Un intégrateur géométrique pour la thermo-visco-élastodynamique

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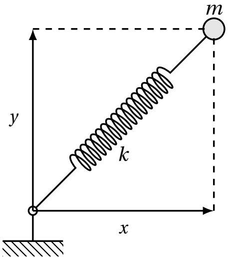
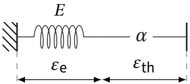
Partie A
Introduction aux intégrateurs géométriques



Partie B
Thermo-élastodynamique couplée

Petites transformations, 1D

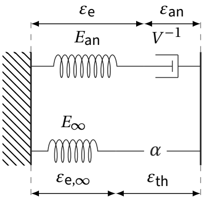
$$\boldsymbol{z} = \begin{pmatrix} u & \varepsilon & p & T \end{pmatrix}$$



Partie C
Thermo-visco-élastodynamique couplée

Petites transformations, 1D

$$\boldsymbol{z} = \begin{pmatrix} u & \varepsilon & p & T & \varepsilon_{\mathrm{an}} \end{pmatrix}$$



Partie A
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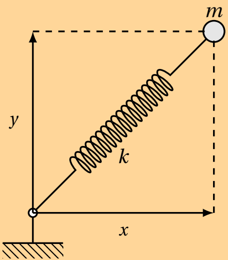

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Thermo-élastodynamique couplée

Petites transformations, 1D

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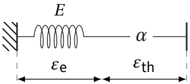

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Petites transformations, 1D

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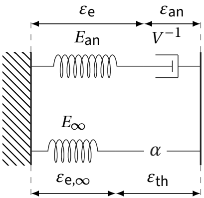
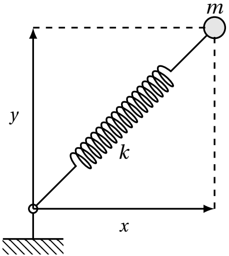


Partie A

Pour ce problème, l'Hamiltonien est

$$L^\star(x, y, p_x, p_y) = \frac{1}{2} m^{-1}(p_x^2 + p_y^2) + \frac{1}{2} k\left(\sqrt{x^2 + y^2} - l_0\right)^2$$



En appliquant le principe d'Hamilton

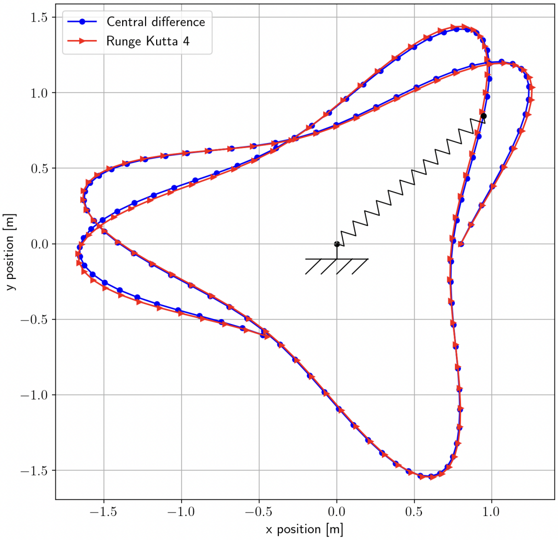
$$\forall \delta \boldsymbol{q} | \delta \boldsymbol{q}(t_\mathrm{i}) = \delta \boldsymbol{q}(t_\mathrm{f}) = 0 \forall \delta \boldsymbol{p}, \delta \mathscr{A}^\star[\boldsymbol{q}, \boldsymbol{p}] = \delta \int_{t_\mathrm{i}}^{t_\mathrm{f}} \left(\boldsymbol{p} \cdot \dot{\boldsymbol{q}} - L^\star(\boldsymbol{q}, \boldsymbol{p})\right) \mathrm{d}t = 0$$

on obtient les équations d'Hamilton

$$p_x = m\dot{x} \qquad p_y = m\dot{y}$$

$$\dot{p}_x + kx\left(1 - \frac{l_0}{\sqrt{x^2 + y^2}}\right) = 0 \qquad \dot{p}_y + ky\left(1 - \frac{l_0}{\sqrt{x^2 + y^2}}\right) = 0$$

Partie A


Central difference
Runge Kutta 4
y position [m]
x position [m]
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
−1.5
−1.0
−0.5
0.0
0.5
1.0

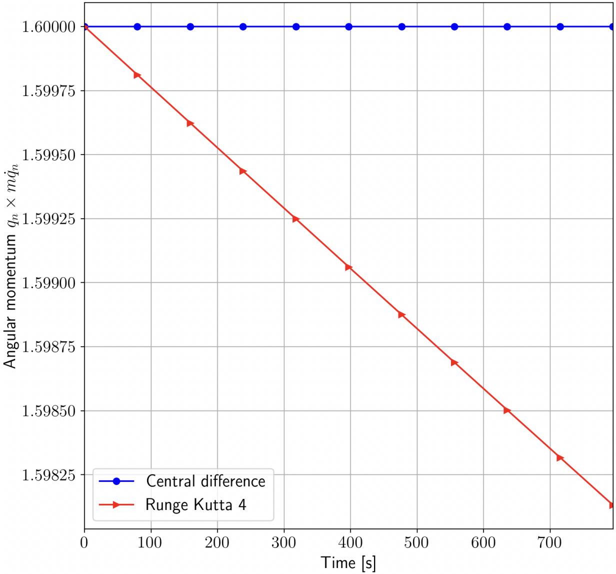
Principe variationnel
Définition d'un Hamiltonien
Principe d'Hamilton

Équations de conservation
Conservation de la quantité de mouvement
Conservation du moment angulaire

$$\frac{\mathrm{d}(\boldsymbol{q}\times\boldsymbol{p})}{\mathrm{d}t}=\boldsymbol{0}$$

Discrétisation

Partie A



1.60000
1.59975
1.59950
1.59925
1.59900
1.59875
1.59850
1.59825
Angular momentum $q_n \times m\dot{q}_n$
0
100
200
300
400
500
600
700
Time [s]
Central difference
Runge Kutta 4

Principe variationnel
Definition d’un Hamiltonien
Principe d’Hamilton

Équations de conservation
Conservation de la quantité de mouvement
Conservation du moment angulaire

Discrétisation

Discrétisation

Principe variationnel discrétisé
Définition d’un Hamiltonien discret
Principe d’Hamilton discret

Équations de conservation discrètes
Conservation de la quantité de mouvement
Conservation du moment angulaire

Principe variationnel
Definition d’un Hamiltonien
Principe d’Hamilton

→

Équations de conservation
Conservation de la quantité de mouvement
Conservation du moment angulaire

? ↓ Discrétisation

Principe variationnel discrétisé
Définition d’un Hamiltonien discret
Principe d’Hamilton discret

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Équations de conservation discrètes
Conservation de la quantité de mouvement
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Discrétisation

Principe variationnel discrétisé
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Principe d’Hamilton discret

Équations de conservation discrètes
Conservation de la quantité de mouvement
Conservation du moment angulaire

Partie A

Les intégrateurs géométriques sont une famille de schémas numériques capables de conserver, au sens discret, les propriétés géométriques du problème considéré (Hairer, Lubich, and Wanner 2006).

1. Sur quelle base construire un intégrateur géométrique pour la thermo-visco-élastodynamique ?
2. Dans ce cadre, comment **étendre la différence centrée** à la thermo-visco-élastodynamique ?
3. Autrement dit, quel intégrateur sur la température et sur les variables dissipatives internes est compatible avec la différence centrée ?
4. Quels sont les principes de conservation discrets vérifiés par un tel intégrateur ?

Structure géométrique faible

Équations de conservation

Conservation de la quantité de mouvement
Conservation du moment angulaire
Premier principe de la thermodynamique
Second principe de la thermodynamique

Discrétisation

Structure géométrique discrétisée

Équations de conservation discrètes

Conservation de la quantité de mouvement
Conservation du moment angulaire
Premier principe de la thermodynamique
Second principe de la thermodynamique

Partie A
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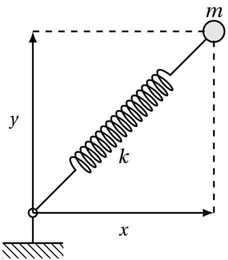
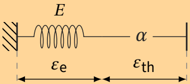


Partie B
Thermo-élastodynamique couplée

Petites transformations, 1D

$$\boldsymbol{z} = \begin{pmatrix} u & \varepsilon & p & T \end{pmatrix}$$



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Thermo-visco-élastodynamique couplée

Petites transformations, 1D

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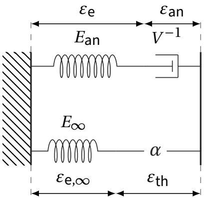

Structure géométrique faible

DGBA bracket structure

$$\frac{\mathrm{d}\mathscr{F}}{\mathrm{d}t} = \{\mathscr{F}, \hat{\mathscr{E}}_{\text{tot}}\} + (\mathscr{F}, \mathscr{S}) + \text{boundary terms}$$

→

Équations de conservation

Conservation de la quantité de mouvement

Premier principe de la thermodynamique

Second principe de la thermodynamique

Discrétisation ↓

Structure géométrique discrétisée

→

Équations de conservation discrètes

Conservation de la quantité de mouvement

Premier principe de la thermodynamique

Second principe de la thermodynamique

Partie B - Introduction par le crochet de Poisson

Soit $F = F(q,p)$, avec (q,p) vérifiant les équations d'Hamilton

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \frac{\partial F}{\partial q}\dot{q} + \frac{\partial F}{\partial p}\dot{p} \overset{\boxed{\dot{q}=\frac{\partial L^\star}{\partial p},\,\dot{p}=-\frac{\partial L^\star}{\partial q}}}{=} \frac{\partial F}{\partial q}\frac{\partial L^\star}{\partial p} - \frac{\partial F}{\partial p}\frac{\partial L^\star}{\partial q} = \begin{pmatrix} \frac{\partial F}{\partial q} \\ \frac{\partial F}{\partial p} \end{pmatrix}^{\mathsf{T}} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial L^\star}{\partial q} \\ \frac{\partial L^\star}{\partial p} \end{pmatrix} = \{F, L^\star\}$$

On retrouve les équations d'Hamilton **et** la conservation de l'énergie totale

$$F = q \Rightarrow \dot{q} = \frac{\partial L^\star}{\partial p} \qquad F = p \Rightarrow \dot{p} = -\frac{\partial L^\star}{\partial q} \qquad F = L^\star \Rightarrow \frac{\mathrm{d}L^\star}{\mathrm{d}t} = 0$$

Comment faire pour un système avec dissipation?

$$F = F(q,p,T) \qquad \frac{\mathrm{d}F}{\mathrm{d}t} = \frac{\partial F}{\partial q}\dot{q} + \frac{\partial F}{\partial p}\dot{p} + \frac{\partial F}{\partial T}\dot{T} = \{F, L^\star\} + \cdots$$

Partie B – Introduction à la DGBA bracket structure

est un crochet **antisymétrique**, représentant l'**évolution réversible** du système

$$\frac{\mathrm{d}\mathscr{F}}{\mathrm{d}t} = \left\{\mathscr{F}, \hat{\mathscr{E}}_{\text{tot}}\right\} + \left(\mathscr{F}, \mathscr{S}\right) + \text{boundary terms}$$

est un crochet **positif, symétrique**, représentant l'**évolution dissipative** du système.

On retrouve **les équations de conservation** pour différents choix de $\mathscr{F}$.

(Premier principe de la thermodynamique) $\mathscr{F} = \hat{\mathscr{E}}_{\text{tot}} \Rightarrow \dfrac{\mathrm{d}\hat{\mathscr{E}}_{\text{tot}}}{\mathrm{d}t} = \underbrace{\left\{\hat{\mathscr{E}}_{\text{tot}}, \hat{\mathscr{E}}_{\text{tot}}\right\}}_{=0} + \left(\hat{\mathscr{E}}_{\text{tot}}, \mathscr{S}\right) + \text{boundary terms}$

(Second principe de la thermodynamique) $\mathscr{F} = \mathscr{S} \Rightarrow \dfrac{\mathrm{d}\mathscr{S}}{\mathrm{d}t} - \left\{\mathscr{S}, \hat{\mathscr{E}}_{\text{tot}}\right\} - \text{boundary terms} \geq 0 \text{ and } \left(\mathscr{S}, \mathscr{S}\right) \geq 0$

Partie B – Introduction à la DGBA bracket structure

est un crochet **antisymétrique**, représentant l'**évolution réversible** du système

$$\frac{\mathrm{d}\mathscr{F}}{\mathrm{d}t} = \left\{\mathscr{F}, \hat{\mathscr{E}}_{\text{tot}}\right\} + \left(\mathscr{F}, \mathscr{S}\right) + \text{boundary terms}$$

est un crochet **positif, symétrique**, représentant l'**évolution dissipative** du système.

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1. Sur quelle base construire un intégrateur géométrique pour la thermo-visco-élastodynamique ?

Partie B – Choix du vecteur d'état - lois d'état

Le vecteur d'état pour ce problème est

$$\boldsymbol{z} = \begin{pmatrix} u & \varepsilon & p & T \end{pmatrix}$$

où

- u est le déplacement;
- ε est la partie symétrique du gradient du déplacement, ici $\varepsilon = \partial u/\partial X$;
- p est la quantité de mouvement;
- T est la température absolue.

Puis, on dérive par rapport au temps $\mathscr{F}$

$$\frac{\mathrm{d}\mathscr{F}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\int_0^L F(u, \varepsilon, p, T)\,\mathrm{d}X = \int_0^L \left(\frac{\partial F}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial F}{\partial \varepsilon}\frac{\partial \varepsilon}{\partial t} + \frac{\partial F}{\partial p}\frac{\partial p}{\partial t} + \frac{\partial F}{\partial T}\frac{\partial T}{\partial t}\right)\mathrm{d}X$$

Partie B – Choix du vecteur d'état - lois d'état

Le vecteur d'état pour ce problème est $\boldsymbol{z} = \begin{pmatrix} u & \varepsilon & p & T \end{pmatrix}$

La DGBA bracket structure est **obtenue via les lois d'état**

$$\frac{\mathrm{d}\mathscr{F}}{\mathrm{d}t} = \int_0^L \left(\frac{\partial F}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial F}{\partial \varepsilon}\frac{\partial \varepsilon}{\partial t} + \frac{\partial F}{\partial p}\frac{\partial p}{\partial t} + \frac{\partial F}{\partial T}\frac{\partial T}{\partial t} \right) \mathrm{d}X$$

Potentiels $\hat{E}_{\text{tot}}$ et S généraux

$$\frac{\partial u}{\partial t} = \frac{\partial \hat{E}_{\text{tot}}}{\partial p}$$

$$\frac{\partial \varepsilon}{\partial t} = \frac{\partial}{\partial X}\left(\frac{\partial \hat{E}_{\text{tot}}}{\partial p}\right)$$

$$\frac{\partial p}{\partial t} = -\frac{\partial \hat{E}_{\text{tot}}}{\partial u} + \frac{\partial}{\partial X}\left(\frac{\partial \hat{E}_{\text{tot}}}{\partial \varepsilon} - \left(\frac{\partial \hat{E}_{\text{tot}}}{\partial T}\right)\left(\frac{\partial S}{\partial T}\right)^{-1}\frac{\partial S}{\partial \varepsilon}\right)$$

$$\rho c \frac{\partial T}{\partial t} + \rho c \left(\frac{\partial S}{\partial T}\right)^{-1}\frac{\partial S}{\partial \varepsilon}\frac{\partial}{\partial X}\left(\frac{\partial \hat{E}_{\text{tot}}}{\partial p}\right) = \frac{\partial}{\partial X}\left(K\frac{\partial T}{\partial X}\right)$$

Thermo-élastodynamique linéaire

$$\frac{\partial u}{\partial t} = \frac{1}{\rho} p$$

$$\frac{\partial \varepsilon}{\partial t} = \frac{1}{\rho}\frac{\partial p}{\partial X}$$

$$\frac{\partial p}{\partial t} = \frac{\partial}{\partial X}\left((\lambda + 2\mu)\varepsilon - k(T - T_0)\right)$$

$$\rho c \frac{\partial T}{\partial t} + k T_0 \frac{\partial}{\partial X}\left(\frac{1}{\rho} p\right) = \frac{\partial}{\partial X}\left(K\frac{\partial T}{\partial X}\right)$$

K: conductivité; ρ: masse volumique; c: capacité thermique; $\hat{E}_{\text{int}}$: énergie interne

Partie B – Choix du vecteur d'état - lois d'état

Le vecteur d'état pour ce problème est $\boldsymbol{z} = \begin{pmatrix} u & \varepsilon & p & T \end{pmatrix}$

La DGBA bracket structure est **obtenue via les lois d'état**

$$\frac{\mathrm{d}\mathscr{F}}{\mathrm{d}t} = \int_0^L \left(\frac{\partial F}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial F}{\partial \varepsilon}\frac{\partial \varepsilon}{\partial t} + \frac{\partial F}{\partial p}\frac{\partial p}{\partial t} + \frac{\partial F}{\partial T}\frac{\partial T}{\partial t} \right) \mathrm{d}X$$

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$$\frac{\partial u}{\partial t} = \frac{\partial \hat{E}_{\text{tot}}}{\partial p}$$

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$$\rho c\frac{\partial T}{\partial t} + \rho c\left(\frac{\partial S}{\partial T}\right)^{-1}\frac{\partial S}{\partial \varepsilon}\frac{\partial}{\partial X}\left(\frac{\partial \hat{E}_{\text{tot}}}{\partial p}\right) = \frac{\partial}{\partial X}\left(K\frac{\partial T}{\partial X}\right)$$

Thermo-élastodynamique linéaire

$$\frac{\partial u}{\partial t} = \frac{1}{\rho}p$$

$$\frac{\partial \varepsilon}{\partial t} = \frac{1}{\rho}\frac{\partial p}{\partial X}$$

$$\frac{\partial p}{\partial t} = \frac{\partial}{\partial X}\left((\lambda + 2\mu)\varepsilon - k(T - T_0)\right)$$

$$\rho c\frac{\partial T}{\partial t} + kT_0\frac{\partial}{\partial X}\left(\frac{1}{\rho}p\right) = \frac{\partial}{\partial X}\left(K\frac{\partial T}{\partial X}\right)$$

K: conductivité; ρ: masse volumique; c: capacité thermique; $\hat{E}_{\text{int}}$: énergie interne

Partie B – Expression de la DGBA bracket structure

La DGBA bracket structure **émerge des lois d'état**

$$\frac{\mathrm{d}\mathscr{F}}{\mathrm{d}t} = \left\{\mathscr{F}, \hat{\mathscr{E}}_{\mathrm{tot}}\right\} + (\mathscr{F}, \mathscr{S}) + \\ + \left[\frac{\partial F}{\partial p}\left(\frac{\partial \hat{E}_{\mathrm{tot}}}{\partial \varepsilon} - \left(\frac{\partial S}{\partial T}\right)^{-1}\frac{\partial S}{\partial \varepsilon}\frac{\partial \hat{E}_{\mathrm{tot}}}{\partial T}\right)\right]_0^L - \left[\frac{\partial F}{\partial T} K \left(\frac{T}{\rho c}\right)^2 \frac{\partial}{\partial X}\left(\frac{\partial S}{\partial T}\right)\right]_0^L$$

Partie B – Expression de la DGBA bracket structure

Détail : crochet réversible

$$
\begin{aligned}
\{\mathscr{F},\hat{\mathscr{E}}_{\text{tot}}\} &= \int_0^L \left(\frac{\partial F}{\partial u}\frac{\partial \hat{E}_{\text{tot}}}{\partial p} - \frac{\partial F}{\partial p}\frac{\partial \hat{E}_{\text{tot}}}{\partial u} + \frac{\partial F}{\partial \varepsilon}\frac{\partial}{\partial X}\left(\frac{\partial \hat{E}_{\text{tot}}}{\partial p}\right) - \frac{\partial}{\partial X}\left(\frac{\partial F}{\partial p}\right)\frac{\partial \hat{E}_{\text{tot}}}{\partial \varepsilon} + \right. \\
&\quad \left. + \left(\frac{\partial S}{\partial T}\right)^{-1}\frac{\partial S}{\partial \varepsilon}\left(\frac{\partial \hat{E}_{\text{tot}}}{\partial T}\frac{\partial}{\partial X}\left(\frac{\partial F}{\partial p}\right) - \frac{\partial F}{\partial T}\frac{\partial}{\partial X}\left(\frac{\partial \hat{E}_{\text{tot}}}{\partial p}\right)\right)\right)\mathrm{d}X \\
&= \int_0^L \begin{pmatrix} \frac{\partial F}{\partial u} \\ \frac{\partial F}{\partial \varepsilon} \\ \frac{\partial F}{\partial p} \\ \frac{\partial F}{\partial T} \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{\partial \square_{\text{right}}}{\partial X} & 0 \\ -1 & -\frac{\partial \square_{\text{left}}}{\partial X} & 0 & \left(\frac{\partial S}{\partial T}\right)^{-1}\frac{\partial S}{\partial \varepsilon}\frac{\partial \square_{\text{left}}}{\partial X} \\ 0 & 0 & -\left(\frac{\partial S}{\partial T}\right)^{-1}\frac{\partial S}{\partial \varepsilon}\frac{\partial \square_{\text{right}}}{\partial X} & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial \hat{E}_{\text{tot}}}{\partial u} \\ \frac{\partial \hat{E}_{\text{tot}}}{\partial \varepsilon} \\ \frac{\partial \hat{E}_{\text{tot}}}{\partial p} \\ \frac{\partial \hat{E}_{\text{tot}}}{\partial T} \end{pmatrix} \mathrm{d}X
\end{aligned}
$$

