (Zeno paradox)
(restitution coefficient e=0.8)

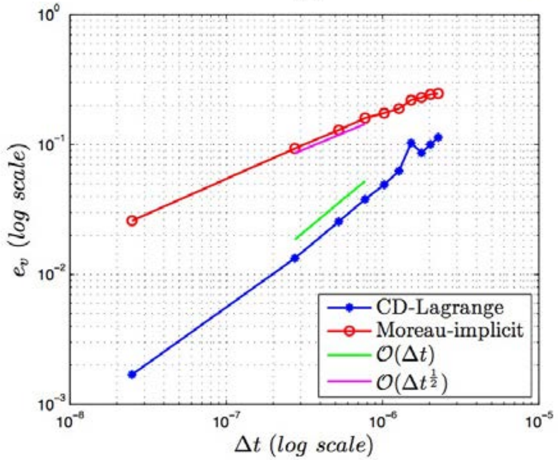
Contact / impact of two identical elastic bars



$$E = 2.1\ 10^{11}\ Pa$$
$$\rho = 7847\ kg/m^3$$
$$L = 0.254\ m$$
$$d = 0.2\ 10^{-3}\ m$$
$$v_0 = 5\ m/s$$
$$A = 0.645\ 10^{-3}\ m^2$$





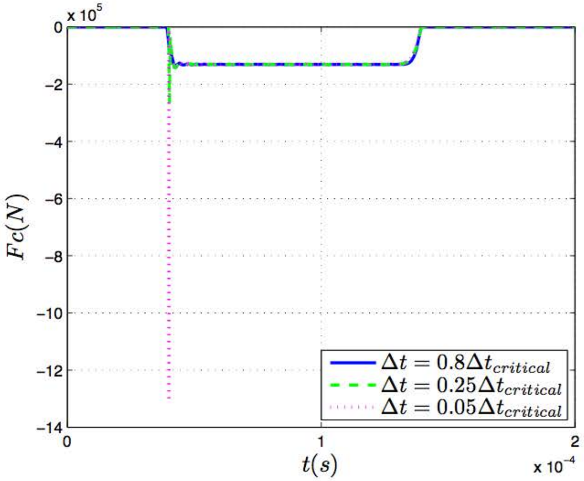


Space-time global error indicator (Hausdorff measure) ***[Acary 2012]***

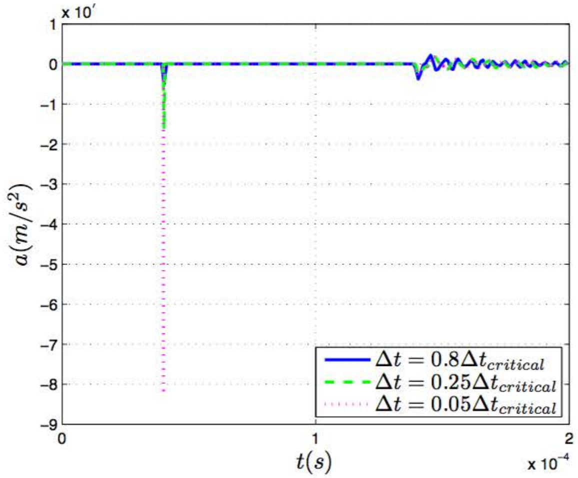
Moreau implicit of order ½, CD-Lagrange of order 1 ***[Fekak 2017]***

Contact / impact of two identical elastic bars

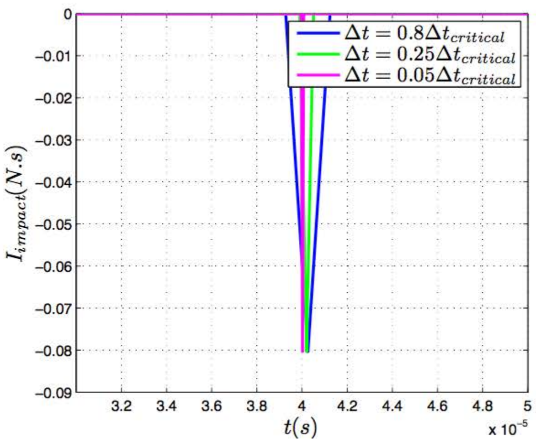
Contact force



Contact acceleration



Impact impulse



At impact event, acceleration and contact forces do not converge, however impulse converges.

