BALANCE OF CONFIGURATIONAL FORCES

DEFINITION OF A REFERENCE CONFIGURATION ?

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GDR-DGM, Lyon, June 23th 2026

Present work

We define and formulate the balance of configurational forces

- in Classical Hyperelasticity

Also possible to formulate them

- in Souriau's Variational Relativity (DGK, 2026).

OUTLINE

Outline

4D Formalism for Continuum Mechanics
(Souriau, 1958, 1960, 1964)

Souriau's modeling of perfect matter is inspired by Gauge Theory, where matter fields are described by sections of vector bundles (here trivial).

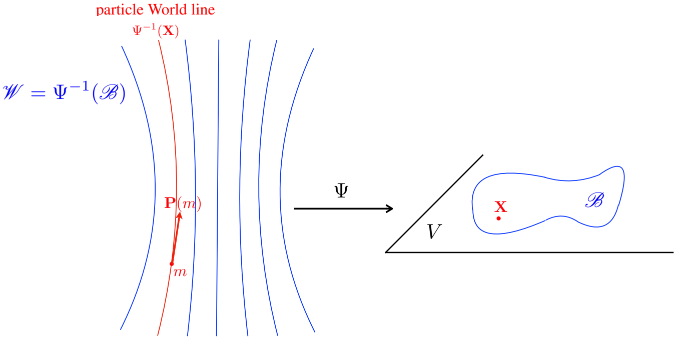
- A perfect matter field is a vector-valued function

$$\Psi : \mathscr{M} \to V = \mathbb{R}^3,$$

  where $\mathscr{M}$, the Universe, is a manifold of dimension 4, equipped with a Lorentzian metric g of signature $(-,+,+,+)$.
- The notation Ψ for the matter field is on purpose,
  as usually used for the wave function in Quantuum Mechanics.

The body $\mathscr{B}$, the mass measure μ and the World tube $\mathscr{W}$

- Matter is described by a compact orientable submanifold with border, of dimension 3, the body $\mathscr{B} \subset \mathbb{R}^3$, which labels the particles.
- The body $\mathscr{B}$ is endowed with a volume form, the mass measure μ (or particle density).
- Ψ is assumed to be a submersion, so that $\mathscr{W} := \Psi^{-1}(\mathscr{B})$ is fibered by the World lines of the particles $\Psi^{-1}(\mathbf{X})$, $\mathbf{X} \in \mathscr{B}$.
- $\mathscr{W}$ is called the matter tube or World tube.



3D FORMALISM : A REVERSE VIEW

- In the classic 3D formalism, a configuration is an embedding

$$p\colon \mathscr{B} \to \mathscr{E}, \qquad \mathbf{X} \mapsto \mathbf{x} = p(\mathbf{X}),$$

of the body $\mathscr{B}$ in (Euclidean) space $\mathscr{E}$.

- In the 4D formalism, a configuration is an application

$$\Psi\colon \mathscr{M} \to \mathscr{B}, \qquad m \mapsto \mathbf{X} = \Psi(m),$$

from the 4D Universe $\mathscr{M}$ to the 3D body $\mathscr{B}$.

An essential difference is that, in classical theory, the embedding p and its differential (the so-called deformation gradient)

$$\mathbf{F} = Tp\colon T\mathscr{B} \to T\mathscr{E}, \qquad \delta\mathbf{X} \mapsto \delta\mathbf{x} = \mathbf{F}\delta\mathbf{X},$$

are invertible, whereas in Variational Relativity Ψ and $T\Psi = \mathrm{d}\Psi$ are not.

Outline

Intrinsic Lagrangian (Noll–Rougée framework)

For first gradient hyperelasticity, the Lagrangian (also called the Action) is

$$\mathscr{L}[\breve{p}] = \int_{t_0}^{t_1} \left(\int_{\mathscr{B}} \ell\left(\mathbf{X}, p(t)(\mathbf{X}), \partial_t p(t)(\mathbf{X}), \mathbf{F}(t)(\mathbf{X})\right) \mu \right) \mathrm{d}t,$$

integrated over the body $\mathscr{B}$

- $\mathscr{C} := \mathrm{Emb}^{\infty}(\mathscr{B}, \mathscr{E})$: configuration space of smooth embeddings $\mathscr{B}$ to $\mathscr{E}$
- $\breve{\kappa} = \breve{p} = p = (p(t))$: path of configurations
- μ : volume form mass measure $\left(m = \int_{\mathscr{B}} \mu \right)$
- ℓ : specific energy

Remark

Conservative body forces can be included in the Lagrangian density (Ball, 1977), but also boundary terms such as prescribed pressure (when non holonomic conditions are satisfied, Sewell, 1965, 1967, Beatty, 1970).

Let $p_0 \in \mathrm{Emb}^\infty(\mathscr{B}, \mathscr{E})$ be a reference configuration and

$$\breve{\phi} = \phi = (\phi(t))$$

the deformation path parameterized by time (from $\Omega_0 \to \Omega_{p(t)}$),

$$\phi(t) = p(t) \circ p_0^{-1}, \qquad \partial_t p(t) = \partial_t \phi(t) \circ p_0 \qquad \mathbf{F}(t) = T\phi(t).Tp_0 = \mathbf{F}_\phi(t).\mathbf{F}_0,$$

one can recast this integral on the reference configuration.