Partie B – Expression de la DGBA bracket structure

Détail : crochet réversible

$$
\begin{aligned}
\{\mathscr{F}, \hat{\mathscr{E}}_{\text{tot}}\} &= \int_0^L \left(\frac{\partial F}{\partial u}\frac{\partial \hat{E}_{\text{tot}}}{\partial p} - \frac{\partial F}{\partial p}\frac{\partial \hat{E}_{\text{tot}}}{\partial u} + \frac{\partial F}{\partial \varepsilon}\frac{\partial}{\partial X}\left(\frac{\partial \hat{E}_{\text{tot}}}{\partial p}\right) - \frac{\partial}{\partial X}\left(\frac{\partial F}{\partial p}\right)\frac{\partial \hat{E}_{\text{tot}}}{\partial \varepsilon} + \right. \\
&\quad \left. + \left(\frac{\partial S}{\partial T}\right)^{-1}\frac{\partial S}{\partial \varepsilon}\left(\frac{\partial \hat{E}_{\text{tot}}}{\partial T}\frac{\partial}{\partial X}\left(\frac{\partial F}{\partial p}\right) - \frac{\partial F}{\partial T}\frac{\partial}{\partial X}\left(\frac{\partial \hat{E}_{\text{tot}}}{\partial p}\right)\right)\right) \mathrm{d}X \\
&= \int_0^L \begin{pmatrix} \frac{\partial F}{\partial u} \\ \frac{\partial F}{\partial \varepsilon} \\ \frac{\partial F}{\partial p} \\ \frac{\partial F}{\partial T} \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{\partial \square_{\text{right}}}{\partial X} & 0 \\ -1 & -\frac{\partial \square_{\text{left}}}{\partial X} & 0 & \left(\frac{\partial S}{\partial T}\right)^{-1}\frac{\partial S}{\partial \varepsilon}\frac{\partial \square_{\text{left}}}{\partial X} \\ 0 & 0 & -\left(\frac{\partial S}{\partial T}\right)^{-1}\frac{\partial S}{\partial \varepsilon}\frac{\partial \square_{\text{right}}}{\partial X} & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial \hat{E}_{\text{tot}}}{\partial u} \\ \frac{\partial \hat{E}_{\text{tot}}}{\partial \varepsilon} \\ \frac{\partial \hat{E}_{\text{tot}}}{\partial p} \\ \frac{\partial \hat{E}_{\text{tot}}}{\partial T} \end{pmatrix} \mathrm{d}X
\end{aligned}
$$

Partie B – Expression de la DGBA bracket structure

Détail : crochet réversible

$$\{\mathscr{F}, \hat{\mathscr{E}}_{\text{tot}}\} = \int_0^L \left(\frac{\partial F}{\partial u} \frac{\partial \hat{E}_{\text{tot}}}{\partial p} - \frac{\partial F}{\partial p} \frac{\partial \hat{E}_{\text{tot}}}{\partial u} + \frac{\partial F}{\partial \varepsilon} \frac{\partial}{\partial X} \left(\frac{\partial \hat{E}_{\text{tot}}}{\partial p} \right) - \frac{\partial}{\partial X} \left(\frac{\partial F}{\partial p} \right) \frac{\partial \hat{E}_{\text{tot}}}{\partial \varepsilon} + \right.$$

$$\left. + \left(\frac{\partial S}{\partial T} \right)^{-1} \frac{\partial S}{\partial \varepsilon} \left(\frac{\partial \hat{E}_{\text{tot}}}{\partial T} \frac{\partial}{\partial X} \left(\frac{\partial F}{\partial p} \right) - \frac{\partial F}{\partial T} \frac{\partial}{\partial X} \left(\frac{\partial \hat{E}_{\text{tot}}}{\partial p} \right) \right) \right) \mathrm{d}X$$

$$= \int_0^L \begin{pmatrix} \frac{\partial F}{\partial u} \\ \frac{\partial F}{\partial \varepsilon} \\ \frac{\partial F}{\partial p} \\ \frac{\partial F}{\partial T} \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{\partial \square_{\text{right}}}{\partial X} & 0 \\ -1 & -\frac{\partial \square_{\text{left}}}{\partial X} & 0 & \left(\frac{\partial S}{\partial T} \right)^{-1} \frac{\partial S}{\partial \varepsilon} \frac{\partial \square_{\text{left}}}{\partial X} \\ 0 & 0 & -\left(\frac{\partial S}{\partial T} \right)^{-1} \frac{\partial S}{\partial \varepsilon} \frac{\partial \square_{\text{right}}}{\partial X} & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial \hat{E}_{\text{tot}}}{\partial u} \\ \frac{\partial \hat{E}_{\text{tot}}}{\partial \varepsilon} \\ \frac{\partial \hat{E}_{\text{tot}}}{\partial p} \\ \frac{\partial \hat{E}_{\text{tot}}}{\partial T} \end{pmatrix} \mathrm{d}X$$

Partie B – Expression de la DGBA bracket structure

Détail : crochet réversible

$$\{\mathscr{F},\hat{\mathscr{E}}_{\text{tot}}\} = \int_0^L \left(\frac{\partial F}{\partial u}\frac{\partial \hat{E}_{\text{tot}}}{\partial p} - \frac{\partial F}{\partial p}\frac{\partial \hat{E}_{\text{tot}}}{\partial u} + \frac{\partial F}{\partial \varepsilon}\frac{\partial}{\partial X}\left(\frac{\partial \hat{E}_{\text{tot}}}{\partial p}\right) - \frac{\partial}{\partial X}\left(\frac{\partial F}{\partial p}\right)\frac{\partial \hat{E}_{\text{tot}}}{\partial \varepsilon} + \right.$$

$$\left. + \left(\frac{\partial S}{\partial T}\right)^{-1}\frac{\partial S}{\partial \varepsilon}\left(\frac{\partial \hat{E}_{\text{tot}}}{\partial T}\frac{\partial}{\partial X}\left(\frac{\partial F}{\partial p}\right) - \frac{\partial F}{\partial T}\frac{\partial}{\partial X}\left(\frac{\partial \hat{E}_{\text{tot}}}{\partial p}\right)\right)\right)\mathrm{d}X$$

$$= \int_0^L \begin{pmatrix} \frac{\partial F}{\partial u} \\ \frac{\partial F}{\partial \varepsilon} \\ \frac{\partial F}{\partial p} \\ \frac{\partial F}{\partial T} \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{\partial \square_{\text{right}}}{\partial X} & 0 \\ -1 & -\frac{\partial \square_{\text{left}}}{\partial X} & 0 & \left(\frac{\partial S}{\partial T}\right)^{-1}\frac{\partial S}{\partial \varepsilon}\frac{\partial \square_{\text{left}}}{\partial X} \\ 0 & 0 & -\left(\frac{\partial S}{\partial T}\right)^{-1}\frac{\partial S}{\partial \varepsilon}\frac{\partial \square_{\text{right}}}{\partial X} & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial \hat{E}_{\text{tot}}}{\partial u} \\ \frac{\partial \hat{E}_{\text{tot}}}{\partial \varepsilon} \\ \frac{\partial \hat{E}_{\text{tot}}}{\partial p} \\ \frac{\partial \hat{E}_{\text{tot}}}{\partial T} \end{pmatrix} \mathrm{d}X$$

Partie B – Expression de la DGBA bracket structure

Détail : crochet dissipatif

$$(\mathscr{F},\mathscr{S}) = \int_0^L \left(\frac{\partial}{\partial X}\left(\frac{\partial F}{\partial T}\right) K \left(\frac{T}{\rho c}\right)^2 \frac{\partial}{\partial X}\left(\frac{\partial S}{\partial T}\right)\right) \mathrm{d}X$$

$$= \int_0^L \begin{pmatrix} \frac{\partial F}{\partial u} \\ \frac{\partial F}{\partial \varepsilon} \\ \frac{\partial F}{\partial p} \\ \frac{\partial F}{\partial T} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\partial \square_{\text{left}}}{\partial X} K \left(\frac{T}{\rho c}\right)^2 \frac{\partial \square_{\text{right}}}{\partial X} \end{pmatrix} \begin{pmatrix} \frac{\partial S}{\partial u} \\ \frac{\partial S}{\partial \varepsilon} \\ \frac{\partial S}{\partial p} \\ \frac{\partial S}{\partial T} \end{pmatrix} \mathrm{d}X$$

Partie B – Expression de la DGBA bracket structure

Détail : crochet dissipatif

$$(\mathscr{F},\mathscr{S}) = \int_0^L \left(\frac{\partial}{\partial X}\left(\frac{\partial F}{\partial T}\right) K \left(\frac{T}{\rho c}\right)^2 \frac{\partial}{\partial X}\left(\frac{\partial S}{\partial T}\right)\right) \mathrm{d}X$$

$$= \int_0^L \begin{pmatrix} \frac{\partial F}{\partial u} \\ \frac{\partial F}{\partial \varepsilon} \\ \frac{\partial F}{\partial p} \\ \frac{\partial F}{\partial T} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\partial \square_{\text{left}}}{\partial X} K \left(\frac{T}{\rho c}\right)^2 \frac{\partial \square_{\text{right}}}{\partial X} \end{pmatrix} \begin{pmatrix} \frac{\partial S}{\partial u} \\ \frac{\partial S}{\partial \varepsilon} \\ \frac{\partial S}{\partial p} \\ \frac{\partial S}{\partial T} \end{pmatrix} \mathrm{d}X$$

Partie B – Retrouver les équations de conservation par la structure

La **conservation de la quantité de mouvement** est obtenue via

$$\mathscr{F} = \int_0^L p \, \mathrm{d}X \Rightarrow (\ldots) \Rightarrow \boxed{\frac{\partial p}{\partial t} = -\frac{\partial \hat{E}_{\text{tot}}}{\partial u} + \mathrm{div}(\sigma)}$$

Le **premier principe de la thermodynamique** est obtenu par

$$\mathscr{F} = \hat{\mathscr{E}}_{\text{tot}} = \int_0^L \left(\frac{1}{2} \frac{1}{\rho} p p + \hat{E}_{\text{int}}(\varepsilon, T) \right) \mathrm{d}X \Rightarrow (\cdots) \Rightarrow \boxed{\frac{\partial \hat{E}_{\text{int}}}{\partial t} = \frac{\partial \varepsilon}{\partial t} \sigma - \frac{\partial q}{\partial X}}$$

Enfin, le **second principe de la thermodynamique** s’obtient comme:

$$\mathscr{F} = \mathscr{S} = \int_0^L S \, \mathrm{d}X, \Rightarrow (\ldots) \Rightarrow \boxed{\frac{\partial S}{\partial t} + \frac{\partial}{\partial X}\left(\frac{q}{T}\right) \geq 0 \Leftrightarrow \frac{1}{T}\left(-\frac{1}{T} q \frac{\partial T}{\partial X}\right) \geq 0}$$

σ: tenseur des contraintes 1D; q: flux de chaleur.

Structure géométrique faible

DGBA bracket structure

$$\frac{\mathrm{d}\mathscr{F}}{\mathrm{d}t} = \{\mathscr{F}, \hat{\mathscr{E}}_{\text{tot}}\} + (\mathscr{F}, \mathscr{S}) + \text{boundary terms}$$

Équations de conservation

Conservation de la quantité de mouvement

Premier principe de la thermodynamique

Second principe de la thermodynamique

Discrétisation

Structure géométrique discrétisée

Équations de conservation discrètes

Conservation de la quantité de mouvement

Premier principe de la thermodynamique

Second principe de la thermodynamique

Structure géométrique faible

DGBA bracket structure

$$\frac{\mathrm{d}\mathscr{F}}{\mathrm{d}t} = \{\mathscr{F}, \hat{\mathscr{E}}_{\text{tot}}\} + (\mathscr{F}, \mathscr{S}) + \text{boundary terms}$$

Équations de conservation

Conservation de la quantité de mouvement

Premier principe de la thermodynamique

Second principe de la thermodynamique

Discrétisation

Structure géométrique discrétisée

Équations de conservation discrètes

Conservation de la quantité de mouvement

Premier principe de la thermodynamique

Second principe de la thermodynamique

Partie B – Méthodologie de discrétisation

DGBA bracket structure

$$\frac{\mathrm{d}\mathscr{F}}{\mathrm{d}t} = \left\{\mathscr{F}, \hat{\mathscr{E}}_{\text{tot}}\right\} + \left(\mathscr{F}, \mathscr{S}\right) + \text{boundary terms}$$

$$\left\{\mathscr{F}, \hat{\mathscr{E}}_{\text{tot}}\right\} = \int_0^L \left(\frac{\partial F}{\partial u}\frac{\partial \hat{E}_{\text{tot}}}{\partial p} - \frac{\partial F}{\partial p}\frac{\partial \hat{E}_{\text{tot}}}{\partial u} + \frac{\partial F}{\partial \varepsilon}\frac{\partial}{\partial X}\left(\frac{\partial \hat{E}_{\text{tot}}}{\partial p}\right) - \frac{\partial}{\partial X}\left(\frac{\partial F}{\partial p}\right)\frac{\partial \hat{E}_{\text{tot}}}{\partial \varepsilon} + \left(\frac{\partial S}{\partial T}\right)^{-1}\frac{\partial S}{\partial \varepsilon}\left(\frac{\partial \hat{E}_{\text{tot}}}{\partial T}\frac{\partial}{\partial X}\left(\frac{\partial F}{\partial p}\right) - \frac{\partial F}{\partial T}\frac{\partial}{\partial X}\left(\frac{\partial \hat{E}_{\text{tot}}}{\partial p}\right)\right)\right)\mathrm{d}X$$

$$\left(\mathscr{F}, \mathscr{S}\right) = \int_0^L \left(\frac{\partial}{\partial X}\left(\frac{\partial F}{\partial T}\right) K \left(\frac{T}{\rho c}\right)^2 \frac{\partial}{\partial X}\left(\frac{\partial S}{\partial T}\right)\right)\mathrm{d}X$$

$$\text{boundary terms} = \left[\frac{\partial F}{\partial p}\left(\frac{\partial \hat{E}_{\text{tot}}}{\partial \varepsilon} - \left(\frac{\partial S}{\partial T}\right)^{-1}\frac{\partial S}{\partial \varepsilon}\frac{\partial \hat{E}_{\text{tot}}}{\partial T}\right)\right]_0^L - \left[\frac{\partial F}{\partial T} K \left(\frac{T}{\rho c}\right)^2 \frac{\partial}{\partial X}\left(\frac{\partial S}{\partial T}\right)\right]_0^L$$

1. Choix d'un schéma d'intégration temporelle
2. Discrétisation spatiale par éléments finis

DGBA bracket structure discrète

$$\frac{1}{\Delta t}\left(\left\langle \frac{\partial \mathsf{F}_{\mathsf{d}}}{\partial \mathbf{z}_k}\right\rangle_n^{n+1}\right)^{\mathsf{T}} [\mathbf{z}_k]_n^{n+1} = \left\{\mathsf{F}_{\mathsf{d}}, \hat{\mathsf{E}}_{\mathsf{d}}^{\mathsf{tot}}\right\} + \left(\mathsf{F}_{\mathsf{d}}, \mathsf{S}_{\mathsf{d}}\right) + \text{boundary terms}$$

Partie B – Méthodologie de discrétisation

1. **Choix d'un schéma d'intégration temporelle**
   On intègre la structure sur tout l'intervalle temporel $[t_0, t_N]$

$$\sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \left(\frac{\mathrm{d}\mathscr{F}}{\mathrm{d}t} - \{\mathscr{F}, \hat{\mathscr{E}}_{\text{tot}}\} - (\mathscr{F}, \mathscr{S}) - \text{boundary terms} \right) \mathrm{d}t = 0$$

   On choisit une approximation de l'intégrale, **choix basé sur le calcul variationnel discret** (Marsden et al., 1994)

$$\int_{t_n}^{t_{n+1}} \int_0^L G(u, p, \varepsilon, T)\,\mathrm{d}X \approx \int_0^L \frac{\Delta t}{2} \left(G(u_n, \varepsilon_n, p_{n+1/2}, T_{n+1/2}) + G(u_{n+1}, \varepsilon_{n+1}, p_{n+1/2}, T_{n+1/2}) \right) \mathrm{d}X$$

   Cette méthode étend la différence centrée à la thermo-élastodynamique couplée.

2. Dans ce cadre, comment étendre la différence centrée à la thermo-élastodynamique couplée ?

Partie B – Méthodologie de discrétisation

1. **Choix d'un schéma d'intégration temporelle**
   Le couplage thermo-mécanique couple le choix d'intégration de l'équation d'évolution de la température avec le choix d'intégration de l'équation d'évolution de la quantité de mouvement via la différence centrée.

$$\frac{\mathrm{d}\mathscr{F}}{\mathrm{d}t} = \int_0^L \left(\boldsymbol{\frac{\partial F}{\partial u}\frac{\partial u}{\partial t}} + \frac{\partial F}{\partial \varepsilon}\frac{\partial \varepsilon}{\partial t} + \frac{\partial F}{\partial p}\frac{\partial p}{\partial t} + \frac{\partial F}{\partial T}\frac{\partial T}{\partial t} \right) \mathrm{d}X$$

$$\{\mathscr{F}, \hat{\mathscr{E}}_{\text{tot}}\} = \int_0^L \begin{pmatrix} \boldsymbol{\partial F/\partial u} \\ \partial F/\partial \varepsilon \\ \partial F/\partial p \\ \partial F/\partial T \end{pmatrix} \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ 0 & 0 & \frac{\partial \square_{\text{right}}}{\partial X} & 0 \\ -1 & -\frac{\partial \square_{\text{left}}}{\partial X} & 0 & \left(\frac{\partial S}{\partial T}\right)^{-1} \frac{\partial S}{\partial \varepsilon} \frac{\partial \square_{\text{left}}}{\partial X} \\ 0 & 0 & -\left(\frac{\partial S}{\partial T}\right)^{-1} \frac{\partial S}{\partial \varepsilon} \frac{\partial \square_{\text{right}}}{\partial X} & 0 \end{pmatrix} \begin{pmatrix} \partial \hat{E}_{\text{tot}}/\partial u \\ \partial \hat{E}_{\text{tot}}/\partial \varepsilon \\ \boldsymbol{\partial \hat{E}_{\text{tot}}/\partial p} \\ \partial \hat{E}_{\text{tot}}/\partial T \end{pmatrix} \mathrm{d}X$$

$$(\mathscr{F}, \mathscr{S}) = \int_0^L \begin{pmatrix} \boldsymbol{\partial F/\partial u} \\ \partial F/\partial \varepsilon \\ \partial F/\partial p \\ \partial F/\partial T \end{pmatrix} \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\partial \square_{\text{left}}}{\partial X} K \left(\frac{T}{\rho c}\right)^2 \frac{\partial \square_{\text{right}}}{\partial X} \end{pmatrix} \begin{pmatrix} \partial S/\partial u \\ \partial S/\partial \varepsilon \\ \partial S/\partial p \\ \partial S/\partial T \end{pmatrix} \mathrm{d}X$$

Partie B – Méthodologie de discrétisation

1. **Choix d'un schéma d'intégration temporelle**
   Le couplage thermo-mécanique couple le choix d'intégration de l'équation d'évolution de la température avec le choix d'intégration de l'équation d'évolution de la quantité de mouvement via la différence centrée.

$$\frac{\mathrm{d}\mathscr{F}}{\mathrm{d}t} = \int_0^L \left(\frac{\partial F}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial F}{\partial \varepsilon}\frac{\partial \varepsilon}{\partial t} + \boldsymbol{\frac{\partial F}{\partial p}\frac{\partial p}{\partial t}} + \frac{\partial F}{\partial T}\frac{\partial T}{\partial t} \right) \mathrm{d}X$$

$$\{\mathscr{F}, \hat{\mathscr{E}}_{\text{tot}}\} = \int_0^L \begin{pmatrix} \partial F/\partial u \\ \partial F/\partial \varepsilon \\ \boldsymbol{\partial F/\partial p} \\ \partial F/\partial T \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{\partial \square_{\text{right}}}{\partial X} & 0 \\ \boldsymbol{-1} & \boldsymbol{-\frac{\partial \square_{\text{left}}}{\partial X}} & \boldsymbol{0} & \boldsymbol{\left(\frac{\partial S}{\partial T}\right)^{-1} \frac{\partial S}{\partial \varepsilon}\frac{\partial \square_{\text{left}}}{\partial X}} \\ 0 & 0 & -\left(\frac{\partial S}{\partial T}\right)^{-1} \frac{\partial S}{\partial \varepsilon}\frac{\partial \square_{\text{right}}}{\partial X} & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{\partial \hat{E}_{\text{tot}}/\partial u} \\ \boldsymbol{\partial \hat{E}_{\text{tot}}/\partial \varepsilon} \\ \partial \hat{E}_{\text{tot}}/\partial p \\ \boldsymbol{\partial \hat{E}_{\text{tot}}/\partial T} \end{pmatrix} \mathrm{d}X$$

$$(\mathscr{F}, \mathscr{S}) = \int_0^L \begin{pmatrix} \partial F/\partial u \\ \partial F/\partial \varepsilon \\ \boldsymbol{\partial F/\partial p} \\ \partial F/\partial T \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ 0 & 0 & 0 & \frac{\partial \square_{\text{left}}}{\partial X} K \left(\frac{T}{\rho c}\right)^2 \frac{\partial \square_{\text{right}}}{\partial X} \end{pmatrix} \begin{pmatrix} \partial S/\partial u \\ \partial S/\partial \varepsilon \\ \partial S/\partial p \\ \partial S/\partial T \end{pmatrix} \mathrm{d}X$$

Partie B – Méthodologie de discrétisation

1. **Choix d'un schéma d'intégration temporelle**
   Le couplage thermo-mécanique couple le choix d'intégration de l'équation d'évolution de la température avec le choix d'intégration de l'équation d'évolution de la quantité de mouvement via la différence centrée.