Contact / impact of two identical elastic bars

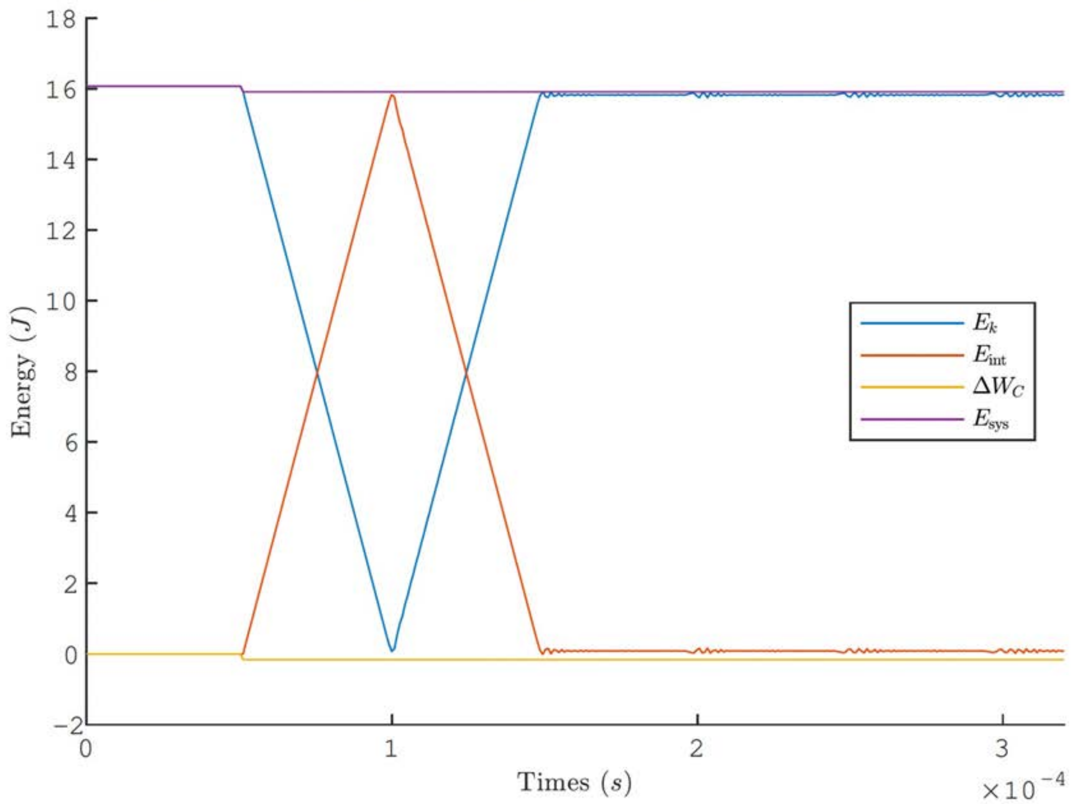


Figure: The impacting bar: energy balance of CD-Lagrange

Contact / impact of two identical elastic bars

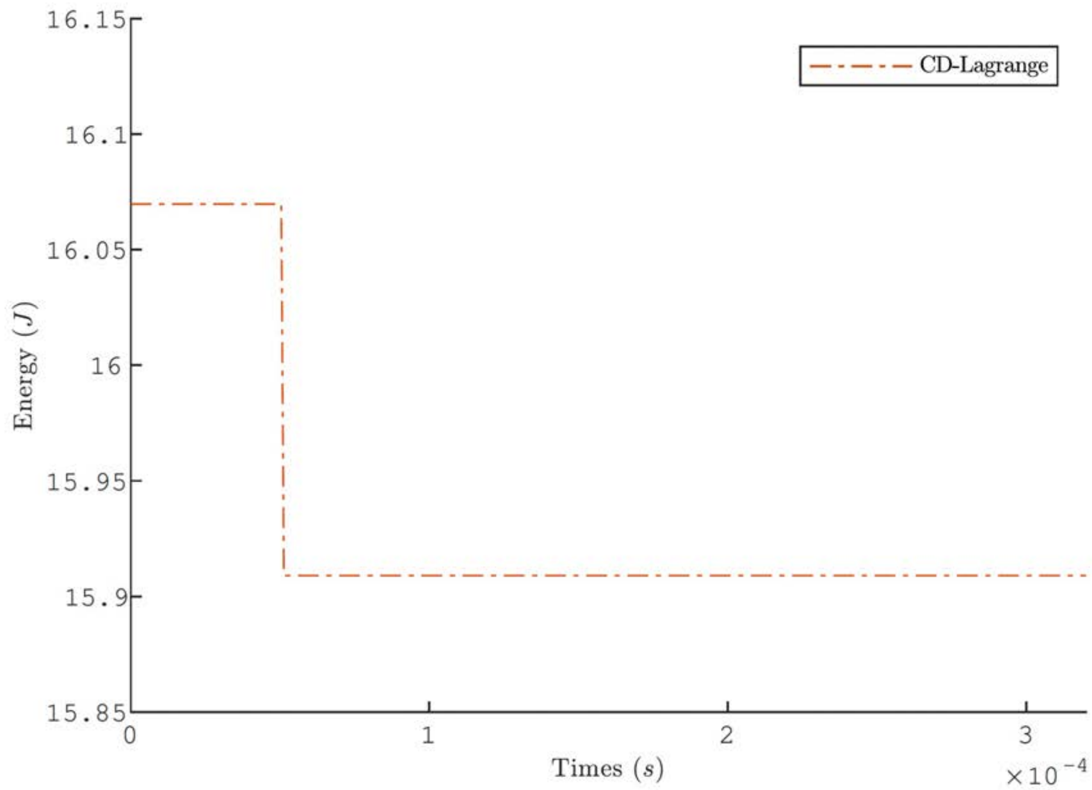


Figure: The impacting bar: system energy of CD-Lagrange

Contact / impact of two identical elastic bars

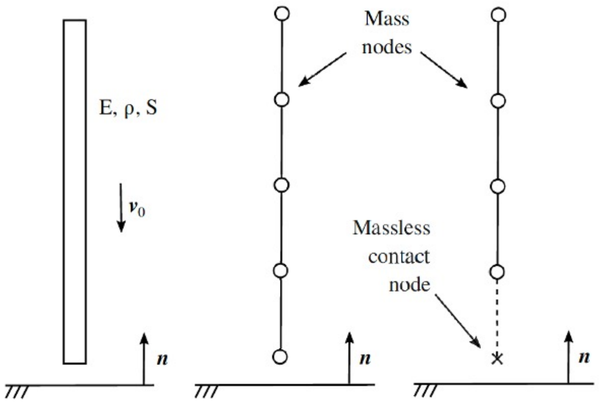


Figure: Consistent and singular spatial discretization

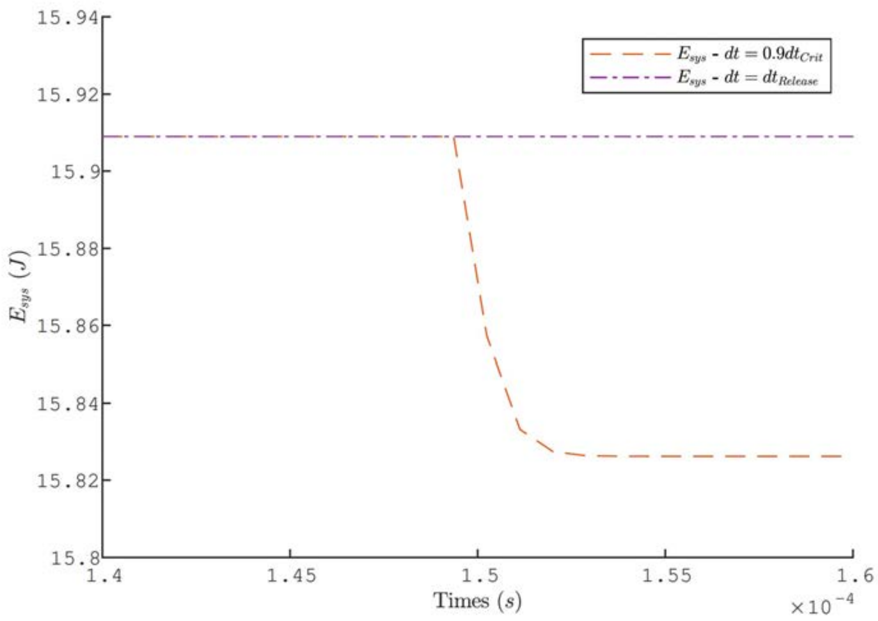


Figure: Contact node - $\Delta E_k = \Delta \bar{W}_{\mathrm{int}} + \Delta \widetilde{W}_{\mathrm{int}}$ for different t_{Release}

Discretized energy balance with singular mass / adaptive time step

[Renard 2008, Di Stasio et al 2021]

The rotating spring: a focus on discrete angular momentum