Intrinsic Lagrangian expressed on reference configuration

$$\begin{aligned}\mathscr{L}[p] &= \int_{t_0}^{t_1} \left(\int_{\Omega_0} (p_0)_* \left[\ell\left(\mathbf{X}, p(t)(\mathbf{X}), \partial_t p(t)(\mathbf{X}), \mathbf{F}_\phi(t).\mathbf{F}_0(\mathbf{X})\right) \mu \right] \right) \mathrm{d}t \\ &= \int_{t_0}^{t_1} \left(\int_{\Omega_0} \ell\left(p_0^{-1}(\mathbf{x}_0), \phi(t)(\mathbf{x}_0), \partial_t \phi(t)(\mathbf{x}_0), \mathbf{F}_\phi(t).\mathbf{F}_0(p_0^{-1}(\mathbf{x}_0))\right) \rho_0 \mathrm{vol}_{\mathbf{q}} \right) \mathrm{d}t,\end{aligned}$$

using mass conservation

$$p_{0*}\mu = \rho_0 \mathrm{vol}_{\mathbf{q}}$$

Reference configuration of Continuum Mechanics ?

- $\partial_t \phi = \boldsymbol{V}$: the Lagrangian velocity
- $T\phi = \mathbf{F}_\phi$: the deformation gradient,

$$L_0\big(\mathbf{x}_0, \phi(t,\mathbf{x}_0), \partial_t\phi(t,\mathbf{x}_0), \mathbf{F}_\phi(t,\mathbf{x}_0)\big)$$
$$:= \rho_0 \ell\big(p_0^{-1}(\mathbf{x}_0), \phi(t)(\mathbf{x}_0), \partial_t\phi(t)(\mathbf{x}_0), \mathbf{F}_\phi(t)\mathbf{F}_0(p_0^{-1}(\mathbf{x}_0))\big),$$

so that

$$\mathscr{L}[p] = \int_{t_0}^{t_1} \left(\int_{\Omega_0} L_0(\mathbf{x}_0, \phi, \partial_t\phi, \mathbf{F}_\phi)\, \mathrm{vol}_{\mathbf{q}} \right) \mathrm{d}t,$$

Rewritten

$$\mathscr{L}[p] = \mathscr{L}_0[\phi; p_0] = \int_{t_0}^{t_1} \left(\int_{\mathscr{B}} L_0\big(p_0, \phi \circ p_0, \partial_t\phi \circ p_0, \mathbf{F}_\phi \circ p_0\big)\, p_0^* \,\mathrm{vol}_{\mathbf{q}} \right) \mathrm{d}t,$$

so that the constitutive tensors are defined on the reference configuration

Outline

Variational extremization of the Lagrangian

For all $\delta\phi, \delta p_0$

$$\delta\mathscr{L}[p] = \delta\mathscr{L}_0[\phi; p_0] = \delta_\phi\mathscr{L}_0 + \delta_{p_0}\mathscr{L}_0 = D_\phi\mathscr{L}_0.\delta\phi + D_{p_0}\mathscr{L}_0.\delta p_0 = 0$$

$$\delta_\phi\mathscr{L}_0 = \int_{t_0}^{t_1}\left(\int_{\Omega_0}\left(\frac{\partial L_0}{\partial\phi} - \operatorname{div}\left(\frac{\partial L_0}{\partial \mathbf{F}_\phi}\right) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L_0}{\partial(\partial_t\phi)}\right)\right)\cdot\delta\phi\,\mathrm{vol}_{\mathbf{q}}\right)\mathrm{d}t$$
$$+ \int_{t_0}^{t_1}\left(\int_{\partial\Omega_0}\left\langle\delta\phi\cdot\frac{\partial L_0}{\partial\mathbf{F}_\phi}, \mathbf{n}_0\right\rangle\mathrm{d}\mathbf{a}_0\right)\mathrm{d}t + \int_{\Omega_0}\left[\frac{\partial L_0}{\partial(\partial_t\phi)}\cdot\delta\phi\right]_{t_0}^{t_1}\mathrm{vol}_{\mathbf{q}},$$

$$\delta_{p_0}\mathscr{L}_0 = \int_{t_0}^{t_1}\left(\int_{\Omega_0}\left(\frac{\partial L_0}{\partial\mathbf{x}_0} + \frac{\partial L_0}{\partial\phi}\cdot\mathbf{F}_\phi + \frac{\partial L_0}{\partial\mathbf{F}_\phi} : T\mathbf{F}_\phi + \frac{\partial L_0}{\partial(\partial_t\phi)}\cdot(\partial_t\mathbf{F}_\phi)\right.\right.$$
$$\left.\left. - \operatorname{div}(L_0\,\mathbf{I}_3^\star)\right)\cdot\delta p_0\circ {p_0}^{-1}\,\mathrm{vol}_{\mathbf{q}}\right)\mathrm{d}t + \int_{t_0}^{t_1}\left(\int_{\partial\Omega_0} L_0\left\langle\delta p_0\circ {p_0}^{-1}, \mathbf{n}_0\right\rangle\mathrm{d}\mathbf{a}_0\right)\mathrm{d}t.$$

- Geometric interpretation of the horizontal variations by Zielonka et al (2008)
- The variation w.r.t. the reference configuration is performed
  at fixed constitutive tensors, defined on the reference configuration system Ω_0.