$$\frac{\mathrm{d}\mathscr{F}}{\mathrm{d}t} = \int_0^L \left(\frac{\partial F}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial F}{\partial \varepsilon}\frac{\partial \varepsilon}{\partial t} + \frac{\partial F}{\partial p}\frac{\partial p}{\partial t} + \boldsymbol{\frac{\partial F}{\partial T}\frac{\partial T}{\partial t}} \right) \mathrm{d}X$$

$$\{\mathscr{F}, \hat{\mathscr{E}}_{\text{tot}}\} = \int_0^L \begin{pmatrix} \partial F/\partial u \\ \partial F/\partial \varepsilon \\ \partial F/\partial p \\ \boldsymbol{\partial F/\partial T} \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{\partial \square_{\text{right}}}{\partial X} & 0 \\ -1 & -\frac{\partial \square_{\text{left}}}{\partial X} & 0 & \left(\frac{\partial S}{\partial T}\right)^{-1} \frac{\partial S}{\partial \varepsilon}\frac{\partial \square_{\text{left}}}{\partial X} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{-\left(\frac{\partial S}{\partial T}\right)^{-1} \frac{\partial S}{\partial \varepsilon}\frac{\partial \square_{\text{right}}}{\partial X}} & \boldsymbol{0} \end{pmatrix} \begin{pmatrix} \partial \hat{E}_{\text{tot}}/\partial u \\ \partial \hat{E}_{\text{tot}}/\partial \varepsilon \\ \boldsymbol{\partial \hat{E}_{\text{tot}}/\partial p} \\ \partial \hat{E}_{\text{tot}}/\partial T \end{pmatrix} \mathrm{d}X$$

$$(\mathscr{F}, \mathscr{S}) = \int_0^L \begin{pmatrix} \partial F/\partial u \\ \partial F/\partial \varepsilon \\ \partial F/\partial p \\ \boldsymbol{\partial F/\partial T} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{\frac{\partial \square_{\text{left}}}{\partial X} K \left(\frac{T}{\rho c}\right)^2 \frac{\partial \square_{\text{right}}}{\partial X}} \end{pmatrix} \begin{pmatrix} \partial S/\partial u \\ \partial S/\partial \varepsilon \\ \partial S/\partial p \\ \boldsymbol{\partial S/\partial T} \end{pmatrix} \mathrm{d}X$$

Partie B – Méthodologie de discrétisation

1. **Choix d'un schéma d'intégration temporelle**
   Le couplage thermo-mécanique couple le choix d'intégration de l'équation d'évolution de la température avec le choix d'intégration de l'équation d'évolution de la quantité de mouvement via la différence centrée.

$$\frac{\mathrm{d}\mathscr{F}}{\mathrm{d}t} = \int_0^L \left(\frac{\partial F}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial F}{\partial \varepsilon}\frac{\partial \varepsilon}{\partial t} + \boldsymbol{\frac{\partial F}{\partial p}\frac{\partial p}{\partial t}} + \boldsymbol{\frac{\partial F}{\partial T}\frac{\partial T}{\partial t}} \right) \mathrm{d}X$$

$$\{\mathscr{F}, \hat{\mathscr{E}}_{\text{tot}}\} = \int_0^L \begin{pmatrix} \partial F/\partial u \\ \partial F/\partial \varepsilon \\ \boldsymbol{\partial F/\partial p} \\ \boldsymbol{\partial F/\partial T} \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{\partial \square_{\text{right}}}{\partial X} & 0 \\ \boldsymbol{-1} & \boldsymbol{-\frac{\partial \square_{\text{left}}}{\partial X}} & \boldsymbol{0} & \boldsymbol{\left(\frac{\partial S}{\partial T}\right)^{-1} \frac{\partial S}{\partial \varepsilon} \frac{\partial \square_{\text{left}}}{\partial X}} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{-\left(\frac{\partial S}{\partial T}\right)^{-1} \frac{\partial S}{\partial \varepsilon} \frac{\partial \square_{\text{right}}}{\partial X}} & \boldsymbol{0} \end{pmatrix} \begin{pmatrix} \partial \hat{E}_{\text{tot}}/\partial u \\ \partial \hat{E}_{\text{tot}}/\partial \varepsilon \\ \boldsymbol{\partial \hat{E}_{\text{tot}}/\partial p} \\ \boldsymbol{\partial \hat{E}_{\text{tot}}/\partial T} \end{pmatrix} \mathrm{d}X$$

$$(\mathscr{F}, \mathscr{S}) = \int_0^L \begin{pmatrix} \partial F/\partial u \\ \partial F/\partial \varepsilon \\ \partial F/\partial p \\ \boldsymbol{\partial F/\partial T} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{\frac{\partial \square_{\text{left}}}{\partial X} K \left(\frac{T}{\rho c}\right)^2 \frac{\partial \square_{\text{right}}}{\partial X}} \end{pmatrix} \begin{pmatrix} \partial S/\partial u \\ \partial S/\partial \varepsilon \\ \partial S/\partial p \\ \boldsymbol{\partial S/\partial T} \end{pmatrix} \mathrm{d}X$$

Partie B – Méthodologie de discrétisation

1. **Choix d'un schéma d'intégration temporelle**
2. **Discrétisation spatiale par éléments finis**
   On applique **la méthode des éléments finis**, sur chacun des champs.

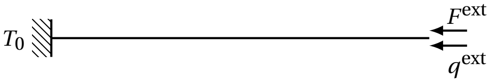
$$u(X, t_n) \approx \sum_a \mathsf{N}_a^u(X)\mathsf{u}_n^a \qquad p(X, t_n) \approx \sum_a \mathsf{N}_a^p(X)\mathsf{p}_n^a \qquad T(X, t_n) \approx \sum_a \mathsf{N}_a^T(X)\mathsf{T}_n^a$$

Partie B – Méthodologie de discrétisation

DGBA bracket structure discrète

$$\frac{1}{\Delta t}\left(\left\langle \frac{\partial \mathsf{F}_\mathsf{d}}{\partial \mathbf{z}_k}\right\rangle_n^{n+1}\right)^{\mathsf{T}} [\mathbf{z}_k]_n^{n+1} = \left\{\mathsf{F}_\mathsf{d}, \hat{\mathsf{E}}_\mathsf{d}^{\mathsf{tot}}\right\} + \left(\mathsf{F}_\mathsf{d}, \mathsf{S}_\mathsf{d}\right) + \text{boundary terms}$$

Problème particulier

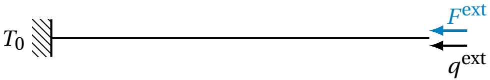


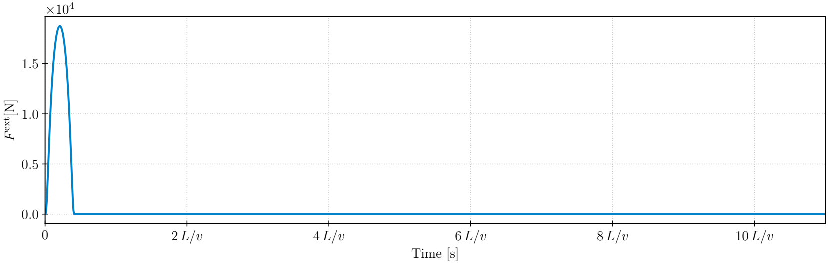
Partie B – Méthodologie de discrétisation

DGBA bracket structure discrète

$$\frac{1}{\Delta t}\left(\left\langle \frac{\partial \mathrm{F}_\mathrm{d}}{\partial \mathbf{z}_k}\right\rangle_n^{n+1}\right)^{\mathrm{T}} [\mathbf{z}_k]_n^{n+1} = \{\mathrm{F}_\mathrm{d}, \hat{\mathrm{E}}_\mathrm{d}^{\mathrm{tot}}\} + (\mathrm{F}_\mathrm{d}, \mathrm{S}_\mathrm{d}) + \text{boundary terms}$$

Problème particulier



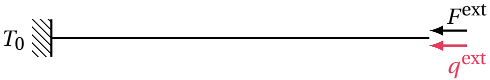


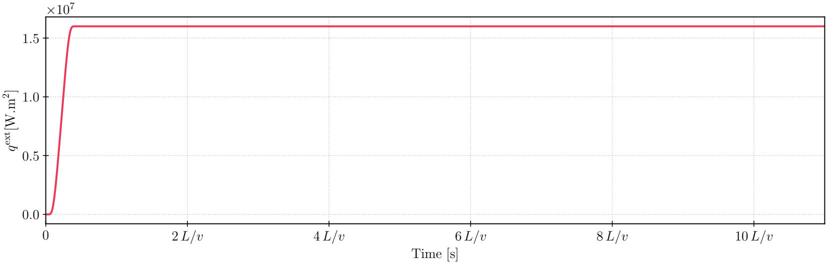
Partie B – Méthodologie de discrétisation

DGBA bracket structure discrète

$$\frac{1}{\Delta t}\left(\left\langle \frac{\partial \mathsf{F}_\mathrm{d}}{\partial \mathbf{z}_k} \right\rangle_n^{n+1}\right)^\mathsf{T} [\mathbf{z}_k]_n^{n+1} = \{\mathsf{F}_\mathrm{d}, \hat{\mathsf{E}}_\mathrm{d}^\mathrm{tot}\} + (\mathsf{F}_\mathrm{d}, \mathsf{S}_\mathrm{d}) + \text{boundary terms}$$

Problème particulier





Partie B – Méthodologie de discrétisation

DGBA bracket structure discrète

$$\frac{1}{\Delta t}\left(\left\langle \frac{\partial \mathsf{F}_{\mathsf{d}}}{\partial \mathbf{z}_k}\right\rangle_n^{n+1}\right)^{\mathsf{T}} [\mathbf{z}_k]_n^{n+1} = \left\{\mathsf{F}_{\mathsf{d}}, \hat{\mathsf{E}}_{\mathsf{d}}^{\mathsf{tot}}\right\} + \left(\mathsf{F}_{\mathsf{d}}, \mathsf{S}_{\mathsf{d}}\right) + \text{boundary terms}$$

Problème particulier

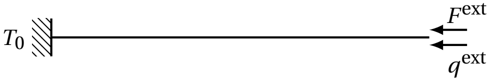


Schéma d'intégration espace-temps

$$\mathbf{u}_{n+1} = \cdots, \mathbf{p}_{n+3/2} = \cdots, \mathbf{T}_{n+1} = \cdots$$

Partie B – Schéma d'intégration

$$\boldsymbol{I}\mathbf{u}_{n+1} = \boldsymbol{I}\mathbf{u}_n + \Delta t \boldsymbol{M}^{-1}\mathbf{p}_{n+1/2}$$

$$\boldsymbol{I}\mathbf{p}_{n+3/2} = \boldsymbol{I}\mathbf{p}_{n+1/2} + \Delta t \left(\mathbf{F}^{\text{ext}}_{n+1} - (\boldsymbol{H}\mathbf{u}_{n+1} - \boldsymbol{L}(\mathbf{T}_{n+1} - \mathbf{T}_0))\right)$$

$$\rho c \boldsymbol{I}\frac{\mathbf{T}_{n+1} - \mathbf{T}_n}{\Delta t} = -T_0 \boldsymbol{L}^\mathsf{T} \rho^{-1}\mathbf{p}_{n+1/2} - \boldsymbol{K}\frac{\mathbf{T}_{n+1} + \mathbf{T}_n}{2} - \mathbf{q}^{\text{ext}}_{n+1/2}$$

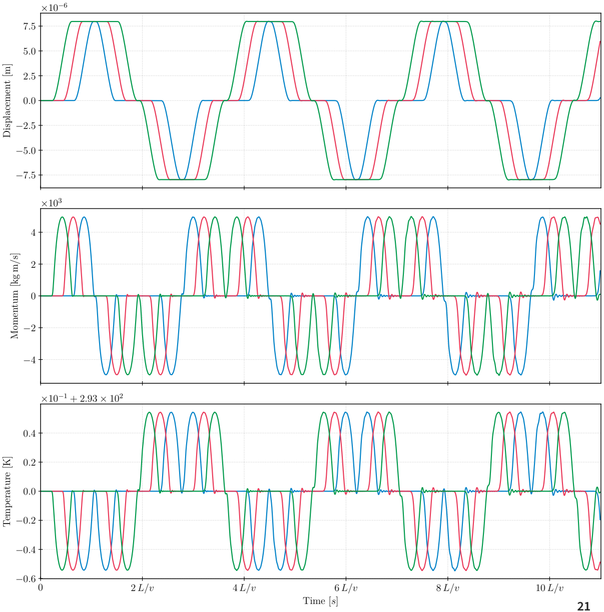
3. Autrement dit, quel intégrateur sur la **température** et les variables dissipatives est compatible avec la différence centrée ?

Partie B – Schéma d'intégration

$$\boldsymbol{I}\mathbf{u}_{n+1} = \boldsymbol{I}\mathbf{u}_n + \Delta t \boldsymbol{M}^{-1}\mathbf{p}_{n+1/2}$$

$$\boldsymbol{I}\mathbf{p}_{n+3/2} = \boldsymbol{I}\mathbf{p}_{n+1/2} + \Delta t \left(\mathbf{F}^{\text{ext}}_{n+1} - (\boldsymbol{H}\mathbf{u}_{n+1} - \boldsymbol{L}(\mathbf{T}_{n+1} - \mathbf{T}_0))\right)$$

$$\rho c \boldsymbol{I}\frac{\mathbf{T}_{n+1} - \mathbf{T}_n}{\Delta t} = -T_0 \boldsymbol{L}^{\mathsf{T}} \rho^{-1}\mathbf{p}_{n+1/2} - \boldsymbol{K}\frac{\mathbf{T}_{n+1} + \mathbf{T}_n}{2} - \mathbf{q}^{\text{ext}}_{n+1/2}$$



Partie B – Principes de la thermodynamique

Pour obtenir les principes de la thermodynamique, on utilise directement l'expression de la DGBA bracket structure discrétisée.

$$\frac{1}{\Delta t}\left(\left\langle \frac{\partial \mathsf{F}_{\mathsf{d}}}{\partial \mathbf{z}_k} \right\rangle_n^{n+1}\right)^{\mathsf{T}} [\mathbf{z}_k]_n^{n+1} = \left\{\mathsf{F}_{\mathsf{d}}, \hat{\mathsf{E}}_{\mathsf{d}}^{\mathsf{tot}}\right\} + \left(\mathsf{F}_{\mathsf{d}}, \mathsf{S}_{\mathsf{d}}\right) + \text{boundary terms}$$

$$\mathsf{F}_{\mathsf{d}} = \hat{\mathsf{E}}_{\mathsf{d}}^{\mathsf{tot}} \Rightarrow \frac{1}{\Delta t}\left(\left\langle \frac{\partial \hat{\mathsf{E}}_{\mathsf{d}}^{\mathsf{tot}}}{\partial \mathbf{z}_k} \right\rangle_n^{n+1}\right)^{\mathsf{T}} [\mathbf{z}_k]_n^{n+1} = \overbrace{\left\{\hat{\mathsf{E}}_{\mathsf{d}}^{\mathsf{tot}}, \hat{\mathsf{E}}_{\mathsf{d}}^{\mathsf{tot}}\right\}}^{=0} + \left(\hat{\mathsf{E}}_{\mathsf{d}}^{\mathsf{tot}}, \mathsf{S}_{\mathsf{d}}\right) + \text{boundary terms}$$

$$\mathsf{F}_{\mathsf{d}} = \hat{\mathsf{S}}_{\mathsf{d}}^{\mathsf{tot}} \Rightarrow \frac{1}{\Delta t}\left(\left\langle \frac{\partial \mathsf{S}_{\mathsf{d}}}{\partial \mathbf{z}_k} \right\rangle_n^{n+1}\right)^{\mathsf{T}} [\mathbf{z}_k]_n^{n+1} - \left\{\mathsf{S}_{\mathsf{d}}, \hat{\mathsf{E}}_{\mathsf{d}}^{\mathsf{tot}}\right\} - \text{boundary terms} \geq 0 \Leftrightarrow \left(\mathsf{S}_{\mathsf{d}}, \mathsf{S}_{\mathsf{d}}\right) \geq 0$$

Partie B – Premier principe de la thermodynamique

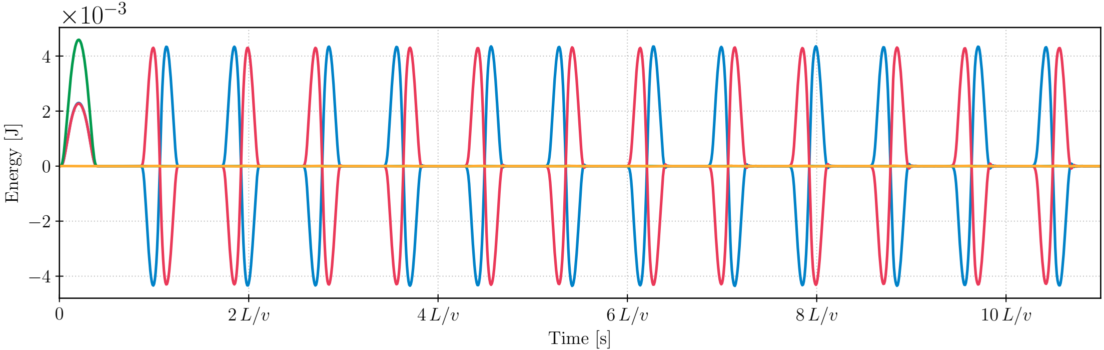
$$
\left[\frac{1}{2}\mathbf{p}_k^\top \boldsymbol{M}^{-1}\mathbf{p}_k + \frac{1}{2}\mathbf{u}_k^\top \boldsymbol{H}\mathbf{u}_k + \mathbf{T}_0^\top \boldsymbol{L}^\top \mathbf{u}_k + \frac{1}{2}\frac{\rho c}{T_0}\mathbf{T}_k^\top \boldsymbol{I}\mathbf{T}_k\right]_n^{n+1} - \frac{\Delta t^2}{4}\left[\frac{1}{2}\ddot{\mathbf{u}}_k^\top \boldsymbol{M}\ddot{\mathbf{u}}_k\right]_n^{n+1}
$$

$$
= \left(\left\langle \mathbf{F}_k^{\text{ext}}\right\rangle_n^{n+1}\right)^\top [\mathbf{u}_k]_n^{n+1} - \frac{\Delta t}{T_0}\left\langle \mathbf{T}_k{}^\top\right\rangle_n^{n+1}\boldsymbol{K}\langle \mathbf{T}_k\rangle_n^{n+1} - \frac{\Delta t}{T_0}(\mathbf{q}_{n+1/2}^{\text{ext}})^\top \langle \mathbf{T}_k\rangle_n^{n+1}
$$

Partie B – Premier principe de la thermodynamique

$$\left[\frac{1}{2}\mathbf{p}_k^{\mathsf{T}}\boldsymbol{M}^{-1}\mathbf{p}_k + \frac{1}{2}\mathbf{u}_k^{\mathsf{T}}\boldsymbol{H}\mathbf{u}_k + \mathbf{T}_0^{\mathsf{T}}\boldsymbol{L}^{\mathsf{T}}\mathbf{u}_k + \frac{1}{2}\frac{\rho c}{T_0}\mathbf{T}_k^{\mathsf{T}}\boldsymbol{I}\mathbf{T}_k\right]_n^{n+1} - \frac{\Delta t^2}{4}\left[\frac{1}{2}\ddot{\mathbf{u}}_k^{\mathsf{T}}\boldsymbol{M}\ddot{\mathbf{u}}_k\right]_n^{n+1}$$

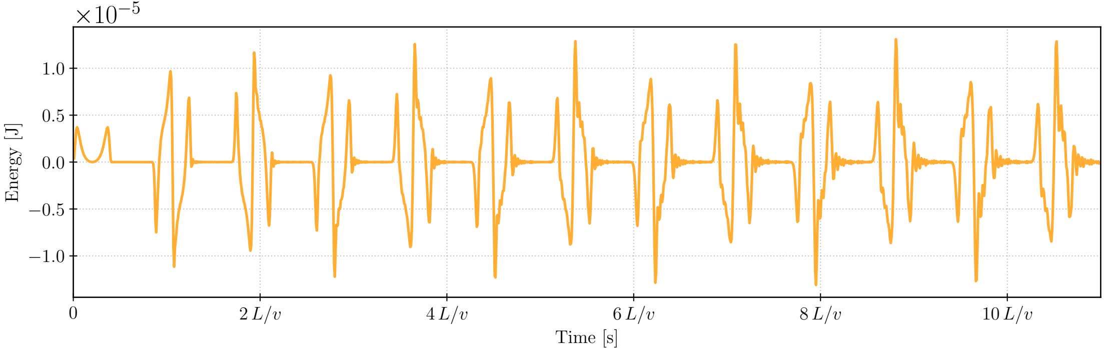
$$= \left(\left\langle \mathbf{F}_k^{\mathrm{ext}}\right\rangle_n^{n+1}\right)^{\mathsf{T}}[\mathbf{u}_k]_n^{n+1} - \frac{\Delta t}{T_0}\left\langle \mathbf{T}_k{}^{\mathsf{T}}\right\rangle_n^{n+1}\boldsymbol{K}\langle\mathbf{T}_k\rangle_n^{n+1} - \frac{\Delta t}{T_0}(\mathbf{q}_{n+1/2}^{\mathrm{ext}})^{\mathsf{T}}\langle\mathbf{T}_k\rangle_n^{n+1}$$



Partie B – Premier principe de la thermodynamique

$$\left[\frac{1}{2}\mathbf{p}_k^\top \boldsymbol{M}^{-1}\mathbf{p}_k + \frac{1}{2}\mathbf{u}_k^\top \boldsymbol{H}\mathbf{u}_k + \mathbf{T}_0^\top \boldsymbol{L}^\top \mathbf{u}_k + \frac{1}{2}\frac{\rho c}{T_0}\mathbf{T}_k^\top \boldsymbol{I}\mathbf{T}_k\right]_n^{n+1} - \frac{\Delta t^2}{4}\left[\frac{1}{2}\ddot{\mathbf{u}}_k^\top \boldsymbol{M}\ddot{\mathbf{u}}_k\right]_n^{n+1}$$