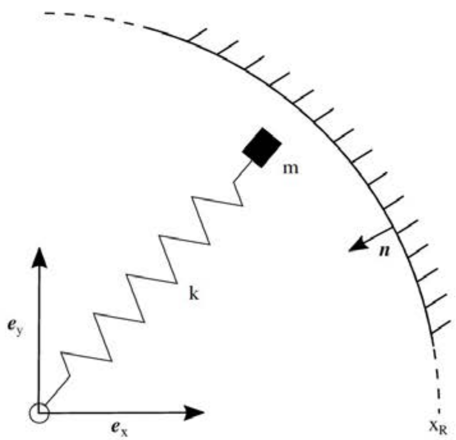


Figure: The rotating spring

$$m\mathrm{d}\dot{\boldsymbol{x}} + k\left(1 - \frac{l_0}{\|\boldsymbol{x}\|}\right)\boldsymbol{x} = \mathrm{d}r\boldsymbol{n} \qquad (12)$$

A mass-spring system in rotation:

- impact with a restitution coefficient e_c;
- large rotations;

What are the highlighted properties ?

- conservation of discrete momentum in smooth phase;
- conservation of discrete momentum through non-smooth events.

A case which challenges the symplecticity under non-smooth events.

The Moreau-Jean scheme[2] as a reference (time-integration with $\theta = 1/2$).

[2] Jean, M. The non-smooth contact dynamics method. *Computer Methods in Applied Mechanics and Engineering* **177**, 235–257 (1999).