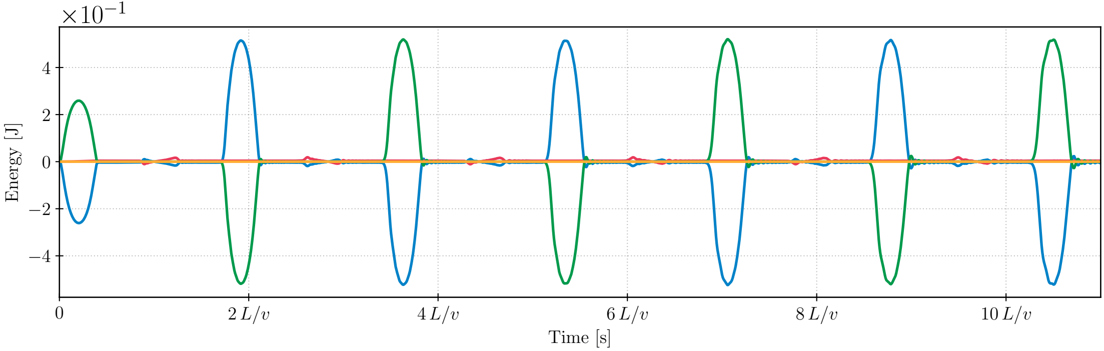
$$= \left(\left\langle \mathbf{F}_k^{\mathrm{ext}}\right\rangle_n^{n+1}\right)^\top [\mathbf{u}_k]_n^{n+1} - \frac{\Delta t}{T_0}\left\langle \mathbf{T}_k{}^\top\right\rangle_n^{n+1} \boldsymbol{K}\langle \mathbf{T}_k\rangle_n^{n+1} - \frac{\Delta t}{T_0}(\mathbf{q}_{n+1/2}^{\mathrm{ext}})^\top \langle \mathbf{T}_k\rangle_n^{n+1}$$



Partie B – Premier principe de la thermodynamique

$$\left[\frac{1}{2}\mathbf{p}_k^\top \boldsymbol{M}^{-1}\mathbf{p}_k + \frac{1}{2}\mathbf{u}_k^\top \boldsymbol{H}\mathbf{u}_k + \mathbf{T}_0^\top \boldsymbol{L}^\top \mathbf{u}_k + \frac{1}{2}\frac{\rho c}{T_0}\mathbf{T}_k^\top \boldsymbol{I}\mathbf{T}_k\right]_n^{n+1} - \frac{\Delta t^2}{4}\left[\frac{1}{2}\ddot{\mathbf{u}}_k^\top \boldsymbol{M}\ddot{\mathbf{u}}_k\right]_n^{n+1}$$

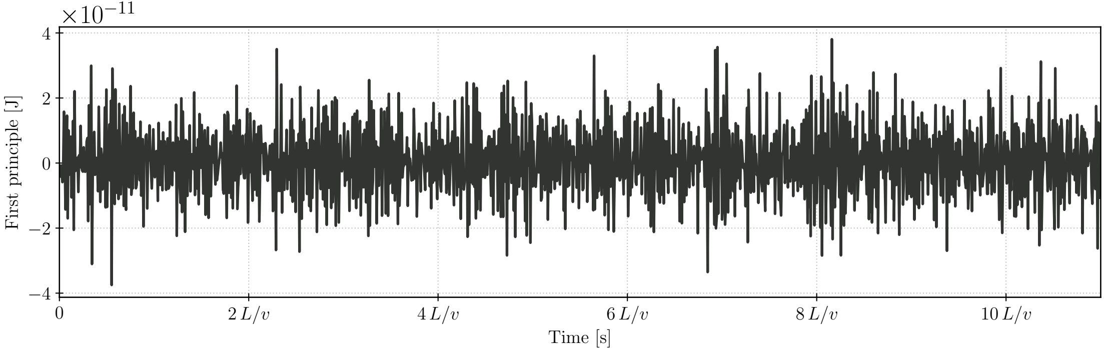
$$= \left(\left\langle \mathbf{F}_k^{\text{ext}} \right\rangle_n^{n+1}\right)^\top [\mathbf{u}_k]_n^{n+1} - \frac{\Delta t}{T_0}\left\langle \mathbf{T}_k{}^\top \right\rangle_n^{n+1} \boldsymbol{K}\langle \mathbf{T}_k\rangle_n^{n+1} - \frac{\Delta t}{T_0}(\mathbf{q}_{n+1/2}^{\text{ext}})^\top \langle \mathbf{T}_k\rangle_n^{n+1}$$



Partie B – Premier principe de la thermodynamique

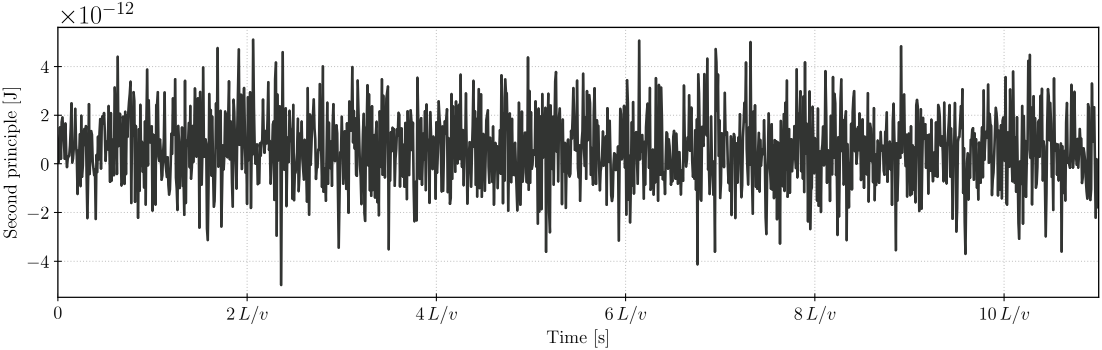
$$\left[\frac{1}{2}\mathbf{p}_k^\top \boldsymbol{M}^{-1}\mathbf{p}_k + \frac{1}{2}\mathbf{u}_k^\top \boldsymbol{H}\mathbf{u}_k + \mathbf{T}_0^\top \boldsymbol{L}^\top \mathbf{u}_k + \frac{1}{2}\frac{\rho c}{T_0}\mathbf{T}_k^\top \boldsymbol{I}\mathbf{T}_k\right]_n^{n+1} - \frac{\Delta t^2}{4}\left[\frac{1}{2}\ddot{\mathbf{u}}_k^\top \boldsymbol{M}\ddot{\mathbf{u}}_k\right]_n^{n+1}$$

$$= \left(\left\langle \mathbf{F}_k^{\mathrm{ext}}\right\rangle_n^{n+1}\right)^\top [\mathbf{u}_k]_n^{n+1} - \frac{\Delta t}{T_0}\left\langle {\mathbf{T}_k}^\top\right\rangle_n^{n+1} \boldsymbol{K}\langle \mathbf{T}_k\rangle_n^{n+1} - \frac{\Delta t}{T_0}(\mathbf{q}_{n+1/2}^{\mathrm{ext}})^\top \langle \mathbf{T}_k\rangle_n^{n+1}$$

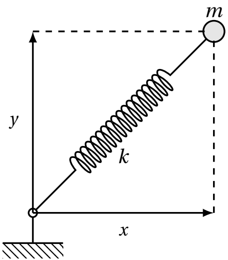


Partie B – Second principe de la thermodynamique

$$\mathbf{T}_0^\top \boldsymbol{L}^\top [\mathbf{u}_k]_n^{n+1} + \rho c \mathbf{T}_0^\top \boldsymbol{I} [\mathbf{T}_k]_n^{n+1} + \Delta t \mathbf{T}_0^\top \mathbf{q}_{n+1/2}^{\text{ext}} \geq 0$$



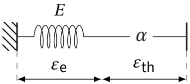
Partie A
Introduction aux intégrateurs géométriques



Partie B
Thermo-élastodynamique couplée

Petites transformations, 1D

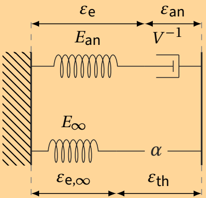
$$\boldsymbol{z} = \begin{pmatrix} u & \varepsilon & p & T \end{pmatrix}$$



Partie C
Thermo-visco-élastodynamique couplée

Petites transformations, 1D

$$\boldsymbol{z} = \begin{pmatrix} u & \varepsilon & p & T & \varepsilon_{\mathrm{an}} \end{pmatrix}$$



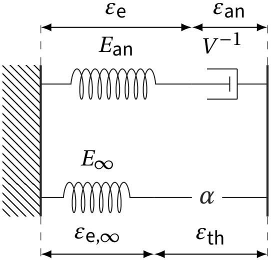
Partie C – Choix du vecteur d'état

Le vecteur d'état pour ce problème est

$$\boldsymbol{z} = \begin{pmatrix} u & \varepsilon & p & T & \varepsilon_{\text{an}} \end{pmatrix}$$

où

- u est le déplacement;
- ε est la partie symétrique du gradient du déplacement, ici $\varepsilon = \partial u / \partial X$;
- p est la quantité de mouvement;
- T est la température absolue;
- ε_{an} est la partie anélastique (visqueuse) du gradient du déplacement.



Partie C – lois d'état

La DGBA bracket structure est **obtenue via les lois d'état**

$$\frac{\mathrm{d}\mathscr{F}}{\mathrm{d}t} = \int_0^L \left(\frac{\partial F}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial F}{\partial p}\frac{\partial p}{\partial t} + \frac{\partial F}{\partial \varepsilon}\frac{\partial \varepsilon}{\partial t} + \frac{\partial F}{\partial T}\frac{\partial T}{\partial t} + \left(\frac{\partial F}{\partial \varepsilon_{\text{an}}}\right)\frac{\partial \varepsilon_{\text{an}}}{\partial t} \right) \mathrm{d}X$$

$$\frac{\partial u}{\partial t} = \frac{\partial \hat{E}_{\text{tot}}}{\partial p}$$

$$\frac{\partial \varepsilon}{\partial t} = \frac{\partial}{\partial X}\left(\frac{\partial \hat{E}_{\text{tot}}}{\partial p}\right)$$

$$\frac{\partial p}{\partial t} = -\frac{\partial \hat{E}_{\text{tot}}}{\partial u} + \frac{\partial}{\partial X}\left(\frac{\partial \hat{E}_{\text{tot}}}{\partial \varepsilon} - \left(\frac{\partial \hat{E}_{\text{tot}}}{\partial T}\right)\left(\frac{\partial S}{\partial T}\right)^{-1}\frac{\partial S}{\partial \varepsilon}\right)$$

$$\rho c \frac{\partial T}{\partial t} + \rho c \left(\frac{\partial S}{\partial T}\right)^{-1}\frac{\partial S}{\partial \varepsilon}\frac{\partial}{\partial X}\left(\frac{\partial \hat{E}_{\text{tot}}}{\partial p}\right)$$

$$= \frac{\partial}{\partial X}\left(K\frac{\partial T}{\partial X}\right) - \frac{\partial(\rho \hat{E}_{\text{int}})}{\partial \varepsilon_{\text{an}}}\frac{\partial \varepsilon_{\text{an}}}{\partial t}$$

$$\frac{\partial \varepsilon_{\text{an}}}{\partial t} = -\frac{T}{\rho c}V^{-1}\frac{\partial \hat{E}_{\text{int}}}{\partial \varepsilon_{\text{an}}}\frac{\partial S}{\partial T} + TV^{-1}\frac{\partial S}{\partial \varepsilon_{\text{an}}}$$

$$\frac{\partial u}{\partial t} = \frac{1}{\rho}p$$

$$\frac{\partial \varepsilon}{\partial t} = \frac{1}{\rho}\frac{\partial p}{\partial X}$$

$$\frac{\partial p}{\partial t} = \frac{\partial}{\partial X}\left((E_\infty + E_{\text{an}})\varepsilon - E_{\text{an}}\varepsilon_{\text{an}} - k(T - T_0)\right)$$

$$\rho c \frac{\partial T}{\partial t} + T_0 k \frac{\partial}{\partial X}\left(\frac{p}{\rho}\right)$$

$$= \frac{\partial}{\partial X}\left(K\frac{\partial T}{\partial X}\right) + \rho E_{\text{an}}(\varepsilon - \varepsilon_{\text{an}})\left[E_{\text{an}} V^{-1}(\varepsilon - \varepsilon_{\text{an}})\right]$$

$$\frac{\partial \varepsilon_{\text{an}}}{\partial t} = V^{-1}E_{\text{an}}(\varepsilon - \varepsilon_{\text{an}})$$

Partie C – Expression de la DGBA bracket structure

L’expression de la DGBA bracket structure est

$$\frac{\mathrm{d}\mathscr{F}}{\mathrm{d}t} = \left\{\mathscr{F}, \hat{\mathscr{E}}_{\text{tot}}\right\} + (\mathscr{F}, \mathscr{S}) + \\ - \left[\frac{\partial F}{\partial T} K \left(\frac{T}{\rho c}\right) \frac{\partial}{\partial X}\left(\frac{\partial S}{\partial T}\right)\right]_0^L + \left[\frac{\partial F}{\partial p}\left(\frac{\partial \hat{E}_{\text{tot}}}{\partial \varepsilon} - \left(\frac{\partial S}{\partial T}\right)^{-1} \frac{\partial S}{\partial \varepsilon}\frac{\partial \hat{E}_{\text{tot}}}{\partial T}\right)\right]_0^L$$

avec

$$\left\{\mathscr{F}, \hat{\mathscr{E}}_{\text{tot}}\right\} = \int_0^L \left(\frac{\partial F}{\partial u}\frac{\partial \hat{E}_{\text{tot}}}{\partial p} - \frac{\partial F}{\partial p}\frac{\partial \hat{E}_{\text{tot}}}{\partial u} + \frac{\partial F}{\partial \varepsilon}\frac{\partial}{\partial X}\left(\frac{\partial \hat{E}_{\text{tot}}}{\partial p}\right) - \frac{\partial}{\partial X}\left(\frac{\partial F}{\partial p}\right)\frac{\partial \hat{E}_{\text{tot}}}{\partial \varepsilon} + \right. \\ \left. + \left(\frac{\partial S}{\partial T}\right)^{-1} \frac{\partial S}{\partial \varepsilon}\left(\frac{\partial \hat{E}_{\text{tot}}}{\partial T}\frac{\partial}{\partial X}\left(\frac{\partial F}{\partial p}\right) - \frac{\partial F}{\partial T}\frac{\partial}{\partial X}\left(\frac{\partial \hat{E}_{\text{tot}}}{\partial p}\right)\right)\right)\mathrm{d}X$$

$$(\mathscr{F}, \mathscr{S}) = \int_0^L \left(\frac{\partial}{\partial X}\left(\frac{\partial F}{\partial T}\right) K \left(\frac{T}{\rho c}\right)^2 \frac{\partial}{\partial X}\left(\frac{\partial S}{\partial T}\right) + \frac{\partial F}{\partial T}\frac{T}{(\rho c)^2}\frac{\partial \hat{E}_{\text{int}}}{\partial \varepsilon_{\text{an}}} V^{-1} \frac{\partial \hat{E}_{\text{int}}}{\partial \varepsilon_{\text{an}}}\frac{\partial S}{\partial T} + \right. \\ \left. - \frac{\partial F}{\partial T}\frac{T}{\rho c}\frac{\partial \hat{E}_{\text{int}}}{\partial \varepsilon_{\text{an}}} V^{-1} \frac{\partial S}{\partial \varepsilon_{\text{an}}} - \frac{T}{\rho c}\frac{\partial F}{\partial \varepsilon_{\text{an}}} V^{-1}\frac{\partial \hat{E}_{\text{int}}}{\partial \varepsilon_{\text{an}}}\frac{\partial S}{\partial T} + T\frac{\partial F}{\partial \varepsilon_{\text{an}}} V^{-1}\frac{\partial S}{\partial \varepsilon_{\text{an}}}\right)\mathrm{d}X$$

Partie C – Expression de la DGBA bracket structure

$$(\mathscr{F}, \mathscr{S}) = \int_0^L \frac{\partial F}{\partial \boldsymbol{z}} \cdot \boldsymbol{M}(\boldsymbol{z}) \cdot \frac{\partial S}{\partial \boldsymbol{z}} \, \mathrm{d}X$$

$$\text{avec } \boldsymbol{M}(\boldsymbol{z}) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\partial \square_{\text{left}}}{\partial X} K \left(\frac{T}{\rho c}\right)^2 \frac{\partial \square_{\text{right}}}{\partial X} + \frac{1}{(\rho c)^2} \left(\frac{\partial \hat{E}_{\text{int}}}{\partial \varepsilon_{\text{an}}}\right)^{\mathsf{T}} V^{-1} \frac{\partial \hat{E}_{\text{int}}}{\partial \varepsilon_{\text{an}}} T & -\frac{T}{\rho c} \left(\frac{\partial \hat{E}_{\text{int}}}{\partial \varepsilon_{\text{an}}}\right)^{\mathsf{T}} V^{-1} \\ 0 & 0 & 0 & -\frac{T}{\rho c} V^{-1} \frac{\partial \hat{E}_{\text{int}}}{\partial \varepsilon_{\text{an}}} & T V^{-1} \end{pmatrix}$$

$$\frac{\partial F}{\partial \boldsymbol{z}} = \begin{pmatrix} \partial F / \partial u & \partial F / \partial \varepsilon & \partial F / \partial p & \partial F / \partial T & \partial F / \partial \varepsilon_{\text{an}} \end{pmatrix}^{\mathsf{T}}$$

Partie C – Méthodologie de discrétisation

1. **Choix d'un schéma d'intégration temporelle**
   On intègre la structure sur tout l'intervalle temporel $[t_0, t_N]$

$$\sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \left(\frac{\mathrm{d}\mathscr{F}}{\mathrm{d}t} - \{\mathscr{F}, \hat{\mathscr{E}}_{\text{tot}}\} - (\mathscr{F}, \mathscr{S}) - \text{boundary terms} \right) \mathrm{d}t = 0$$

   On choisit une approximation de l'intégrale, **choix basé sur le calcul variationnel discret** (Marsden et al., 1994)

$$\int_{t_n}^{t_{n+1}} \int_0^L G(u, p, \varepsilon, T, \varepsilon_{\text{an}})\, \mathrm{d}X$$
$$\approx \int_0^L \frac{\Delta t}{2} \left(G(u_n, \varepsilon_n, p_{n+1/2}, T_{n+1/2}, (\varepsilon_{\text{an}})_{n+1/2}) + G(u_{n+1}, \varepsilon_{n+1}, p_{n+1/2}, T_{n+1/2}, (\varepsilon_{\text{an}})_{n+1/2}) \right) \mathrm{d}X$$

Partie C – Schéma d'intégration

$$\boldsymbol{I}^{\varepsilon\varepsilon}\mathbf{e}_{n+1} = \boldsymbol{I}^{\varepsilon\varepsilon}\mathbf{e}_{n} + \Delta t \boldsymbol{M}^{-1,\varepsilon p'}\mathbf{p}_{n+1/2}$$

$$\boldsymbol{I}^{pp}\mathbf{p}_{n+3/2} = \boldsymbol{I}^{pp}\mathbf{p}_{n+1/2} + \Delta t \left(\mathbf{F}^{\mathrm{ext}}_{n+1} - \left((E_\infty + E_{\mathrm{an}})\boldsymbol{I}\mathbf{e}_{n+1} - E_{\mathrm{an}}\boldsymbol{I}^{p'\varepsilon_{\mathrm{an}}}(\mathbf{e}_{\mathrm{an}})_{n+1} - \boldsymbol{L}^{p'T}(\mathbf{T}_{n+1} - \mathbf{T}_0) \right) \right)$$

$$\begin{aligned}
&\boldsymbol{I}^{TT}\frac{\mathbf{T}_{n+1} - \mathbf{T}_n}{\Delta t} + \frac{1}{\rho c}\mathbf{q}^{\mathrm{ext}}_{n+1/2} - \left(-\frac{T_0}{\rho c}\boldsymbol{L}^{Tp'}\rho^{-1}\mathbf{p}_{n+1/2} + \frac{1}{\rho c}\boldsymbol{K}^{T'T'}\langle \mathbf{T}_k \rangle_n^{n+1} + \right.\\
&\quad + \frac{E_{\mathrm{an}}^2 V^{-1}}{c}\tilde{\tilde{\mathrm{I}}}^{T\varepsilon\varepsilon}\langle \mathbf{e}_k \rangle_n^{n+1}\langle \mathbf{e}_k \rangle_n^{n+1} - \frac{E_{\mathrm{an}}^2 V^{-1}}{c}\tilde{\tilde{\mathrm{I}}}^{T\varepsilon\varepsilon_{\mathrm{an}}}\langle \mathbf{e}_k \rangle_n^{n+1}\left\langle (\mathbf{e}_{\mathrm{an}})_k \right\rangle_n^{n+1} +\\
&\quad \left. - \frac{E_{\mathrm{an}}^2 V^{-1}}{c}\tilde{\tilde{\mathrm{I}}}^{T\varepsilon_{\mathrm{an}}\varepsilon}\left\langle (\mathbf{e}_{\mathrm{an}})_k \right\rangle_n^{n+1}\langle \mathbf{e}_k \rangle_n^{n+1} + \frac{E_{\mathrm{an}}^2 V^{-1}}{c}\tilde{\tilde{\mathrm{I}}}^{T\varepsilon_{\mathrm{an}}\varepsilon_{\mathrm{an}}}\left\langle (\mathbf{e}_{\mathrm{an}})_k \right\rangle_n^{n+1}\left\langle (\mathbf{e}_{\mathrm{an}})_k \right\rangle_n^{n+1} \right) = \mathbf{0}
\end{aligned}$$

$$\boldsymbol{I}^{\varepsilon_{\mathrm{an}}\varepsilon_{\mathrm{an}}}(\mathbf{e}_{\mathrm{an}})_{n+1} = \boldsymbol{I}^{\varepsilon_{\mathrm{an}}\varepsilon_{\mathrm{an}}}(\mathbf{e}_{\mathrm{an}})_{n} + \Delta t E_{\mathrm{an}} V^{-1}\left(\boldsymbol{I}^{\varepsilon_{\mathrm{an}}\varepsilon}\langle \mathbf{e}_k \rangle_n^{n+1} - \boldsymbol{I}^{\varepsilon_{\mathrm{an}}\varepsilon_{\mathrm{an}}}\left\langle (\mathbf{e}_{\mathrm{an}})_k \right\rangle_n^{n+1} \right)$$

3. Autrement dit, quel intégrateur sur la **température** et les **variables dissipatives** est compatible avec la différence centrée ?

Partie C – Premier principe de la thermodynamique discret

$$
\left[\frac{1}{2}\mathbf{p}_k^{\mathsf{T}}\boldsymbol{M}^{-1,pp}\mathbf{p}_k + \frac{1}{2}\mathbf{e}_k^{\mathsf{T}}(E_\infty + E_{\text{an}})\boldsymbol{I}^{\varepsilon\varepsilon}\mathbf{e}_k + \mathbf{T}_0^{\mathsf{T}}\boldsymbol{L}^{T\varepsilon}\mathbf{e}_k + \frac{\rho c}{T_0}\frac{1}{2}\mathbf{T}_k^{\mathsf{T}}\boldsymbol{I}^{TT}\mathbf{T}_k\right]_n^{n+1} - \frac{\Delta t^2}{4}\left[\frac{1}{2}\ddot{\boldsymbol{u}}_k^{\mathsf{T}}\boldsymbol{M}^{pp}\ddot{\boldsymbol{u}}_k\right]_n^{n+1}
$$

$$
= \left(\left\langle \mathbf{F}_k^{\text{ext}}\right\rangle_n^{n+1}\right)^{\mathsf{T}} [\mathbf{u}_k]_n^{n+1} - \frac{\Delta t}{T_0}(\mathbf{q}_{n+1/2}^{\text{ext}})^{\mathsf{T}}\langle \mathbf{T}_k\rangle_n^{n+1} - \frac{\Delta t}{T_0}\left\langle \mathbf{T}_k^{\mathsf{T}}\right\rangle_n^{n+1}\boldsymbol{K}^{T'T'}\langle \mathbf{T}_k\rangle_n^{n+1} +
$$

$$
+\left\langle (\mathbf{e}_{\text{an}})_k^{\mathsf{T}}\right\rangle_n^{n+1} E_{\text{an}}\boldsymbol{I}^{\varepsilon_{\text{an}}\varepsilon}[\mathbf{e}_k]_n^{n+1} + \left\langle \mathbf{e}_k^{\mathsf{T}}\right\rangle_n^{n+1} E_{\text{an}}\boldsymbol{I}^{\varepsilon\varepsilon_{\text{an}}}\left[(\mathbf{e}_{\text{an}})_k\right]_n^{n+1} - \left[\frac{1}{2}(\mathbf{e}_{\text{an}})_k^{\mathsf{T}}E_{\text{an}}\boldsymbol{I}^{\varepsilon_{\text{an}}\varepsilon_{\text{an}}}(\mathbf{e}_{\text{an}})_k\right]_n^{n+1} +
$$

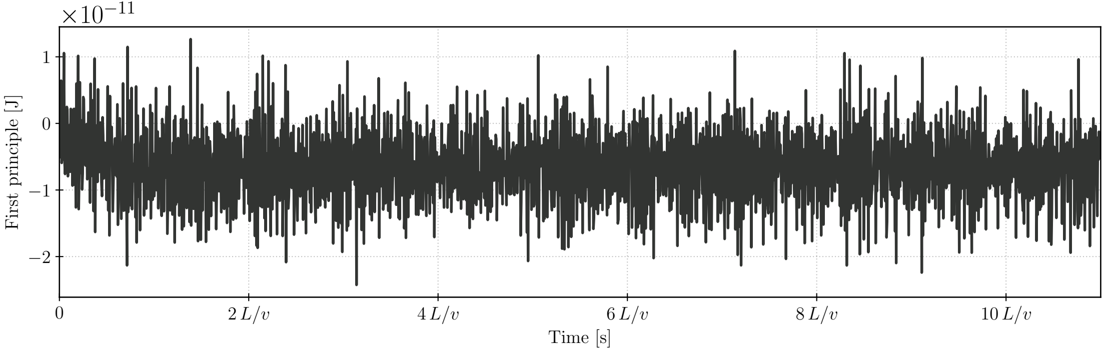
$$
+\Delta t\frac{\langle \mathbf{T}_k\rangle_n^{n+1}}{T_0}V^{-1}E_{\text{an}}^2\Big(\tilde{\tilde{\mathsf{I}}}^{T\varepsilon\varepsilon}\langle \mathbf{e}_k\rangle_n^{n+1}\langle \mathbf{e}_k\rangle_n^{n+1} + \tilde{\tilde{\mathsf{I}}}^{T\varepsilon_{\text{an}}\varepsilon_{\text{an}}}\left\langle (\mathbf{e}_{\text{an}})_k\right\rangle_n^{n+1}\left\langle (\mathbf{e}_{\text{an}})_k\right\rangle_n^{n+1} +
$$

$$
-\tilde{\tilde{\mathsf{I}}}^{T\varepsilon\varepsilon_{\text{an}}}\langle \mathbf{e}_k\rangle_n^{n+1}\left\langle (\mathbf{e}_{\text{an}})_k\right\rangle_n^{n+1} - \tilde{\tilde{\mathsf{I}}}^{T\varepsilon_{\text{an}}\varepsilon}\left\langle (\mathbf{e}_{\text{an}})_k\right\rangle_n^{n+1}\langle \mathbf{e}_k\rangle_n^{n+1}\Big) +
$$

$$
-\Delta t V^{-1}E_{\text{an}}^2\Big(\boldsymbol{I}^{\varepsilon\varepsilon}\langle \mathbf{e}_k\rangle_n^{n+1}\langle \mathbf{e}_k\rangle_n^{n+1} + \boldsymbol{I}^{\varepsilon_{\text{an}}\varepsilon_{\text{an}}}\left\langle (\mathbf{e}_{\text{an}})_k\right\rangle_n^{n+1}\left\langle (\mathbf{e}_{\text{an}})_k\right\rangle_n^{n+1} +
$$

$$
-\boldsymbol{I}^{\varepsilon\varepsilon_{\text{an}}}\langle \mathbf{e}_k\rangle_n^{n+1}\left\langle (\mathbf{e}_{\text{an}})_k\right\rangle_n^{n+1} - \boldsymbol{I}^{\varepsilon_{\text{an}}\varepsilon}\left\langle (\mathbf{e}_{\text{an}})_k\right\rangle_n^{n+1}\langle \mathbf{e}_k\rangle_n^{n+1}\Big)
$$

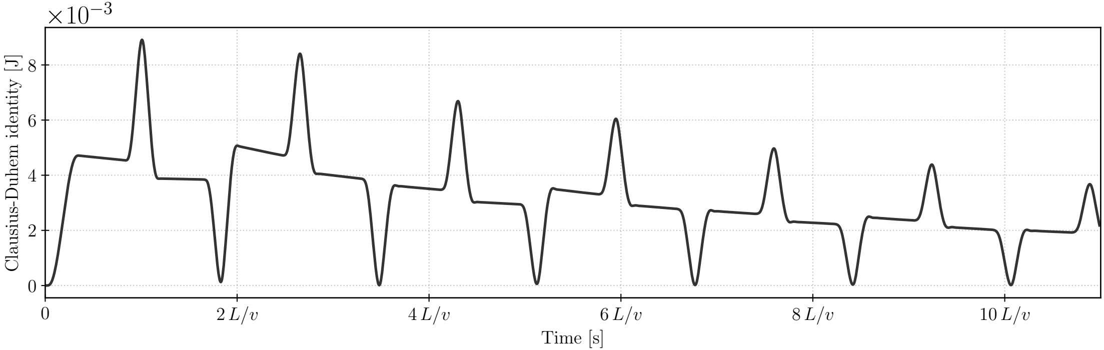
Partie C – Premier principe de la thermodynamique



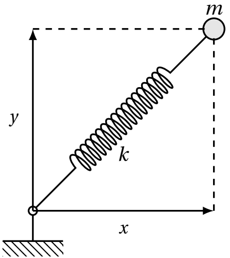
Partie C – Second principe de la thermodynamique – Inégalité de Clausius-Duhem

$$V^{-1} E_{\mathsf{an}}^2 \Big(\boldsymbol{I}^{\varepsilon\varepsilon} \langle \mathbf{e}_k \rangle_n^{n+1} \langle \mathbf{e}_k \rangle_n^{n+1} + \boldsymbol{I}^{\varepsilon_{\mathsf{an}}\varepsilon_{\mathsf{an}}} \big\langle (\mathbf{e}_{\mathsf{an}})_k \big\rangle_n^{n+1} \big\langle (\mathbf{e}_{\mathsf{an}})_k \big\rangle_n^{n+1} +$$

$$- \boldsymbol{I}^{\varepsilon\varepsilon_{\mathsf{an}}} \langle \mathbf{e}_k \rangle_n^{n+1} \big\langle (\mathbf{e}_{\mathsf{an}})_k \big\rangle_n^{n+1} - \boldsymbol{I}^{\varepsilon_{\mathsf{an}}\varepsilon} \big\langle (\mathbf{e}_{\mathsf{an}})_k \big\rangle_n^{n+1} \langle \mathbf{e}_k \rangle_n^{n+1} \Big) \geq 0$$



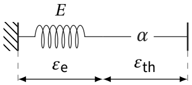
Partie A
Introduction aux intégrateurs géométriques



Partie B
Thermo-élastodynamique couplée

Petites transformations, 1D

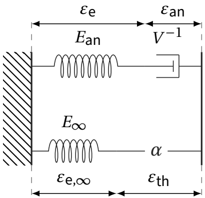
$$\boldsymbol{z} = \begin{pmatrix} u & \varepsilon & p & T \end{pmatrix}$$



Partie C
Thermo-visco-élastodynamique couplée

Petites transformations, 1D

$$\boldsymbol{z} = \begin{pmatrix} u & \varepsilon & p & T & \varepsilon_{\mathrm{an}} \end{pmatrix}$$



Conclusion

1. Sur quelle base construire un intégrateur géométrique pour la thermo-visco-élastodynamique ?
2. Dans ce cadre, comment **étendre la différence centrée** à la thermo-visco-élastodynamique ?
3. Autrement dit, quel intégrateur sur la température et sur les variables dissipatives internes est compatible avec la différence centrée ?
4. Quels sont les principes de conservation discrets vérifiés par un tel intégrateur ?

Conclusion

1. Sur quelle base construire un intégrateur géométrique pour la thermo-visco-élastodynamique ?

est un crochet **antisymétrique**, représentant l'**évolution réversible** du système

$$\frac{\mathrm{d}\mathscr{F}}{\mathrm{d}t} = \left\{\mathscr{F}, \hat{\mathscr{E}}_{\text{tot}}\right\} + \left(\mathscr{F}, \mathscr{S}\right) + \text{boundary terms}$$

est un crochet **positif, symétrique**, représentant l'**évolution dissipative** du système.

Conclusion

2. Dans ce cadre, comment **étendre la différence centrée** à la thermo-élastodynamique couplée ?

On intègre la structure sur tout l'intervalle temporel $[t_0, t_N]$

$$\sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \left(\frac{\mathrm{d}\mathscr{F}}{\mathrm{d}t} - \{\mathscr{F}, \hat{\mathscr{E}}_{\text{tot}}\} - (\mathscr{F}, \mathscr{S}) - \text{boundary terms} \right) \mathrm{d}t = 0$$

On choisit une approximation de l'intégrale, **choix basé sur le calcul variationnel discret** (Marsden et al., 1994)

$$\int_{t_n}^{t_{n+1}} \int_0^L G(u, p, \varepsilon, T, \varepsilon_{\text{an}}) \,\mathrm{d}X$$
$$\approx \int_0^L \frac{\Delta t}{2} \left(G(u_n, \varepsilon_n, p_{n+1/2}, T_{n+1/2}, (\varepsilon_{\text{an}})_{n+1/2}) + G(u_{n+1}, \varepsilon_{n+1}, p_{n+1/2}, T_{n+1/2}, (\varepsilon_{\text{an}})_{n+1/2}) \right) \mathrm{d}X$$

Conclusion

3. Autrement dit, quel intégrateur sur la température et sur les variables dissipatives internes est compatible avec la différence centrée ?

$$
\begin{aligned}
&\boldsymbol{I}^{\varepsilon\varepsilon}\mathbf{e}_{n+1} = \boldsymbol{I}^{\varepsilon\varepsilon}\mathbf{e}_n + \Delta t \boldsymbol{M}^{-1,\varepsilon p'}\mathbf{p}_{n+1/2} \\
&\boldsymbol{I}^{pp}\mathbf{p}_{n+3/2} = \boldsymbol{I}^{pp}\mathbf{p}_{n+1/2} + \Delta t\left(\mathbf{F}^{\text{ext}}_{n+1} - \left((E_\infty + E_{\text{an}})\boldsymbol{I}\mathbf{e}_{n+1} - E_{\text{an}}\boldsymbol{I}^{p'\varepsilon_{\text{an}}}(\mathbf{e}_{\text{an}})_{n+1} - \boldsymbol{L}^{p'T}(\mathbf{T}_{n+1} - \mathbf{T}_0)\right)\right) \\
&\boldsymbol{I}^{TT}\frac{\mathbf{T}_{n+1} - \mathbf{T}_n}{\Delta t} + \frac{1}{\rho c}\mathbf{q}^{\text{ext}}_{n+1/2} - \left(-\frac{T_0}{\rho c}\boldsymbol{L}^{Tp'}\rho^{-1}\mathbf{p}_{n+1/2} + \frac{1}{\rho c}\boldsymbol{K}^{T'T'}\langle\mathbf{T}_k\rangle_n^{n+1} + \right. \\
&\qquad + \frac{E_{\text{an}}^2 V^{-1}}{c}\tilde{\tilde{\tilde{\mathsf{I}}}}^{T\varepsilon\varepsilon}\langle\mathbf{e}_k\rangle_n^{n+1}\langle\mathbf{e}_k\rangle_n^{n+1} - \frac{E_{\text{an}}^2 V^{-1}}{c}\tilde{\tilde{\tilde{\mathsf{I}}}}^{T\varepsilon\varepsilon_{\text{an}}}\langle\mathbf{e}_k\rangle_n^{n+1}\left\langle(\mathbf{e}_{\text{an}})_k\right\rangle_n^{n+1} + \\
&\qquad \left. - \frac{E_{\text{an}}^2 V^{-1}}{c}\tilde{\tilde{\tilde{\mathsf{I}}}}^{T\varepsilon_{\text{an}}\varepsilon}\left\langle(\mathbf{e}_{\text{an}})_k\right\rangle_n^{n+1}\langle\mathbf{e}_k\rangle_n^{n+1} + \frac{E_{\text{an}}^2 V^{-1}}{c}\tilde{\tilde{\tilde{\mathsf{I}}}}^{T\varepsilon_{\text{an}}\varepsilon_{\text{an}}}\left\langle(\mathbf{e}_{\text{an}})_k\right\rangle_n^{n+1}\left\langle(\mathbf{e}_{\text{an}})_k\right\rangle_n^{n+1}\right) = \mathbf{0} \\
&\boldsymbol{I}^{\varepsilon_{\text{an}}\varepsilon_{\text{an}}}(\mathbf{e}_{\text{an}})_{n+1} = \boldsymbol{I}^{\varepsilon_{\text{an}}\varepsilon_{\text{an}}}(\mathbf{e}_{\text{an}})_n + \Delta t E_{\text{an}} V^{-1}\left(\boldsymbol{I}^{\varepsilon_{\text{an}}\varepsilon}\langle\mathbf{e}_k\rangle_n^{n+1} - \boldsymbol{I}^{\varepsilon_{\text{an}}\varepsilon_{\text{an}}}\left\langle(\mathbf{e}_{\text{an}})_k\right\rangle_n^{n+1}\right)
\end{aligned}
$$

Conclusion

4. Quels sont les principes de conservation discrets vérifiés par un tel intégrateur ?

Premier principe de la thermodynamique

$$
\begin{aligned}
&\left[\frac{1}{2}\mathbf{p}_k^{\mathsf{T}}\boldsymbol{M}^{-1,pp}\mathbf{p}_k+\frac{1}{2}\mathbf{e}_k^{\mathsf{T}}(E_\infty+E_{\text{an}})\boldsymbol{I}^{\varepsilon\varepsilon}\mathbf{e}_k+\mathbf{T}_0^{\mathsf{T}}\boldsymbol{L}^{T\varepsilon}\mathbf{e}_k+\frac{\rho c}{T_0}\frac{1}{2}\mathbf{T}_k^{\mathsf{T}}\boldsymbol{I}^{TT}\mathbf{T}_k\right]_n^{n+1}-\frac{\Delta t^2}{4}\left[\frac{1}{2}\ddot{\boldsymbol{u}}_k^{\mathsf{T}}\boldsymbol{M}^{pp}\ddot{\boldsymbol{u}}_k\right]_n^{n+1}\\
&=\left(\left\langle\mathbf{F}_k^{\text{ext}}\right\rangle_n^{n+1}\right)^{\mathsf{T}}\left[\mathbf{u}_k\right]_n^{n+1}-\frac{\Delta t}{T_0}(\mathbf{q}_{n+1/2}^{\text{ext}})^{\mathsf{T}}\left\langle\mathbf{T}_k\right\rangle_n^{n+1}-\frac{\Delta t}{T_0}\left\langle\mathbf{T}_k^{\mathsf{T}}\right\rangle_n^{n+1}\boldsymbol{K}^{T'T'}\left\langle\mathbf{T}_k\right\rangle_n^{n+1}+\\
&\quad+\left\langle(\mathbf{e}_{\text{an}})_k^{\mathsf{T}}\right\rangle_n^{n+1}E_{\text{an}}\boldsymbol{I}^{\varepsilon_{\text{an}}\varepsilon}\left[\mathbf{e}_k\right]_n^{n+1}+\left\langle\mathbf{e}_k^{\mathsf{T}}\right\rangle_n^{n+1}E_{\text{an}}\boldsymbol{I}^{\varepsilon\varepsilon_{\text{an}}}\left[(\mathbf{e}_{\text{an}})_k\right]_n^{n+1}-\left[\frac{1}{2}(\mathbf{e}_{\text{an}})_k^{\mathsf{T}}E_{\text{an}}\boldsymbol{I}^{\varepsilon_{\text{an}}\varepsilon_{\text{an}}}(\mathbf{e}_{\text{an}})_k\right]_n^{n+1}+\\
&\quad+\Delta t\frac{\left\langle\mathbf{T}_k\right\rangle_n^{n+1}}{T_0}V^{-1}E_{\text{an}}^2\Bigg(\tilde{\tilde{\tilde{\mathsf{I}}}}^{T\varepsilon\varepsilon}\left\langle\mathbf{e}_k\right\rangle_n^{n+1}\left\langle\mathbf{e}_k\right\rangle_n^{n+1}+\tilde{\tilde{\tilde{\mathsf{I}}}}^{T\varepsilon_{\text{an}}\varepsilon_{\text{an}}}\left\langle(\mathbf{e}_{\text{an}})_k\right\rangle_n^{n+1}\left\langle(\mathbf{e}_{\text{an}})_k\right\rangle_n^{n+1}+\\
&\qquad\qquad-\tilde{\tilde{\tilde{\mathsf{I}}}}^{T\varepsilon\varepsilon_{\text{an}}}\left\langle\mathbf{e}_k\right\rangle_n^{n+1}\left\langle(\mathbf{e}_{\text{an}})_k\right\rangle_n^{n+1}-\tilde{\tilde{\tilde{\mathsf{I}}}}^{T\varepsilon_{\text{an}}\varepsilon}\left\langle(\mathbf{e}_{\text{an}})_k\right\rangle_n^{n+1}\left\langle\mathbf{e}_k\right\rangle_n^{n+1}\Bigg)+\\
&\quad-\Delta t V^{-1}E_{\text{an}}^2\Bigg(\boldsymbol{I}^{\varepsilon\varepsilon}\left\langle\mathbf{e}_k\right\rangle_n^{n+1}\left\langle\mathbf{e}_k\right\rangle_n^{n+1}+\boldsymbol{I}^{\varepsilon_{\text{an}}\varepsilon_{\text{an}}}\left\langle(\mathbf{e}_{\text{an}})_k\right\rangle_n^{n+1}\left\langle(\mathbf{e}_{\text{an}})_k\right\rangle_n^{n+1}+\\
&\qquad\qquad-\boldsymbol{I}^{\varepsilon\varepsilon_{\text{an}}}\left\langle\mathbf{e}_k\right\rangle_n^{n+1}\left\langle(\mathbf{e}_{\text{an}})_k\right\rangle_n^{n+1}-\boldsymbol{I}^{\varepsilon_{\text{an}}\varepsilon}\left\langle(\mathbf{e}_{\text{an}})_k\right\rangle_n^{n+1}\left\langle\mathbf{e}_k\right\rangle_n^{n+1}\Bigg)
\end{aligned}
$$

Conclusion

4. Quels sont les principes de conservation discrets vérifiés par un tel intégrateur ?

Second principe de la thermodynamique

$$V^{-1}E_{\mathrm{an}}^{2}\Bigg(\boldsymbol{I}^{\varepsilon\varepsilon}\left\langle\mathbf{e}_k\right\rangle_n^{n+1}\left\langle\mathbf{e}_k\right\rangle_n^{n+1}+\boldsymbol{I}^{\varepsilon_{\mathrm{an}}\varepsilon_{\mathrm{an}}}\left\langle(\mathbf{e}_{\mathrm{an}})_k\right\rangle_n^{n+1}\left\langle(\mathbf{e}_{\mathrm{an}})_k\right\rangle_n^{n+1}+$$
$$-\boldsymbol{I}^{\varepsilon\varepsilon_{\mathrm{an}}}\left\langle\mathbf{e}_k\right\rangle_n^{n+1}\left\langle(\mathbf{e}_{\mathrm{an}})_k\right\rangle_n^{n+1}-\boldsymbol{I}^{\varepsilon_{\mathrm{an}}\varepsilon}\left\langle(\mathbf{e}_{\mathrm{an}})_k\right\rangle_n^{n+1}\left\langle\mathbf{e}_k\right\rangle_n^{n+1}\Bigg)\geq 0$$

Conclusion

L’ensemble des résultats continus est publié dans

> *Georgette, B., Dureisseix, D. and Gravouil, A. (2026) “The Double Generator Boundary Augmented bracket structure: a structure-preserving space-time integration framework for coupled thermo-visco-elastodynamics”, Comptes Rendus. Mécanique, 354(G1), pp. 333–364. Available at: https://doi.org/10.5802/crmeca.357.*

→ Extension à la thermo-visco-élastodynamique en petites transformations et en grandes transformations dans le cadre multisymplectique

https://benjamin-georgette.vercel.app/

Un intégrateur géométrique pour la thermo-visco-élastodynamique

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Financement: École Normale Supérieure Paris-Saclay

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- Christöfl, Petra et al. (2021). "Comprehensive Investigation of the Viscoelastic Properties of PMMA by Nanoindentation". In: Polymer Testing 93, p. 106978. ISSN: 01429418. DOI: 10.1016/j.polymertesting.2020.106978.
- Govindjee, Sanjay and Stefanie Reese (July 1997). "A Presentation and Comparison of Two Large Deformation Viscoelasticity Models". In: Journal of Engineering Materials and Technology 119.3, pp. 251–255. ISSN: 0094-4289, 1528-8889. DOI: 10.1115/1.2812252.
- Hairer, Ernst, Christian Lubich, and Gerhard Wanner (2006). Geometric Numerical Integration. Vol. 31. Springer Series in Computational Mathematics. Berlin/Heidelberg: Springer-Verlag. ISBN: 978-3-540-30663-4. DOI: 10.1007/3-540-30666-8.
- Lee, Erastus H. (1969). "Elastic-Plastic Deformation at Finite Strains". In: Journal of Applied Mechanics 36.11, pp. 1–6.
- Marsden, Jerrold E. and Thomas J. R. Hughes (1994). Mathematical Foundations of Elasticity. New York: Dover. ISBN: 978-0-486-67865-8.
- Schiebl, Mark and Peter Betsch (July 2021). "Structure-preserving Space-time Discretization of Large-strain Thermo-viscoelasticity in the Framework of GENERIC". In: International Journal for Numerical Methods in Engineering 122.14, pp. 3448–3488. ISSN: 0029-5981, 1097-0207. DOI: 10.1002/nme.6670.