The rotating spring: a focus on discrete angular momentum

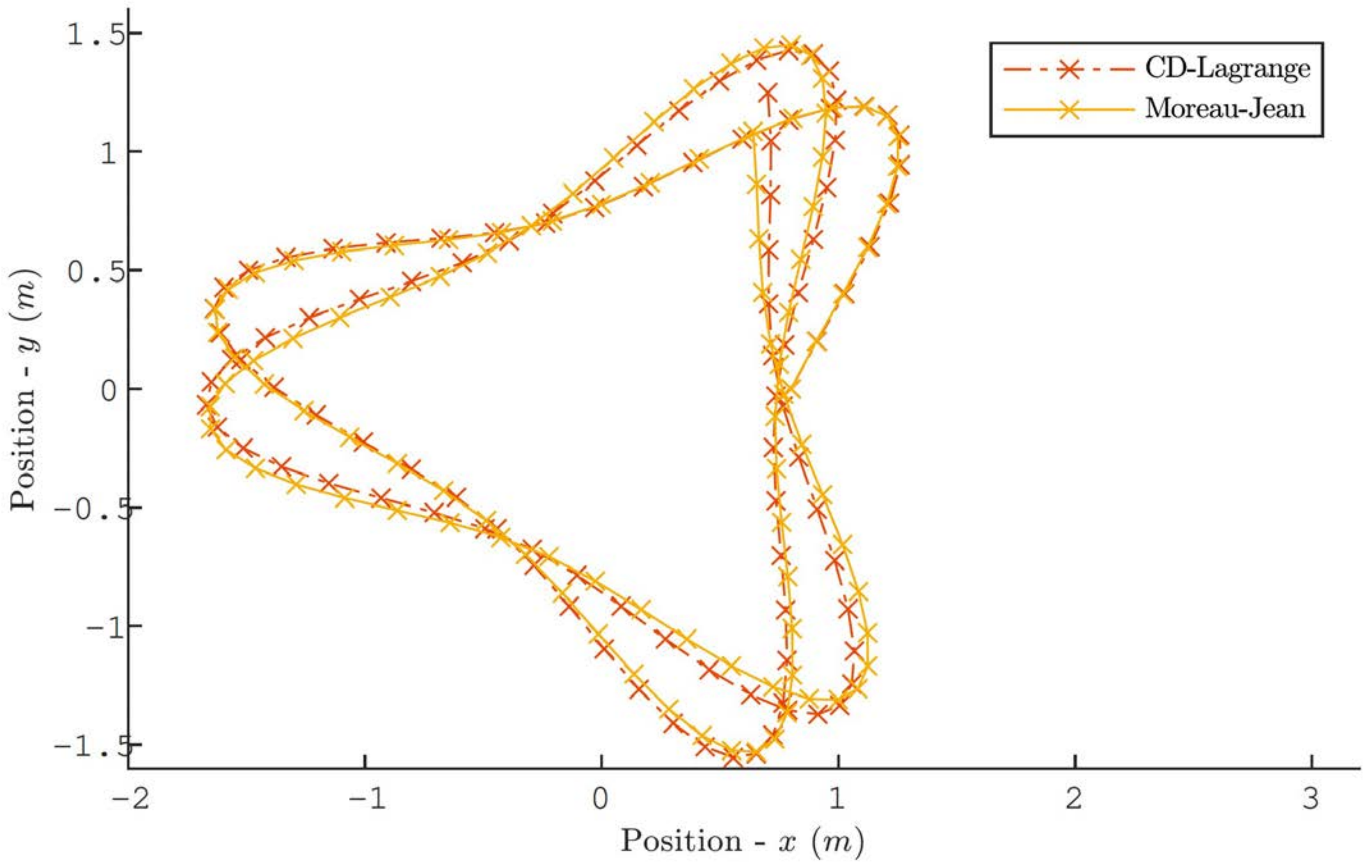


Figure: The rotating spring: position (over 10 s) - $e_c = 1$

[Di Stasio et al 2021]

The rotating spring: a focus on discrete angular momentum

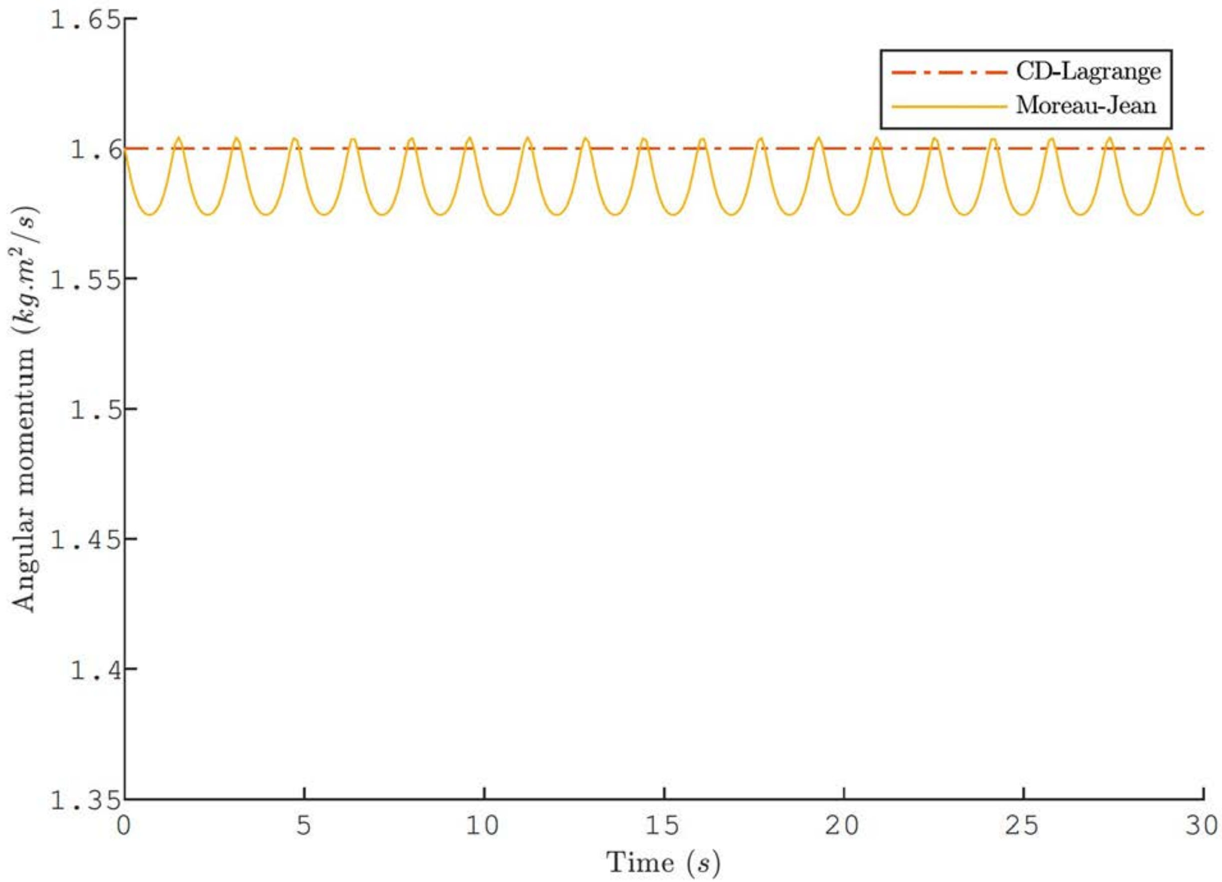


Figure: The rotating spring: angular momentum - $e_c = 1$

The rotating spring: a focus on discrete angular momentum

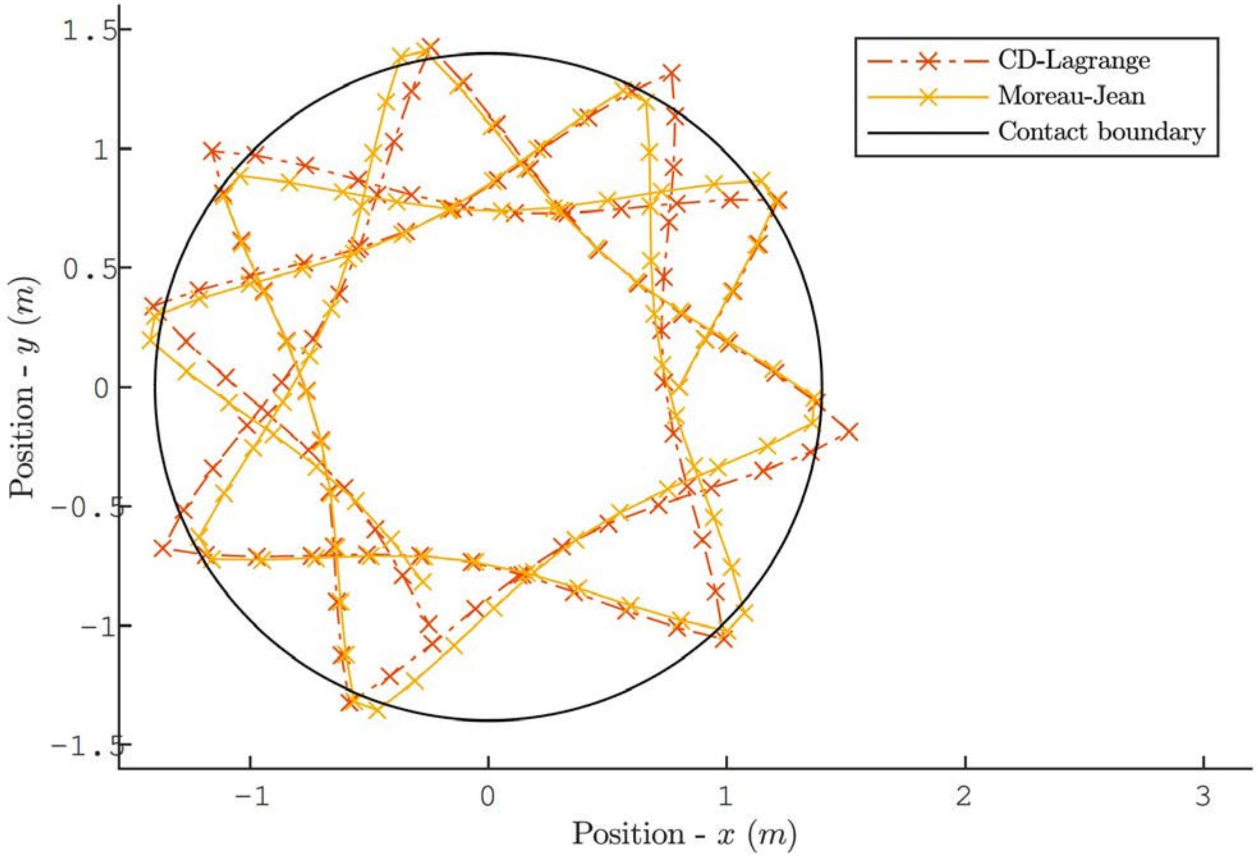


Figure: The rotating spring: position (over 10 s) - $e_c = 1$

[Di Stasio et al 2021]