Sidoroff, F. (1974). “Un Modèle Viscoélastique Non Linéaire Avec Configuration Intermédiaire.”. In: Journal de Mécanique 13, pp. 679–713.

Discrete Variational Calculus on the Hamilton Principle

$$\mathscr{A}^\star[q,p] = \int_{t_0}^{t_N} \big(\dot q p - L^\star(q,p)\big)\mathrm{d}t$$

Define a discrete Hamiltonian

$$\mathscr{A}^\star_{\mathrm{d}} = \sum_{n=0}^{N-1}\left(\frac{q_{n+1}-q_n}{\Delta t} p_{n+1/2} - L^\star_{\mathrm{d}}(q,p)\right)$$

$$L^\alpha_{\mathrm{d}} = p_{n+(1-\alpha)}\frac{q_{n+1}-q_n}{\Delta t} - L^\star_{\mathrm{d}}\big(q_{n+\alpha}, p_{n+(1-\alpha)}\big)$$

$$L^{\alpha=0}_{\mathrm{d}}$$

$$L^{\alpha=1}_{\mathrm{d}}$$

$$L_{\mathrm{d}} = \frac{\Delta t}{2}\left(L^{\alpha=0}_{\mathrm{d}}\Big|_{t\in[t_n,t_{n+1/2}]} + L^{\alpha=1}_{\mathrm{d}}\Big|_{t\in[t_{n+1/2},t_{n+1}]}\right)$$

Discretisation of the Hamilton Equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} q \\ p\end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}\begin{pmatrix} \partial L^\star/\partial q \\ \partial L^\star/\partial p\end{pmatrix}$$

Choose a scheme that preserves the structure $\dot z = \tilde{\mathbb{J}}\nabla L^\star$

$$\frac{z_{n+1}-z_n}{\Delta t} = \tilde{\mathbb{J}}\nabla L^\star(z_{n+\alpha}, z_n)$$

$$\Phi^\alpha : (q_n,p_n) \mapsto \begin{cases} q_{n+1} = q_n + \Delta t \dfrac{\partial L^\star}{\partial p}(q_{n+\alpha}, p_{n+(1-\alpha)}) \\ p_{n+1} = p_n - \Delta t \dfrac{\partial L^\star}{\partial q}(q_{n+\alpha}, p_{n+(1-\alpha)})\end{cases}$$

$$\Phi^{\alpha=0}$$

$$\Phi^{\alpha=1}$$

$$\left(\Phi^{\alpha=0}\Big|_{t\in[t_n,t_{n+1/2}]} + \Phi^{\alpha=1}\Big|_{t\in[t_{n+1/2},t_{n+1}]}\right)$$

Discrete Variational Calculus on the Poisson Bracket

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \begin{pmatrix} \partial F/\partial q \\ \partial F/\partial p\end{pmatrix}^{\top}\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}\begin{pmatrix} \partial L^\star/\partial q \\ \partial L^\star/\partial p\end{pmatrix} \quad \text{or} \quad \frac{\mathrm{d}F}{\mathrm{d}t} = \{F, L^\star\}$$

$$\frac{\partial F}{\partial z}\left(\dot z - \tilde{\mathbb{J}}\nabla L^\star\right)$$

$$\sum_{n=0}^{N-1}\int_{t_n}^{t_{n+1}} \frac{\partial F}{\partial z}\dot z \,\mathrm{d}t = \sum_{n=0}^{N-1}\int_{t_n}^{t_{n+1}} \{F, L^\star\}\,\mathrm{d}t$$

Choose the appropriate interpolation

$$\Delta t \frac{\partial F}{\partial z}\frac{z_{n+1}-z_n}{\Delta t} = \Delta t\{F, L^\star\}$$

$$\begin{pmatrix} \dfrac{\partial F}{\partial q}(q_{n+\alpha}, p_{n+(1-\alpha)}) \\ \dfrac{\partial F}{\partial p}(q_{n+\alpha}, p_{n+(1-\alpha)})\end{pmatrix}\begin{pmatrix} \dfrac{q_{n+1}-q_n}{\Delta t} \\ \dfrac{p_{n+1}-p_n}{\Delta t}\end{pmatrix} = \begin{pmatrix} \dfrac{\partial F}{\partial q}(q_{n+\alpha}, p_{n+(1-\alpha)}) \\ \dfrac{\partial F}{\partial p}(q_{n+\alpha}, p_{n+(1-\alpha)})\end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}\begin{pmatrix} \dfrac{\partial L^\star}{\partial q}(q_{n+\alpha}, p_{n+(1-\alpha)}) \\ \dfrac{\partial L^\star}{\partial p}(q_{n+\alpha}, p_{n+(1-\alpha)})\end{pmatrix}$$

$$F^{\alpha=0}_{\mathrm{d}}$$

$$F^{\alpha=1}_{\mathrm{d}}$$

$$\int_{t_n}^{t_{n+1}} F(q,p)\,\mathrm{d}t = \frac{\Delta t}{2}F(q_{n+\alpha}, p_{n+(1-\alpha)})\Big|_{\alpha=0}^{t\in[t_n,t_{n+1/2}]} + \frac{\Delta t}{2}F(q_{n+\alpha}, p_{n+(1-\alpha)})\Big|_{\alpha=1}^{t\in[t_{n+1/2},t_{n+1}]}$$

α symplectic Euler scheme

$$\begin{cases} \dfrac{q_{n+1}-q_n}{\Delta t} = \dfrac{\partial L^\star}{\partial p}(q_{n+\alpha}, p_{n+(1-\alpha)}) \\ \dfrac{p_{n+1}-p_n}{\Delta t} = -\dfrac{\partial L^\star}{\partial q}(q_{n+\alpha}, p_{n+(1-\alpha)})\end{cases}$$

Implicit symplectic Euler

$$\begin{cases} \dfrac{q_{n+1}-q_n}{\Delta t} = \dfrac{\partial L^\star}{\partial p}(q_n, p_{n+1}) \\ \dfrac{p_{n+1}-p_n}{\Delta t} = -\dfrac{\partial L^\star}{\partial q}(q_n, p_{n+1})\end{cases}$$

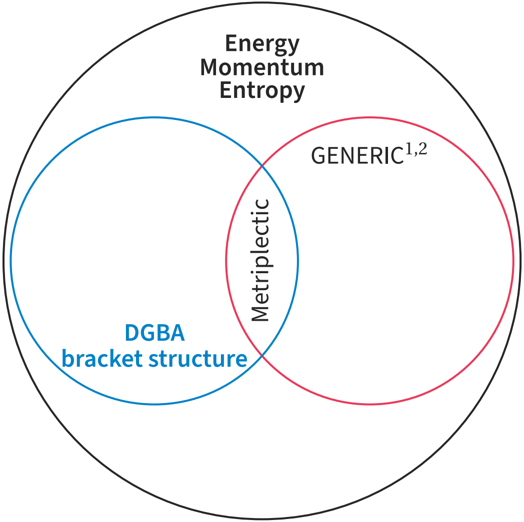
Explicit symplectic Euler

$$\begin{cases} \dfrac{q_{n+1}-q_n}{\Delta t} = \dfrac{\partial L^\star}{\partial p}(q_{n+1}, p_n) \\ \dfrac{p_{n+1}-p_n}{\Delta t} = -\dfrac{\partial L^\star}{\partial q}(q_{n+1}, p_n)\end{cases}$$

Central difference

$$\begin{cases} q_{n+1} = q_n + \dfrac{\Delta t}{2}\left(\dfrac{\partial L^\star}{\partial p}(q_{n+1}, p_{n+1/2}) + \dfrac{\partial L^\star}{\partial p}(q_n, p_{n+1/2})\right) \\ p_{n+3/2} = p_{n+1/2} - \dfrac{\Delta t}{2}\left(\dfrac{\partial L^\star}{\partial q}(q_{n+1}, p_{n+1/2}) + \dfrac{\partial L^\star}{\partial q}(q_{n+1}, p_{n+3/2})\right)\end{cases}$$

Annex - Partie B – Review of discrete bracket formulations



1. Ignacio Romero. 'Algorithms for Coupled Problems That Preserve Symmetries and the Laws of Thermodynamics'. Computer Methods in Applied Mechanics and Engineering 199, no. 25–28 (May 2010): 1841–58.
2. Mark Schiebl, and Peter Betsch. 'Structure–preserving Space–time Discrétisation of Large–strain Thermo–viscoelasticity in the Framework of GENERIC'. International Journal for Numerical Methods in Engineering 122, no. 14 (July 2021): 3448–88.

Annex - Comment étendre le schéma de la différence centrée à la physique dissipative?

On obtient le schéma de la différence centrée par l'approximation suivante

$$\int_{t_n}^{t_{n+1}} F(q,p)\mathrm{d}t = \frac{\Delta t}{2}\left(F(q_n, p_{n+1/2}) + F(q_{n+1}, p_{n+1/2})\right)$$

sur

$$\int_{t_0}^{t_N} \left(\frac{\partial f}{\partial q}\dot{q} + \frac{\partial f}{\partial p}\dot{p} - \left(\frac{\partial f}{\partial q}\frac{\partial l^\star}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial l^\star}{\partial q}\right)\right)\mathrm{d}t = 0$$

Comment faire dans le cas de la physique dissipative, *i.e.* sur

$$\int_{t_0}^{t_N} \left(\frac{\mathrm{d}\mathscr{F}}{\mathrm{d}t} - \{\mathscr{F}, \hat{\mathscr{E}}_{\text{tot}}\} - (\mathscr{F}, \mathscr{S}) - \text{boundary terms}\right)\mathrm{d}t = 0$$

Annex - Comment étendre le schéma de la différence centrée à la physique dissipative?

Variables internes $\boldsymbol{\alpha}$.

- On commence d'abord par tester un intégrateur qui calque la différence centrée.

$$\int_{t_n}^{t_{n+1}} F(\boldsymbol{\alpha})\mathrm{d}t \approx \frac{\Delta t}{2}\left(F(\boldsymbol{\alpha}_{n+\tilde{\gamma}})\Big|_{\tilde{\gamma}=0}^{t\in[t_n,t_{n+1/2}]} + F(\boldsymbol{\alpha}_{n+\tilde{\gamma}})\Big|_{\tilde{\gamma}=1}^{t\in[t_{n+1/2},t_{n+1}]}\right) \approx \frac{\Delta t}{2}\left(F(\boldsymbol{\alpha}_n)+F(\boldsymbol{\alpha}_{n+1})\right)$$

  Ce choix mène à l'expression suivante dans le premier principe de la thermodynamique

$$-\sum_{n=0}^{N-1}\frac{\Delta t}{2}\int_0^L \frac{1}{2}\Big((\boldsymbol{\alpha}_{n+1}+\boldsymbol{\alpha}_n)^\top \boldsymbol{E}^\top \boldsymbol{V}^{-1}\boldsymbol{E}(\boldsymbol{\alpha}_{n+1}+\boldsymbol{\alpha}_n) + (\boldsymbol{\alpha}_{n+1}-\boldsymbol{\alpha}_n)^\top \boldsymbol{E}^\top \boldsymbol{V}^{-1}\boldsymbol{E}(\boldsymbol{\alpha}_{n+1}-\boldsymbol{\alpha}_n)\Big)\mathrm{d}X$$

- On propose alors une généralisation

$$\int_{t_n}^{t_{n+1}} F(\boldsymbol{\alpha})\mathrm{d}t = \frac{\Delta t}{2}\left(F(\boldsymbol{\alpha}_{n+\tilde{\gamma}})+F(\boldsymbol{\alpha}_{n+(1-\tilde{\gamma})})\right).$$

  ce qui mène à l'expression suivante dans le premier principe de la thermodynamique

$$-\sum_{n=0}^{N-1}\frac{\Delta t}{2}\int_0^L \frac{1}{2}\Big((\boldsymbol{\alpha}_{n+1}+\boldsymbol{\alpha}_n)^\top \boldsymbol{E}^\top \boldsymbol{V}^{-1}\boldsymbol{E}(\boldsymbol{\alpha}_{n+1}+\boldsymbol{\alpha}_n) + (1-2\tilde{\gamma})^2(\boldsymbol{\alpha}_{n+1}-\boldsymbol{\alpha}_n)^\top \boldsymbol{E}^\top \boldsymbol{V}^{-1}\boldsymbol{E}(\boldsymbol{\alpha}_{n+1}-\boldsymbol{\alpha}_n)\Big)\mathrm{d}X$$

Annex - Comment étendre le schéma de la différence centrée à la physique dissipative?

Cas thermo-visco-élastodynamique $\boldsymbol{\alpha} = \varepsilon_{\text{an}}$ On applique directement $\tilde{\gamma} = 1/2$

$$\int_{t_n}^{t_{n+1}} F(\varepsilon_{\text{an}})\mathrm{d}t \approx \frac{\Delta t}{2}\left(F((\varepsilon_{\text{an}})_{n+\tilde{\gamma}}) + F((\varepsilon_{\text{an}})_{n+(1-\tilde{\gamma})})\right) = \Delta t F((\varepsilon_{\text{an}})_{n+1/2}).$$

On impose au sens fort

$$(\varepsilon_{\text{e}})_{n+1/2} = \varepsilon_{n+1/2} - (\varepsilon_{\text{an}})_{n+1/2}.$$

D'où la notation

$$\int_{t_n}^{t_{n+1}} \frac{\partial F}{\partial \varepsilon_{\text{an}}}\mathrm{d}t \approx \Delta t \left(\frac{\partial F}{\partial \varepsilon_{\text{an}}}\right)_{n+1/2}$$

Annex - Comment étendre le schéma de la différence centrée à la physique dissipative?

Température T

- On commence d'abord par un intégrateur qui calque le schéma de la différence centrée.

$$\sum_{n=0}^{N-1}\int_{t_n}^{t_{n+1}} F(u,p,T)\mathrm{d}t = \sum_{n=0}^{N-1}\frac{\Delta t}{2}\left(F(u_n,\varepsilon_n,p_{n+1/2},T_n)+F(u_{n+1},\varepsilon_{n+1},p_{n+1/2},T_{n+1})\right).$$

Ce choix mène à l'expression suivante dans le premier principe de la thermodynamique

$$-\int_0^L \frac{\Delta t}{2}\frac{c}{T_0}\left(\left(\frac{\partial T_{n+1}}{\partial X}+\frac{\partial T_n}{\partial X}\right)K\left(\frac{\partial T_{n+1}}{\partial X}+\frac{\partial T_n}{\partial X}\right)+\left(\frac{\partial T_{n+1}}{\partial X}-\frac{\partial T_n}{\partial X}\right)K\left(\frac{\partial T_{n+1}}{\partial X}-\frac{\partial T_n}{\partial X}\right)\right)\mathrm{d}X.$$

Annex - Comment étendre le schéma de la différence centrée à la physique dissipative?

Température T

- On propose alors une généralisation

$$\int_{t_n}^{t_{n+1}} F(T)\mathrm{d}t = \frac{1}{2}\left(F\left(T_{n+\tilde{\beta}}\right) + F\left(T_{n+(1-\tilde{\beta})}\right)\right).$$

  - S'il y a le terme de couplage thermo-mécanique dans le crochet réversible,

$$\int_0^L \begin{pmatrix} \partial F/\partial u \\ \partial F/\partial \varepsilon \\ \partial F/\partial p \\ \partial F/\partial T \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{\partial \square_{\text{right}}}{\partial X} & 0 \\ -1 & -\frac{\partial \square_{\text{left}}}{\partial X} & 0 & \left(\frac{\partial S}{\partial T}\right)^{-1} \frac{\partial S}{\partial \varepsilon} \frac{\partial \square_{\text{left}}}{\partial X} \\ 0 & 0 & -\left(\frac{\partial S}{\partial T}\right)^{-1} \frac{\partial S}{\partial \varepsilon} \frac{\partial \square_{\text{right}}}{\partial X} & 0 \end{pmatrix} \begin{pmatrix} \partial \hat{E}_{\text{tot}}/\partial u \\ \partial \hat{E}_{\text{tot}}/\partial \varepsilon \\ \partial \hat{E}_{\text{tot}}/\partial p \\ \partial \hat{E}_{\text{tot}}/\partial T \end{pmatrix} \mathrm{d}X$$

Annex - Comment étendre le schéma de la différence centrée à la physique dissipative?

Température T

- On propose alors une généralisation

$$\int_{t_n}^{t_{n+1}} F(T)\mathrm{d}t = \frac{1}{2}\left(F\left(T_{n+\tilde{\beta}}\right) + F\left(T_{n+(1-\tilde{\beta})}\right)\right).$$

  - S'il y a le terme de couplage thermo-mécanique dans le crochet réversible, alors on a une approximation du type

$$\int_{t_n}^{t_{n+1}} F(u,\varepsilon,p,T)\mathrm{d}t = \sum_{n=0}^{N-1} \frac{\Delta t}{2}\left(\frac{1}{2}\left(F(u_n,\varepsilon_n,p_{n+1/2},T_{n+\tilde{\beta}}) + F(u_n,\varepsilon_n,p_{n+1/2},T_{n+(1-\tilde{\beta})})\right)\Bigg|^{t\in[t_n,t_{n+1/2}]} + \right.$$
$$\left. + \frac{1}{2}\left(F(u_{n+1},\varepsilon_{n+1},p_{n+1/2},T_{n+\tilde{\beta}}) + F(u_{n+1},\varepsilon_{n+1},p_{n+1/2},T_{n+(1-\tilde{\beta})})\right)\Bigg|^{t\in[t_{n+1/2},t_{n+1}]}\right).$$

qui impose

$$0 \le \tilde{\beta} \le 1/2 \qquad \text{et} \qquad 1/2 \le \tilde{\beta} \le 1 \Rightarrow \boxed{\tilde{\beta} = 1/2}$$

Annex - Comment étendre le schéma de la différence centrée à la physique dissipative?

Température T

- On propose alors une généralisation

$$\int_{t_n}^{t_{n+1}} F(T)\mathrm{d}t = \frac{1}{2}\left(F\left(T_{n+\tilde{\beta}}\right) + F\left(T_{n+(1-\tilde{\beta})}\right)\right).$$

  - S'il y a le terme de couplage thermo-mécanique dans le crochet réversible, alors $\tilde{\beta} = 1/2$
  - Enfin, s"il n'y a pas de couplage thermo-mécanique, alors on a, dans l'expression du premier principe

$$\sum_{n=0}^{N-1}\int_0^L \frac{\Delta t}{2}\frac{1}{\rho}\frac{\rho c}{T_0}K\frac{1}{2}\left(\left(\frac{\partial T_n}{\partial X} + \frac{\partial T_{n+1}}{\partial X}\right)^2 + (1-2\tilde{\beta})^2\left(\frac{\partial T_n}{\partial X} - \frac{\partial T_{n+1}}{\partial X}\right)^2\right)\mathrm{d}X.$$

ce qui amène à

$$\boxed{\tilde{\beta} = 1/2}.$$

Annex - Partie B – Simulation parameters

Material parameters : Steel

- Young modulus : $E = 210\text{GPa}$
- Density : $\rho = 7800\text{kg.m}^{-3}$
- Thermal linear dilatation coefficient : $\alpha = 11 \times 10^{-6}\text{K}^{-1}$
- Poisson coefficient : 0.3
- Heat capacity : $c = 420\text{J.kg}^{-1}.\text{K}^{-1}$
- Thermal conductivity : $K = 55\text{W.m}^{-1}.\text{K}^{-1}$
- length : $L = 25\text{cm}$
- Cross section : $S = (L/10)^2 = 6.25 \times 10^{-4}\text{m}^2$

Simulation parameters

- External force $F = \sigma_0 \times S$ with $\sigma_0 = 300\text{MPa}$
- External heat flux $q = (10^4)/(S)\text{W} = 1.6 \times 10^7\text{W/m}^2$
- Number of elements $N = 100$
- Time step $\Delta t = 0.8\Delta x\sqrt{\frac{\rho}{\lambda+2\mu}} = 3.32 \times 10^{-7}\text{s}$
- Time for the elastic wave to travel the rod $T_{\text{elastic wave}} = 4.15 \times 10^{-5}\text{s}$.