The rotating spring: a focus on discrete angular momentum

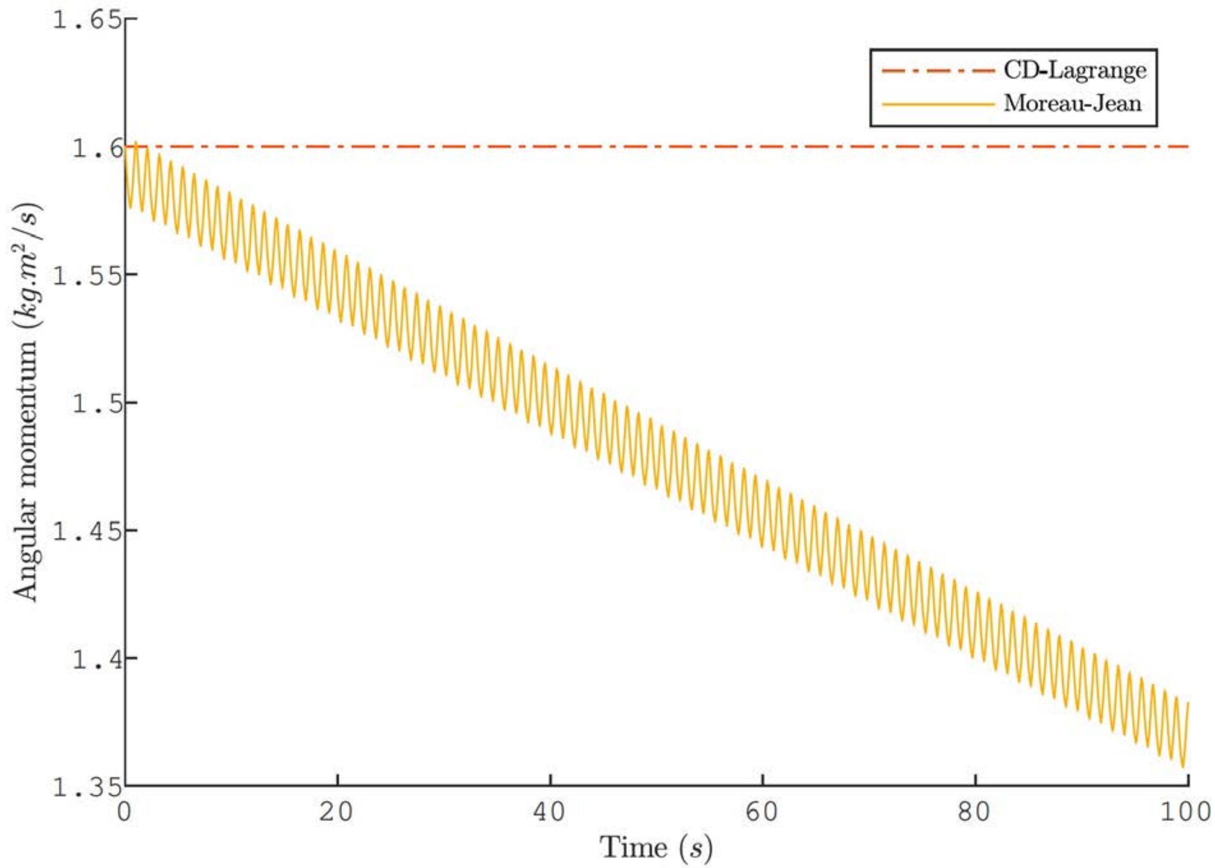


Figure: The rotating spring: angular momentum - $e_c = 1$

Variational formulation discretized in space and time for non-smooth elastodynamics

- Multi-field action discretized in space and time for non-smooth elastodynamics with adaptive time step:

$$\tau_{k+1} = t_{k+1} - t_k$$

- Multi-field action discretized in space and time for non-smooth elastodynamics with variable time step:

$$\tilde{\mathscr{A}}^h \simeq \sum_{k=0}^{N-1} \tilde{\mathscr{A}}^{h\tau}(\tau_{k+1}, \mathbf{U}_{k+1}, \mathbf{P}_{k+\frac{1}{2}}, \mathbf{F}_{\mathrm{int}\,k+1}, \Lambda_{k+\frac{1}{2}})$$

$$\begin{aligned}
&\tilde{\mathscr{A}}^{h\tau}(\tau_{k+1}, \mathbf{U}_{k+1}, \mathbf{P}_{k+\frac{1}{2}}, \mathbf{F}_{\mathrm{int}\,k+1}, \Lambda_{k+\frac{1}{2}}) \\
&= \tfrac{1}{2}(\tau_k + \tau_{k+1})\mathbf{P}^T_{k+\frac{1}{2}} . \frac{(\mathbf{U}_{k+1} - \mathbf{U}_k)}{\tau_{k+1}} - \tfrac{1}{2}[\tau_k \mathbf{F}_{\mathrm{int}}{}^T_k . \mathbf{U}_k + \tau_{k+1} \mathbf{F}_{\mathrm{int}}{}^T_{k+1} . \mathbf{U}_{k+1}] \\
&- \tfrac{1}{2}[\tau_k \mathscr{L}^{h^*}(\mathbf{U}_k, \mathbf{P}_{k+\frac{1}{2}}, \mathbf{F}_{\mathrm{int}\,k}) + \tau_{k+1} \mathscr{L}^{h^*}(\mathbf{U}_{k+1}, \mathbf{P}_{k+\frac{1}{2}}, \mathbf{F}_{\mathrm{int}\,k+1})] \\
&+ \tfrac{1}{2}(\tau_k + \tau_{k+1})\Lambda^T_{k+\frac{1}{2}} . \dot{\mathbf{g}}_{N_{k+\frac{1}{2}}}
\end{aligned}$$

$$+ \ \text{HSM conditions}$$

with:

$$\mathscr{L}^{h^*}(\mathbf{U}_k, \mathbf{P}_{k+\frac{1}{2}}, \mathbf{F}_{\mathrm{int}\,k}) = T^{h^*}(\mathbf{P}_{k+\frac{1}{2}}) - \Psi^{h^*}(\mathbf{F}_{\mathrm{int}\,k}) - \mathbf{F}^T_{\mathrm{ext}\,k} . \mathbf{U}_k$$

$$\dot{\mathbf{g}}_{N_{k+\frac{1}{2}}} = -\mathbf{L}_k \dot{\mathbf{U}}_{k+\frac{1}{2}} = -\mathbf{L}_k \frac{(\mathbf{U}_{k+1} - \mathbf{U}_k)}{\tau_{k+1}}$$

Variational formulation discretized in space and time for non-smooth elastodynamics

- *State vector :*

$$\tilde{\mathbf{m}}^{h\tau} \equiv (t_{k+1}, \mathbf{U}_{k+1}, \mathbf{P}_{k+\frac{1}{2}}, \mathbf{F}_{\mathrm{int}\,k+1}, \Lambda_{k+\frac{1}{2}})$$

- *Discrete Euler-Lagrange equations with adaptive time step:*

$$\begin{aligned}
&\mathbf{U}_{k+1} = \mathbf{U}_k + \tau_{k+1}\mathbf{M}^{-1}\mathbf{P}_{k+\frac{1}{2}} \\
&\mathbf{F}_{\mathrm{int}\,k+1} = \mathbf{K}(\mathbf{U}_{k+1})\mathbf{U}_{k+1} \\
&\frac{1}{2}\frac{\tau_{k+1}+\tau_{k+2}}{\tau_{k+2}}\mathbf{P}_{k+\frac{3}{2}} = \frac{1}{2}\frac{\tau_k+\tau_{k+1}}{\tau_{k+1}}\mathbf{P}_{k+\frac{1}{2}} + \tau_{k+1}(\mathbf{F}_{\mathrm{ext}\,k+1} - \mathbf{F}_{\mathrm{int}\,k+1}) \\
&+\frac{1}{2}\frac{\tau_{k+1}+\tau_{k+2}}{\tau_{k+2}}\mathbf{L}_{k+1}^T\Lambda_{k+\frac{3}{2}} - \frac{1}{2}\frac{\tau_k+\tau_{k+1}}{\tau_{k+1}}\mathbf{L}_k^T\Lambda_{k+\frac{1}{2}} \\
&\tau_{k+2} = \frac{1}{2}\mathbf{P}^T_{k+\frac{3}{2}}.(\mathbf{U}_{k+2} - \mathbf{U}_{k+1}) \\
&.[-\frac{1}{2}\frac{-\tau_k}{\tau_{k+1}^2}\mathbf{P}^T_{k+\frac{1}{2}}.(\mathbf{U}_{k+1} - \mathbf{U}_k) + \mathbf{F}_{\mathrm{int}}{}^T_{k+1}.\mathbf{U}_{k+1} + \frac{1}{2}[T^{h\tau^*}(\mathbf{P}_{k+\frac{3}{2}}) + T^{h^*}(\mathbf{P}_{k+\frac{1}{2}})] \\
&-\Psi^{h^*}(\mathbf{F}_{\mathrm{int}\,k+1}) - \mathbf{F}^T_{\mathrm{ext}\,k+1}.\mathbf{U}_{k+1} + \frac{1}{2}(\Lambda^T_{k+\frac{3}{2}}.\dot{\mathbf{g}}_{N_{k+\frac{3}{2}}} + \Lambda^T_{k+\frac{1}{2}}.\dot{\mathbf{g}}_{N_{k+\frac{1}{2}}})]^{-1} \\
&\text{and HSM conditions}
\end{aligned}$$