Annex - Partie C - Stability analysis of the scheme

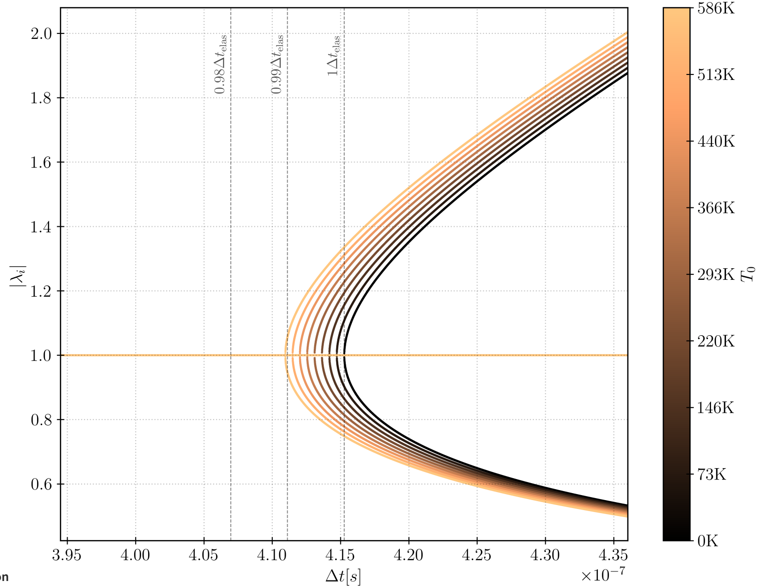
The stability of the scheme is assessed using the spectral stability criterion. The amplification matrix is given by

$$\begin{pmatrix} \mathbf{u}_{n+1} \\ \mathbf{p}_{n+3/2} \\ \mathbf{T}_{n+1} \end{pmatrix} = \boldsymbol{A}(\Delta t, \Delta x; \lambda, \mu, \rho, c, k, K) \begin{pmatrix} \mathbf{u}_{n} \\ \mathbf{p}_{n+1/2} \\ \mathbf{T}_{n} \end{pmatrix}$$

with $\boldsymbol{A}(\Delta t, \Delta x; \lambda, \mu, \rho, c, k, K)$

$$= \begin{pmatrix} \boldsymbol{I}^{uu} & \mathbf{0} & \mathbf{0} \\ \Delta t \boldsymbol{H}^{u'u'} & \boldsymbol{I}^{pp} & -\Delta t \boldsymbol{L}^{p'T} \\ 0 & 0 & \boldsymbol{I}^{TT} + \frac{1}{2}\frac{\Delta t}{\rho c}\boldsymbol{K}^{T'T'} \end{pmatrix}^{-1} \begin{pmatrix} \boldsymbol{I}^{uu} & \Delta t \boldsymbol{M}^{-1,pp} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{I}^{pp} & \mathbf{0} \\ \mathbf{0} & -\frac{\Delta t}{\rho c}\frac{T_0}{\rho}\boldsymbol{L}^{Tp'} & \boldsymbol{I}^{TT} - \frac{1}{2}\frac{\Delta t}{\rho c}\boldsymbol{K}^{T'T'} \end{pmatrix}$$

Annex - Partie C - Stability analysis of the scheme



Annex – Small strain generalised standard material – Choice of the state vector

The state vector for the problem is **chosen** as

$$\boldsymbol{z} = \begin{pmatrix} u & \varepsilon & p & T \end{pmatrix}$$

where

- u is the displacement;
- ε is the symmetric part of the gradient of displacement, here $\varepsilon = \partial u / \partial X$;
- p is the linear momentum;
- T is the absolute temperature;
- $\boldsymbol{\alpha} = \begin{pmatrix} \alpha_1 & \cdots & \cdots & \alpha_N \end{pmatrix}^\mathsf{T}$ is a set of N internal dissipative variables.

Next, we take the time derivative of a functional $\mathscr{F}$ dependent of the state vector

$$\frac{\mathrm{d}\mathscr{F}}{\mathrm{d}t} = \int_0^L \left(\frac{\partial F}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial F}{\partial \varepsilon}\frac{\partial \varepsilon}{\partial t} + \frac{\partial F}{\partial p}\frac{\partial p}{\partial t} + \frac{\partial F}{\partial T}\frac{\partial T}{\partial t} + \left(\frac{\partial F}{\partial \boldsymbol{\alpha}}\right)^\mathsf{T} \frac{\partial \boldsymbol{\alpha}}{\partial t} \right) \mathrm{d}X$$

Annex – Small strain generalised standard material – State laws

The DGBA bracket structure is **obtained from the state laws**

$$\frac{\mathrm{d}\mathscr{F}}{\mathrm{d}t} = \int_0^L \left(\frac{\partial F}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial F}{\partial p}\frac{\partial p}{\partial t} + \frac{\partial F}{\partial \varepsilon}\frac{\partial \varepsilon}{\partial t} + \frac{\partial F}{\partial T}\frac{\partial T}{\partial t} + \left(\frac{\partial F}{\partial \boldsymbol{\alpha}}\right)^{\mathsf{T}} \frac{\partial \boldsymbol{\alpha}}{\partial t} \right) \mathrm{d}X$$

$$\frac{\partial u}{\partial t} = \frac{\partial \hat{E}_{\text{tot}}}{\partial p}$$

First Legendre transformation

$$\frac{\partial p}{\partial t} = -\frac{\partial \hat{E}_{\text{tot}}}{\partial u} + \frac{\partial}{\partial X}\left(\frac{\partial \hat{E}_{\text{tot}}}{\partial \varepsilon} - \left(\frac{\partial \hat{E}_{\text{tot}}}{\partial T}\right)\left(\frac{\partial S}{\partial T}\right)^{-1} \frac{\partial S}{\partial \varepsilon} \right)$$

Conservation of linear momentum

$$\frac{\partial \varepsilon}{\partial t} = \frac{\partial}{\partial X}\left(\frac{\partial \hat{E}_{\text{tot}}}{\partial p}\right)$$

Compatibility equation

$$\rho c \frac{\partial T}{\partial t} + \rho c \left(\frac{\partial S}{\partial T}\right)^{-1} \frac{\partial S}{\partial \varepsilon} \frac{\partial}{\partial X}\left(\frac{\partial \hat{E}_{\text{tot}}}{\partial p}\right) = \frac{\partial}{\partial X}\left(K \frac{\partial T}{\partial X}\right) - \left(\frac{\partial (\rho \hat{E}_{\text{int}})}{\partial \boldsymbol{\alpha}}\right)^{\mathsf{T}} \frac{\partial \boldsymbol{\alpha}}{\partial t}$$

Gibbs law

$$\frac{\partial \boldsymbol{\alpha}}{\partial t} = -\frac{T}{c}\boldsymbol{V}^{-1}\frac{\partial \hat{E}_{\text{int}}}{\partial \boldsymbol{\alpha}}\frac{\partial S}{\partial T} + T\boldsymbol{V}^{-1}\frac{\partial S}{\partial \boldsymbol{\alpha}}$$

Internal dissipative laws

K: material's conductivity; ρ: density; c: specific thermal capacity; $\hat{E}_{\text{int}}$: internal energy

Annex – Small strain generalised standard material – DGBA bracket structure expression

The expression of the DGBA bracket structure is

$$\frac{\mathrm{d}\mathscr{F}}{\mathrm{d}t} = \left\{\mathscr{F}, \hat{\mathscr{E}}_{\text{tot}}\right\} + (\mathscr{F}, \mathscr{S}) - \int_{\partial\omega_X} \frac{1}{\rho}\frac{\partial F}{\partial T} K \frac{T^2}{c^2}\frac{\partial}{\partial X}\left(\frac{\partial S}{\partial T}\right)\mathrm{d}(\partial\omega_X) +$$

$$+\int_{\partial\omega_X} \frac{\partial F}{\partial \varepsilon}\left(\frac{\partial u}{\partial t} - \frac{\partial \hat{E}_{\text{tot}}}{\partial p}\right)\mathrm{d}(\partial\omega_X) + \int_{\partial\omega_X} \frac{\partial F}{\partial p}\left(\frac{\partial \hat{E}_{\text{tot}}}{\partial \varepsilon} - \left(\frac{\partial S}{\partial T}\right)^{-1}\frac{\partial S}{\partial \varepsilon}\frac{\partial \hat{E}_{\text{tot}}}{\partial T}\right)\mathrm{d}(\partial\omega_X)$$

where

$$\left\{\mathscr{F}, \hat{\mathscr{E}}_{\text{tot}}\right\} = \int_0^L \left(\frac{\partial F}{\partial u}\frac{\partial \hat{E}_{\text{tot}}}{\partial p} - \frac{\partial F}{\partial p}\frac{\partial \hat{E}_{\text{tot}}}{\partial u} + \frac{\partial F}{\partial \varepsilon}\frac{\partial}{\partial X}\left(\frac{\partial \hat{E}_{\text{tot}}}{\partial p}\right) - \frac{\partial}{\partial X}\left(\frac{\partial F}{\partial p}\right)\frac{\partial \hat{E}_{\text{tot}}}{\partial \varepsilon} + \right.$$

$$\left. + \left(\frac{\partial S}{\partial T}\right)^{-1}\frac{\partial S}{\partial \varepsilon}\left(\frac{\partial \hat{E}_{\text{tot}}}{\partial T}\frac{\partial}{\partial X}\left(\frac{\partial F}{\partial p}\right) - \frac{\partial F}{\partial T}\frac{\partial}{\partial X}\left(\frac{\partial \hat{E}_{\text{tot}}}{\partial p}\right)\right)\right)\mathrm{d}X$$

$$(\mathscr{F}, \mathscr{S}) = \int_0^L \left(\frac{1}{\rho}\frac{\partial}{\partial X}\left(\frac{\partial F}{\partial T}\right) K \frac{T^2}{c^2}\frac{\partial}{\partial X}\left(\frac{\partial S}{\partial T}\right) + \frac{\partial F}{\partial T}\frac{1}{c}\frac{T}{c}\frac{\partial \hat{E}_{\text{int}}}{\partial \boldsymbol{\alpha}}\boldsymbol{V}^{-1}\frac{\partial \hat{E}_{\text{int}}}{\partial \boldsymbol{\alpha}}\frac{\partial S}{\partial T} + \right.$$

$$\left. - \frac{\partial F}{\partial T}\frac{T}{c}\frac{\partial \hat{E}_{\text{int}}}{\partial \boldsymbol{\alpha}}\boldsymbol{V}^{-1}\frac{\partial S}{\partial \boldsymbol{\alpha}} - \frac{T}{c}\frac{\partial F}{\partial \boldsymbol{\alpha}}\boldsymbol{V}^{-1}\frac{\partial \hat{E}_{\text{int}}}{\partial \boldsymbol{\alpha}}\frac{\partial S}{\partial T} + T\frac{\partial F}{\partial \boldsymbol{\alpha}}\boldsymbol{V}^{-1}\frac{\partial S}{\partial \boldsymbol{\alpha}}\right)\mathrm{d}X$$

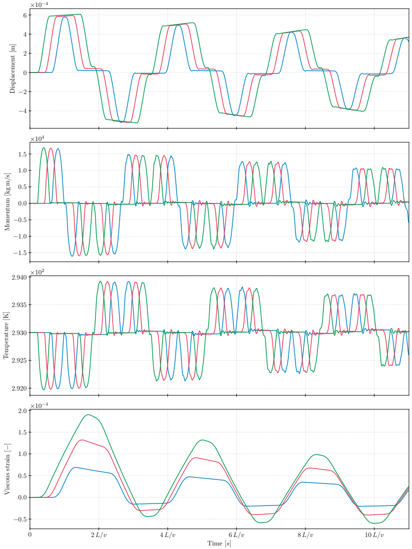
Annex - Small strain thermo-visco-elastodynamics

Parameters of simulation

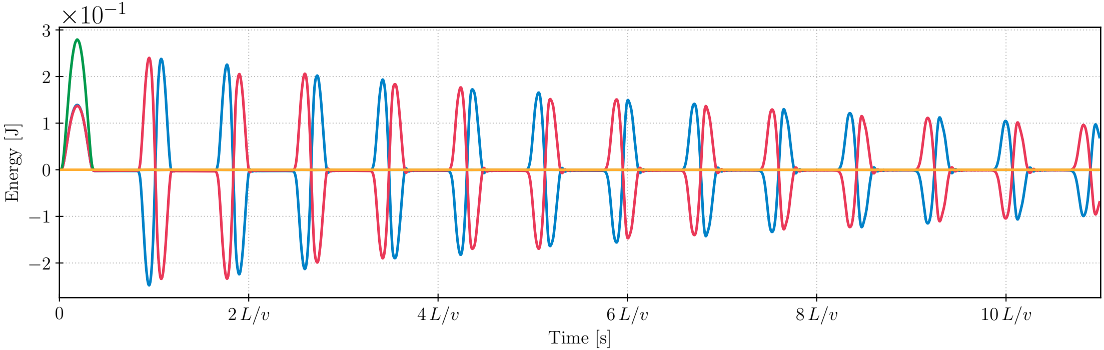
Density	ρ	$1.18 \times 10^3\,\mathrm{kg\,m^{-3}}$
Poisson ratio	ν	0.35
Young modulus - equilibrium branch	E_∞	$2.5\,\mathrm{GPa}$
Young modulus - out-of-equilibrium branch	E_{an}	$1.1\,\mathrm{GPa}$
Viscous coefficient	$V^{-1} = (E_{\mathrm{an}}\tau_1)^{-1}$	$(59.4 \times 10^9)^{-1}\,\mathrm{GPa^{-1}\,s^{-1}}$
Coefficient of linear thermal expansion	α	$90 \times 10^{-6}\,\mathrm{K^{-1}}$
Specific heat capacity	c	$1.47 \times 10^3\,\mathrm{J\,kg^{-1}\,K^{-1}}$
Thermal conductivity	K	$1.19 \times 10^{-1}\,\mathrm{W\,m^{-1}\,K^{-1}}$

Material properties of polyethyl methacrylate (PMMA). The density ρ and the thermal parameters α, K, c are extracted from the folllowing website `https://analyzing-testing.netzsch.com/en/polymers-netzsch-com/engineering-thermoplastics/pmma-polymethylmethacrylate`, while the parameters of the standard linear solid $E_\infty, E_{\mathrm{an}}, \nu, V^{-1}$ are extracted from (Christöfl et al. 2021)

Annex - Small strain thermo-visco-elastodynamics - Fields

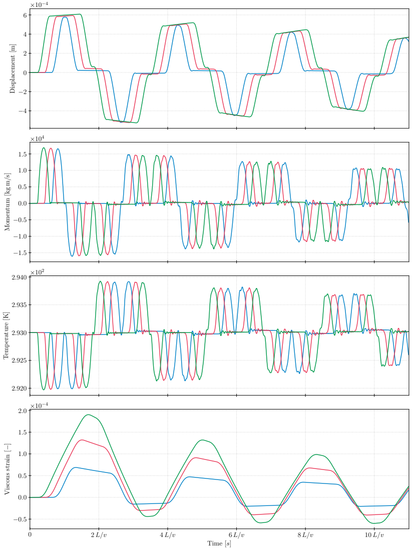


Annex - Small strain thermo-visco-elastodynamics - First principle of thermodynamics

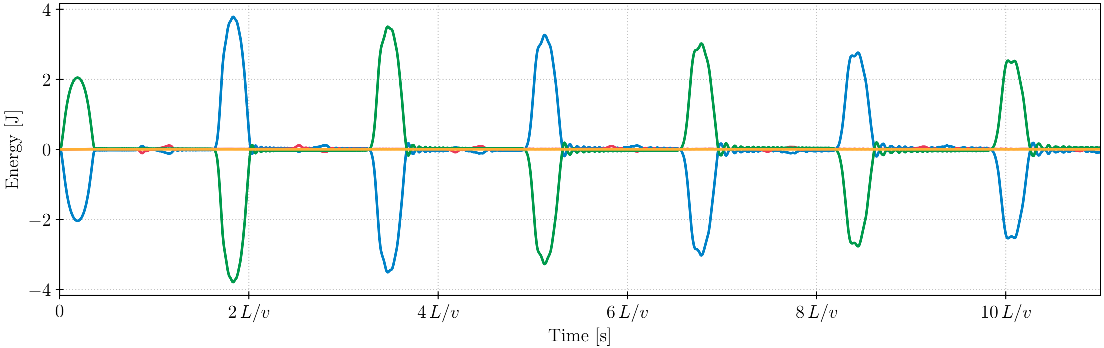


$\left[\frac{1}{2}\tilde{\mathsf{p}}_k^{\mathsf{T}}\tilde{\tilde{\mathsf{M}}}^{-1,pp}\tilde{\mathsf{p}}_k\right]_n^{n+1}$ $\left[\frac{1}{2}\tilde{\mathsf{e}}_k^{\mathsf{T}}(E_\infty + E_{\mathrm{an}})\tilde{\tilde{\mathsf{I}}}^{\varepsilon\varepsilon}\tilde{\mathsf{e}}_k\right]_n^{n+1}$ $\rho^{-1}\Delta t\left\langle(\tilde{\mathsf{F}}_k^{\mathrm{ext}})^{\mathsf{T}}\right\rangle_n^{n+1}\tilde{\mathsf{p}}_{n+1/2}$ $\frac{\Delta t^2}{4}\left[\frac{1}{2}\ddot{\tilde{\mathsf{u}}}_k^{\mathsf{T}}\tilde{\tilde{\mathsf{M}}}^{pp}\ddot{\tilde{\mathsf{u}}}_k\right]_n^{n+1}$

Annex - Small strain thermo-visco-elastodynamics - Fields

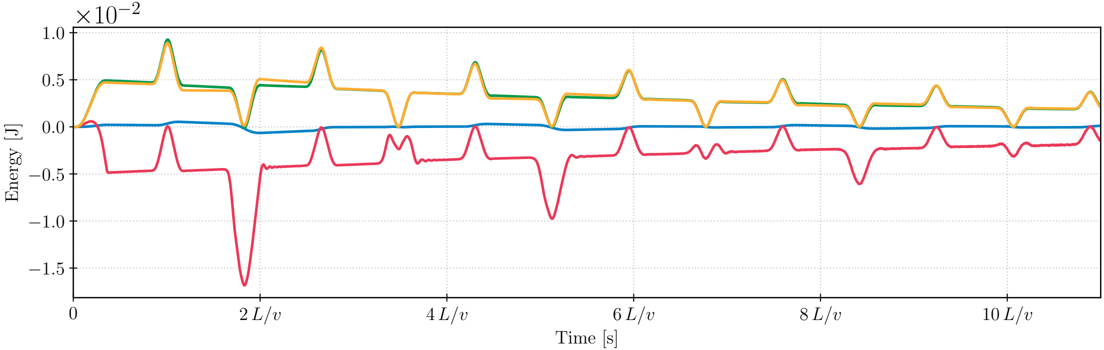


Annex - Small strain thermo-visco-elastodynamics - First principle of thermodynamics



$\left[\frac{1}{2}\frac{\rho c}{T_0}\tilde{\mathsf{T}}_k^{\mathsf{T}}\tilde{\tilde{\mathsf{I}}}^{TT}\tilde{\mathsf{T}}_k\right]_n^{n+1}$ $\frac{1}{T_0}\Delta t(\tilde{\mathsf{q}}^{\mathrm{ext}}_{n+1/2})^{\mathsf{T}}\langle\tilde{\mathsf{T}}_k\rangle_n^{n+1}$ $\left[\tilde{\mathsf{T}}_0^{\mathsf{T}}\tilde{\tilde{\mathsf{L}}}^{Tp'}\tilde{\mathsf{u}}_k\right]_n^{n+1}$ $\frac{1}{T_0}\Delta t\left\langle\tilde{\mathsf{T}}_k^{\mathsf{T}}\right\rangle_n^{n+1}\tilde{\tilde{\mathsf{K}}}^{T'T'}\langle\tilde{\mathsf{T}}_k\rangle_n^{n+1}$

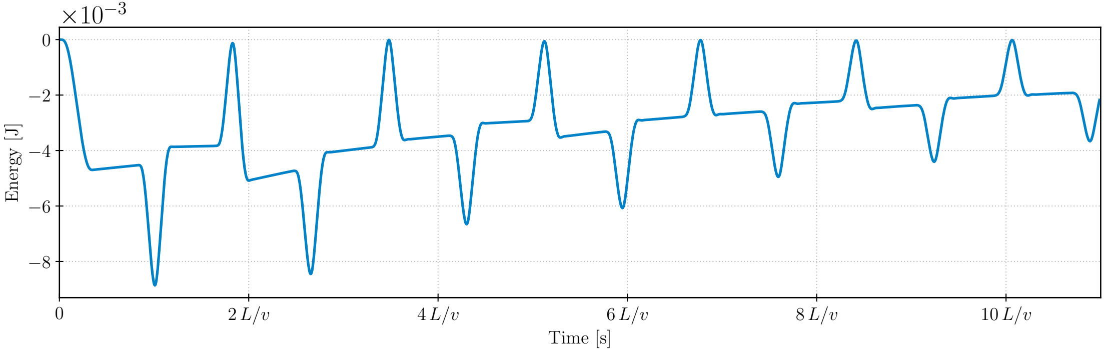
Annex - Small strain thermo-visco-elastodynamics - First principle of thermodynamics



$\left[\frac{1}{2}(\tilde{\mathsf{e}}_{\mathrm{an}})_k^{\mathsf{T}} E_{\mathrm{an}} \tilde{\tilde{I}}^{\varepsilon_{\mathrm{an}}\varepsilon_{\mathrm{an}}} (\tilde{\mathsf{e}}_{\mathrm{an}})_k\right]_n^{n+1}$ $\langle(\tilde{\mathsf{e}}_{\mathrm{an}})_k^{\mathsf{T}}\rangle_n^{n+1} E_{\mathrm{an}} \tilde{\tilde{I}}^{\varepsilon_{\mathrm{an}}\varepsilon} [\tilde{\mathsf{e}}_k]_n^{n+1}$ $\langle\tilde{\mathsf{e}}_k\rangle_n^{n+1} E_{\mathrm{an}} \tilde{\tilde{I}}^{\varepsilon\varepsilon_{\mathrm{an}}} [(\tilde{\mathsf{e}}_{\mathrm{an}})_k^{\mathsf{T}}]_n^{n+1}$

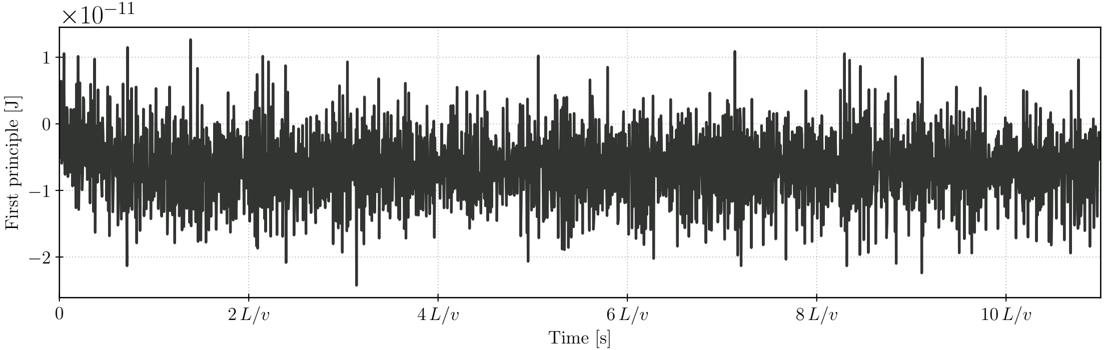
$\rho \Delta t V^{-1} E_{\mathrm{an}}^2 \Big(\tilde{\tilde{\mathsf{I}}}^{\varepsilon\varepsilon} \langle\tilde{\mathsf{e}}_k\rangle_n^{n+1} \langle\tilde{\mathsf{e}}_k\rangle_n^{n+1} + \tilde{\tilde{\mathsf{I}}}^{\varepsilon_{\mathrm{an}}\varepsilon_{\mathrm{an}}} \langle(\tilde{\mathsf{e}}_{\mathrm{an}})_k\rangle_n^{n+1} \langle(\tilde{\mathsf{e}}_{\mathrm{an}})_k\rangle_n^{n+1} - \tilde{\tilde{\mathsf{I}}}^{\varepsilon\varepsilon_{\mathrm{an}}} \langle\tilde{\mathsf{e}}_k\rangle_n^{n+1} \langle(\tilde{\mathsf{e}}_{\mathrm{an}})_k\rangle_n^{n+1} - \tilde{\tilde{\mathsf{I}}}^{\varepsilon_{\mathrm{an}}\varepsilon} \langle(\tilde{\mathsf{e}}_{\mathrm{an}})_k\rangle_n^{n+1} \langle\tilde{\mathsf{e}}_k\rangle_n^{n+1}\Big)$

Annex - Small strain thermo-visco-elastodynamics - First principle of thermodynamics

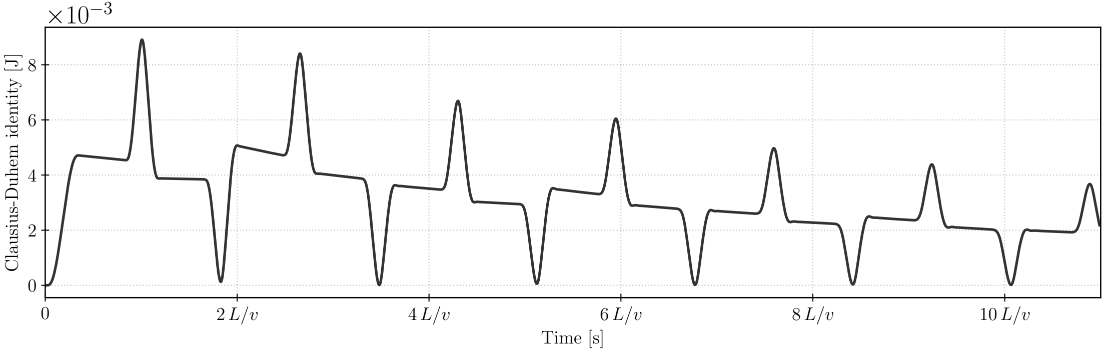


$$-\rho\Delta t\frac{\left\langle\tilde{\mathsf{T}}_k^{\mathsf{T}}\right\rangle_n^{n+1}}{T_0}V^{-1}E_{\mathrm{an}}^2\Big(\tilde{\tilde{\tilde{\mathsf{I}}}}^{T\varepsilon\varepsilon}\langle\tilde{\mathsf{e}}_k\rangle_n^{n+1}\langle\tilde{\mathsf{e}}_k\rangle_n^{n+1}+\tilde{\tilde{\tilde{\mathsf{I}}}}^{T\varepsilon_{\mathrm{an}}\varepsilon_{\mathrm{an}}}\langle(\tilde{\mathsf{e}}_{\mathrm{an}})_k\rangle_n^{n+1}\langle(\tilde{\mathsf{e}}_{\mathrm{an}})_k\rangle_n^{n+1}-$$
$$\tilde{\tilde{\tilde{\mathsf{I}}}}^{T\varepsilon\varepsilon_{\mathrm{an}}}\langle\tilde{\mathsf{e}}_k\rangle_n^{n+1}\langle(\tilde{\mathsf{e}}_{\mathrm{an}})_k\rangle_n^{n+1}-\tilde{\tilde{\tilde{\mathsf{I}}}}^{T\varepsilon_{\mathrm{an}}\varepsilon}\langle(\tilde{\mathsf{e}}_{\mathrm{an}})_k\rangle_n^{n+1}\langle\tilde{\mathsf{e}}_k\rangle_n^{n+1}\Big)$$

Annex - Small strain thermo-visco-elastodynamics - First principle of thermodynamics



Annex - Small strain thermo-visco-elastodynamics - Second principle of thermodynamics



Annex – Large strain Thermo-visco-elastodynamics – Choice of the state vector

Small strain, 1D, Generalised standard material

$$\boldsymbol{z} = \begin{pmatrix} u & \varepsilon & p & T & \boldsymbol{\alpha} \end{pmatrix}$$

where

- u is the displacement;
- p is the linear momentum;
- ε is the symmetric part of the gradient of displacement, here $\varepsilon = \partial u / \partial X$;
- T is the absolute temperature;
- $\boldsymbol{\alpha} = \begin{pmatrix} \alpha_1 & \cdots & \cdots & \alpha_N \end{pmatrix}^\mathsf{T}$ is a set of N internal dissipative variables.

Large strain, 3D, Thermo-visco-elastodynamics

$$\boldsymbol{z} = \begin{pmatrix} \underline{\chi} & \underline{p} & \underline{\underline{\Pi}} & T & \underline{\underline{C}}_{\mathrm{i}}^{-1} \end{pmatrix}$$

where

- $\underline{\chi}$ is the configuration;
- $\underline{p}$ is the linear momentum;
- $\underline{\underline{\Pi}}$ is the first Piola Kirchhoff stress tensor;
- T is the absolute temperature;
- $\underline{\underline{C}}_{\mathrm{i}}^{-1}$ is the inverse of the inelastic Cauchy-Green tensor.

Annex – Large strain Thermo-visco-elastodynamics – Details on finite visco-elasticity – multisymplectic Hamiltonian

Thermo-**visco**-elastodynamics: Sidoroff's decomposition of the gradient of transformation (Lee 1969; Sidoroff 1974; Govindjee and Reese 1997)

$$\underline{\underline{F}} = \underline{\underline{F}}_{\mathrm{e}} \cdot \underline{\underline{F}}_{\mathrm{i}}$$

The **total energy and the entropy are defined in terms of the first Piola Kirchhoff stress tensor** $\underline{\underline{\Pi}}$

$$L^{\star}(\underline{\chi}, \underline{p}, \underline{\underline{\Pi}}, T, \underline{\underline{C}}_{\mathrm{i}}^{-1}) = \underline{p} \cdot \frac{\partial \underline{\chi}}{\partial t} - \underline{\underline{\Pi}} : \underline{\underline{F}} - l\left(\underline{\chi}, \frac{\partial \underline{\chi}}{\partial t}, \underline{\underline{F}}, T, \underline{\underline{C}}_{\mathrm{i}}^{-1}\right), \qquad \underline{p} = \frac{\partial l}{\partial \left(\frac{\partial \underline{\chi}}{\partial t}\right)}, \qquad \underline{\underline{\Pi}} = -\frac{\partial l}{\partial \underline{\underline{F}}}$$

$$\rho T s^{*}(T, \underline{\underline{\Pi}}, \underline{\underline{C}}_{\mathrm{i}}^{-1}) = \underline{\underline{\Pi}} : \underline{\underline{F}} + \rho T s(T, \underline{\underline{F}}, \underline{\underline{C}}_{\mathrm{i}}^{-1})$$

Annex – Large strain Thermo-visco-elastodynamics – Expression de la DGBA bracket structure

After "some" calculus, we obtain

$$\frac{\mathrm{d}\mathscr{F}}{\mathrm{d}t} = \int_0^L \left(\frac{\partial F}{\partial \underline{\chi}} \cdot \frac{\partial \underline{\chi}}{\partial t} + \frac{\partial F}{\partial \underline{p}} \cdot \frac{\partial \underline{p}}{\partial t} - \frac{\partial F}{\partial \underline{\underline{F}}} : \underline{\underline{\dot{F}}} + \frac{\partial F}{\partial T}\left(\left(\frac{\partial T}{\partial t}\right)_{\text{rev}} + \left(\frac{\partial T}{\partial t}\right)_{\text{irr}} \right) + \frac{\partial F}{\partial \underline{\underline{C}}_{\text{i}}^{-1}} : \frac{\partial \underline{\underline{C}}_{\text{i}}^{-1}}{\partial t} \right)$$

$$= \left\{\mathscr{F}, \hat{\mathscr{E}}^*_{\text{tot}}\right\} + \left(\mathscr{F}, \mathscr{S}^*\right) +$$

$$+ \int_{\partial\omega_X} \frac{\partial F}{\partial \underline{p}} \cdot (\underline{\underline{\Pi}} \cdot \underline{N})\, \mathrm{d}(\partial\omega_X) - \int_{\partial\omega_X} \frac{\partial F}{\partial T} \underline{\underline{K}} \left(\frac{T}{\rho c}\right)^2 \cdot \frac{\partial}{\partial \underline{X}} \left(\frac{\partial S^*}{\partial T}\right) \cdot \underline{N}\, \mathrm{d}(\partial\omega_X)$$

- **boundary terms** already present before appear through the calculus;
- use of the **Lie derivative**, or of time derivatives on Lagrangian quantities to derive the reversible bracket;
- remark a derivative of f in terms of the gradient of transformations, which appears naturally within the calculus, through transformations using

$$\underline{\underline{\Pi}} = \underline{\underline{F}} \cdot \underline{\underline{S}}$$

where $\underline{\underline{S}}$ is the second Piola Kirchhoff stress tensor.

Annex – Large strain Thermo-visco-elastodynamics – Details of the brackets

The **reversible bracket** is given by

$$\{\mathscr{F}, \hat{\mathscr{E}}^*_{\text{tot}}\} = \int_0^L \left(\frac{\partial F}{\partial \underline{\chi}} \cdot \frac{\partial \hat{E}^*_{\text{tot}}}{\partial \underline{p}} - \frac{\partial F}{\partial \underline{p}} \cdot \frac{\partial \hat{E}^*_{\text{tot}}}{\partial \underline{\chi}} + \underline{\underline{\nabla}}_{\underline{X}} \frac{\partial F}{\partial \underline{p}} : \frac{\partial \hat{E}^*_{\text{tot}}}{\partial \underline{\underline{\Pi}}} \underline{\underline{S}} - \frac{\partial F}{\partial \underline{\underline{\Pi}}} : \underline{\underline{\nabla}}_{\underline{X}} \frac{\partial \hat{E}^*_{\text{tot}}}{\partial \underline{p}} \underline{\underline{S}} + \right.$$

$$\left. - \frac{\partial \hat{E}^*_{\text{tot}}}{\partial T} \left(\frac{\partial S^*}{\partial T} \right)^{-1} \underline{\underline{\nabla}}_{\underline{X}} \frac{\partial F}{\partial \underline{p}} : \frac{\partial S^*}{\partial \underline{\underline{\Pi}}} \underline{\underline{S}} + \frac{\partial F}{\partial T} \left(\frac{\partial S^*}{\partial T} \right)^{-1} \frac{\partial S^*}{\partial \underline{\underline{\Pi}}} : \underline{\underline{\nabla}}_{\underline{X}} \frac{\partial \hat{E}^*_{\text{tot}}}{\partial \underline{p}} \underline{\underline{S}} \right) \mathrm{d}X$$

- The bracket is skew-symmetric if the second Piola-Kirchhoff stress tensor is symmetric $\underline{\underline{S}} = \underline{\underline{S}}^{\mathsf{T}}$
- Fulfills the Jacobi identity for functionals f satisfying

$$\frac{\partial^2 F}{\partial \underline{\chi} \partial \underline{p}} = \frac{\partial^2 F}{\partial \underline{p} \partial \underline{\chi}} = \underline{\underline{0}} \qquad \frac{\partial^2 F}{\partial \underline{\chi} \partial \underline{\underline{\Pi}}} = \frac{\partial^2 F}{\partial \underline{\underline{\Pi}} \partial \underline{\chi}} = \underline{\underline{0}} \qquad \frac{\partial^2 F}{\partial \underline{p} \partial \underline{\underline{\Pi}}} = \frac{\partial^2 F}{\partial \underline{\underline{\Pi}} \partial \underline{p}} = \underline{\underline{0}}$$

$$\frac{\partial^2 F}{\partial T \partial \underline{p}} = \frac{\partial^2 F}{\partial \underline{p} \partial T} = \underline{0} \qquad \frac{\partial^2 F}{\partial T \partial \underline{\chi}} = \frac{\partial^2 F}{\partial \underline{\chi} \partial T} = \underline{0} \qquad \frac{\partial^3 F}{\partial \underline{X} \partial \underline{p} \partial \underline{p}} = \frac{\partial}{\partial \underline{X}} \left(\frac{\partial^2 F}{\partial \underline{p} \partial \underline{p}} \right) = \underline{\underline{0}}$$

consistent with the expression of the total energy of large strain thermo-elastodynamics

$$\hat{\mathscr{E}}_{\text{tot}} = \int_0^L \hat{E}_{\text{tot}} \, \mathrm{d}X = \int_0^L \left(\frac{1}{2} \frac{1}{\rho} \underline{p} \cdot \underline{p} + w^* (\underline{\underline{\Pi}}, T) - \underline{F}^{\text{ext}} \cdot \underline{\chi} \right) \mathrm{d}X$$

Annex – Large strain Thermo-visco-elastodynamics – Details of the brackets

The **dissipative bracket** is given by (Schiebl and Betsch 2021)

$$
\begin{aligned}
(\mathscr{F},\mathscr{S}^*) = \int_0^L \Bigg[& \frac{\partial}{\partial \underline{X}}\left(\frac{\partial F}{\partial T}\right)\cdot \underline{\underline{K}}\left(\frac{T}{\rho c}\right)^2 \cdot \frac{\partial}{\partial \underline{X}}\left(\frac{\partial S^*}{\partial T}\right) + \frac{\partial F}{\partial T}\frac{4}{(\rho c)^2}\left(\frac{\partial \hat{E}^*_{\mathsf{int}}}{\partial \underline{\underline{C}}_{\mathsf{i}}^{-1}}\cdot \underline{\underline{C}}_{\mathsf{i}}^{-1}\right) : T\underline{\underline{\underline{\underline{N}}}} : \left(\frac{\partial \hat{E}^*_{\mathsf{int}}}{\partial \underline{\underline{C}}_{\mathsf{i}}^{-1}}\cdot \underline{\underline{C}}_{\mathsf{i}}^{-1}\right)\frac{\partial S^*}{\partial T} + \\
& - \frac{\partial F}{\partial T}\frac{4}{\rho c}\left(\frac{\partial S^*}{\partial \underline{\underline{C}}_{\mathsf{i}}^{-1}}\cdot \underline{\underline{C}}_{\mathsf{i}}^{-1}\right) : T\underline{\underline{\underline{\underline{N}}}} : \left(\frac{\partial \hat{E}^*_{\mathsf{int}}}{\partial \underline{\underline{C}}_{\mathsf{i}}^{-1}}\cdot \underline{\underline{C}}_{\mathsf{i}}^{-1}\right) - \frac{\partial S^*}{\partial T}\frac{4}{\rho c}\left(\frac{\partial \hat{E}^*_{\mathsf{int}}}{\partial \underline{\underline{C}}_{\mathsf{i}}^{-1}}\cdot \underline{\underline{C}}_{\mathsf{i}}^{-1}\right) : T\underline{\underline{\underline{\underline{N}}}} : \left(\frac{\partial F}{\partial \underline{\underline{C}}_{\mathsf{i}}^{-1}}\cdot \underline{\underline{C}}_{\mathsf{i}}^{-1}\right) + \\
& + 4\frac{1}{\rho^2}\left(\frac{\partial S^*}{\partial \underline{\underline{C}}_{\mathsf{i}}^{-1}}\cdot \underline{\underline{C}}_{\mathsf{i}}^{-1}\right) : T\underline{\underline{\underline{\underline{N}}}} : \left(\frac{\partial F}{\partial \underline{\underline{C}}_{\mathsf{i}}^{-1}}\cdot \underline{\underline{C}}_{\mathsf{i}}^{-1}\right)\Bigg]\,\mathrm{d}X
\end{aligned}
$$

The bracket is symmetric if the conduction operator $\underline{\underline{K}}$ and the viscous operator $\underline{\underline{\underline{\underline{N}}}}$ are both symmetric

$$\underline{\underline{K}}^{\mathsf{T}} = \underline{\underline{K}} \qquad\qquad \underline{\underline{\underline{\underline{N}}}}^{\mathsf{T}} = \underline{\underline{\underline{\underline{N}}}} \qquad (\Leftrightarrow N_{ABCD} = N_{BADC})$$

The bracket is positive if $\underline{\underline{K}}$, $\underline{\underline{\underline{\underline{N}}}}$ and the temperature are positive

$$\underline{a}\cdot\underline{\underline{K}}\cdot\underline{a} \geq 0 \quad \forall \underline{a} \qquad\qquad \underline{\underline{A}} : \underline{\underline{\underline{\underline{N}}}} : \underline{\underline{A}} \geq 0 \quad \forall \underline{\underline{A}}$$

Annex – Large strain Thermo-visco-elastodynamics – Équations de conservation

The **conservation of linear momentum** can be obtained through

$$\mathscr{F} = \int_0^L \underline{p}\,\mathrm{d}X \Rightarrow (\cdots) \Rightarrow \boxed{\frac{\partial \underline{p}}{\partial t} = -\frac{\partial \hat{E}^*_{\mathrm{tot}}}{\partial \underline{\chi}} + \mathrm{DIV}_{\underline{X}}\,\underline{\underline{\Pi}}}$$

The **conservation of angular momentum** is already fullfilled through the skew-symmetry condition of the reversible bracket

$$\forall \mathscr{F}, \mathscr{G}, \quad \{\mathscr{F}, \mathscr{G}\} = -\{\mathscr{G}, \mathscr{F}\} \Rightarrow \boxed{\underline{\underline{S}} = \underline{\underline{S}}^{\mathsf{T}} (\Leftrightarrow \underline{\underline{\Pi}} \cdot \underline{\underline{F}}^{\mathsf{T}} = \underline{\underline{F}} \cdot \underline{\underline{\Pi}}^{\mathsf{T}})}$$

The **premier principe de la thermodynamique** reads

$$\mathscr{F} = \hat{\mathscr{E}}^*_{\mathrm{tot}} = \int_0^L \left(\frac{1}{2}\frac{1}{\rho}\underline{p} \cdot \underline{p} + \hat{E}^*_{\mathrm{int}}\right)\mathrm{d}X \Rightarrow (\cdots) \Rightarrow \boxed{\rho\frac{\partial \hat{E}^*_{\mathrm{int}}}{\partial t} = \frac{\partial \underline{\underline{F}}}{\partial t} : \underline{\underline{\Pi}} - \mathrm{DIV}_{\underline{X}}\,\underline{Q}}$$

Finally, the **second principle of thermodynamics** is

$$\mathscr{F} = \mathscr{S}^* = \int_0^L s^*\,\mathrm{d}X \Rightarrow (\cdots) \Rightarrow \boxed{\rho\frac{\partial s^*}{\partial t} + \mathrm{DIV}_{\underline{X}}\left(\frac{\underline{Q}}{T}\right) \geq 0 \Leftrightarrow -\frac{1}{T}\underline{Q} \cdot \frac{\partial T}{\partial \underline{X}} - \rho\frac{\partial w}{\partial \underline{\underline{C}}_{\mathrm{i}}^{-1}} : \frac{\partial \underline{\underline{C}}_{\mathrm{i}}^{-1}}{\partial t} \geq 0}$$