Variational formulation discretized in space and time for non-smooth elastodynamics

- *Variational integrator with adaptive time step for non-smooth dynamics:*

0 - time step predictor $\tau_{k+2} = \tau_{k+1}$

1 - compute $\dot{\mathbf{g}}_{N_{\text{free}}} = -\mathbf{L}_{k+1}\mathbf{M}^{-1}\mathbf{P}_{\text{free}}$

and $\frac{1}{2}\frac{\tau_{k+1}+\tau_{k+2}}{\tau_{k+2}}\mathbf{P}_{\text{free}} = \frac{1}{2}\frac{\tau_k+\tau_{k+1}}{\tau_{k+1}}\mathbf{P}_{k+\frac{1}{2}} + \tau_{k+1}(\mathbf{F}_{\text{ext}\,k+1} - \mathbf{F}_{\text{int}\,k+1}) - \frac{1}{2}\frac{\tau_k+\tau_{k+1}}{\tau_{k+1}}\mathbf{L}_k^T\Lambda_{k+\frac{1}{2}}$

2 - compute $\Lambda_{k+\frac{3}{2}}$ with $\Lambda^i_{k+\frac{3}{2}} = \begin{cases} 0 & \text{if} \quad g^i_{N\,k+1} > 0 \\ \text{otherwise} & \max(0, H^{-1}_{ii}\,\dot{g}^i_{N_{\text{free}}}) \end{cases}$

3 - compute $\mathbf{P}_{k+\frac{3}{2}} = \mathbf{P}_{\text{free}} + \mathbf{L}^T_{k+1}\Lambda_{k+\frac{3}{2}}$

4 - compute $\dot{\mathbf{g}}_{N_{k+1}} = -\mathbf{L}_{k+\frac{3}{2}}\mathbf{M}^{-1}\mathbf{P}_{k+\frac{3}{2}}$ (for time step update)

5 - compute $\mathbf{U}_{k+2} = \mathbf{U}_{k+1} + \tau_{k+2}\mathbf{M}^{-1}\mathbf{P}_{k+\frac{3}{2}}$

6 - compute $\mathbf{F}_{\text{int}\,k+2} = \mathbf{K}(\mathbf{U}_{k+2})\mathbf{U}_{k+2}$

7 - update time step τ_{k+2} with equation (102) and go to step 1 if $|\tau_{k+2} - \tau_{k+1}| \geq \varepsilon$

with the fundamental unknowns: $(\tau_{k+2}, \dot{\mathbf{g}}_{N_{k+\frac{3}{2}}}\, \mathbf{U}_{k+2}, \mathbf{P}_{k+\frac{3}{2}}, \mathbf{F}_{\text{int}\,k+2}, \Lambda_{k+\frac{3}{2}})$

Variational formulation discretized in space and time for non-smooth elastodynamics

- Space-time discretized multi-symplectic form for non-smooth elastodynamics with adaptive time step:

$$\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\tau_{k+2}}(\mathbf{U}_{k+2}-\mathbf{U}_{k+1}) \\ \frac{1}{2}\frac{\tau_{k+1}+\tau_{k+2}}{\tau_{k+2}}\mathbf{P}_{k+\frac{3}{2}} - \frac{1}{2}\frac{\tau_{k}+\tau_{k+1}}{\tau_{k+1}}\mathbf{P}_{k+\frac{1}{2}} \\ d_t\mathbf{F}_{\text{int}} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{U}_{k+2} \\ d_{\mathbf{x}}\mathbf{p} \\ \mathbf{F}_{\text{int}\,k+2} \end{bmatrix}$$
$$= \begin{bmatrix} \mathbf{F}_{\text{ext}\,k+2} + \mathbf{F}^{s}_{\text{ext}\,k+2} \\ \mathbf{M}^{-1}\mathbf{P}_{k+\frac{3}{2}} \\ -\mathbf{K}^{-1}(\mathbf{U}_{k+2})\mathbf{F}_{\text{int}\,k+2} \end{bmatrix} + \begin{bmatrix} \frac{1}{2}\frac{\tau_{k+1}+\tau_{k+2}}{\tau_{k+2}}\mathbf{L}^{T}_{k+1}\Lambda_{k+\frac{3}{2}} - \frac{1}{2}\frac{\tau_{k}+\tau_{k+1}}{\tau_{k+1}}\mathbf{L}^{T}_{k}\Lambda_{k+\frac{1}{2}} \\ 0 \\ 0 \end{bmatrix}$$

→ *The proposed variational time integrator checks the following properties:*

- space-time discretized multi-symplectic form (conservation of linear and angular momentum)
- discrete energy conservation
- automatic adaptive time step (with CFL condition checking)
- explicit prediction of the deformed configuration and diagonal mass matrix
- check the numerical behavior of the iterative adaptive time step, the consistency and convergence in space and time (with the Hausdorff measure)

[Gravouil, Dureisseix, Georgette 2026, under review]