Euler–Lagrange equation $\delta_\phi \mathscr{L}_0 = 0$

$$\frac{\partial L_0}{\partial \phi} - \operatorname{div}\left(\frac{\partial L_0}{\partial \mathbf{F}_\phi}\right) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L_0}{\partial(\partial_t \phi)}\right) = 0, \qquad \text{on} \quad \Omega_0. \tag{1}$$

Introducing the mechanical quantities

$$\begin{aligned}
\boldsymbol{f} &:= \frac{\partial L_0}{\partial \phi}, && \text{(Exterior force density)},\\
\boldsymbol{p} &:= \frac{\partial L_0}{\partial(\partial_t \phi)} = \frac{\partial L_0}{\partial \boldsymbol{V}}, && \text{(Linear momentum density)},\\
\hat{\mathbf{P}} &:= -\frac{\partial L_0}{\partial T\phi} = -\frac{\partial L_0}{\partial \mathbf{F}_\phi}, && \text{(First Piola-Kirchhoff stress tensor)},\\
\boldsymbol{V} &:= \partial_t \phi, && \text{(Lagrangian velocity)}.
\end{aligned}$$

the Euler–Lagrange equation recasts as the balance of linear momentum

$$\operatorname{div} \hat{\mathbf{P}} + \boldsymbol{f} = \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t}, \qquad \text{on} \quad \Omega_0.$$

Balance equation of Configurational forces

Many ways to derive the balance of configurational forces (Maugin and Trimarco, 1992, Gurtin 1995, Maugin 2002)

Here, derived as a critical point of the Lagrangian $\mathscr{L}_0[\phi; p_0]$ (expressed on the body $\mathscr{B}$ and in which p_0 acts as a parameter),

$$\delta_\phi \mathscr{L}_0 = 0, \quad \text{and} \quad \delta_{p_0} \mathscr{L}_0 = 0.$$

1. First equation leads back to Euler–Lagrange and equilibrium equations

$$\frac{\partial L_0}{\partial \phi} - \operatorname{div}\left(\frac{\partial L_0}{\partial \mathbf{F}_\phi}\right) - \mathrm{d}t\left(\frac{\partial L_0}{\partial(\partial_t \phi)}\right) = 0, \qquad \text{on} \quad \Omega_0.$$

2. Second equation ($\delta_{p_0}\mathscr{L}_0$ with δp_0 vanishing on $\partial\mathscr{B}$) gives

$$\frac{\partial L_0}{\partial \mathbf{x}_0} + \frac{\partial L_0}{\partial \phi} \cdot \mathbf{F}_\phi + \frac{\partial L_0}{\partial \mathbf{F}_\phi} : T\mathbf{F}_\phi + \frac{\partial L_0}{\partial(\partial_t \phi)} \cdot (\partial_t \mathbf{F}_\phi) - \operatorname{div}(L_0\, \mathbf{Id}^\star) = 0.$$

By the formula

$$\frac{\partial L_0}{\partial \mathbf{F}_\phi} : T\mathbf{F}_\phi = \operatorname{div}\left(\mathbf{F}_\phi^\star \cdot \frac{\partial L_0}{\partial \mathbf{F}_\phi}\right) - \operatorname{div}\left(\frac{\partial L_0}{\partial \mathbf{F}_\phi}\right) \cdot \mathbf{F}_\phi,$$

we get

$$\frac{\partial L_0}{\partial(\partial_t\phi)}\cdot(\partial_t\mathbf{F}_\phi)=\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L_0}{\partial(\partial_t\phi)}\cdot\mathbf{F}_\phi\right)-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L_0}{\partial(\partial_t\phi)}\right)\cdot\mathbf{F}_\phi .$$

We have therefore

$$\frac{\partial L_0}{\partial\mathbf{x}_0}+\frac{\partial L_0}{\partial\phi}\cdot\mathbf{F}_\phi+\left\{\operatorname{div}\left(\mathbf{F}_\phi^\star\cdot\frac{\partial L_0}{\partial\mathbf{F}_\phi}\right)-\operatorname{div}\left(\frac{\partial L_0}{\partial\mathbf{F}_\phi}\right)\cdot\mathbf{F}_\phi\right\}$$
$$+\left\{\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L_0}{\partial(\partial_t\phi)}\cdot\mathbf{F}_\phi\right)-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L_0}{\partial(\partial_t\phi)}\right)\cdot\mathbf{F}_\phi\right\}-\operatorname{div}(L_0\,\mathbf{I}_3^\star)=0.$$

This last equation recasts as

$$\frac{\partial L_0}{\partial\mathbf{x}_0}+\left\{\frac{\partial L_0}{\partial\phi}-\operatorname{div}\left(\frac{\partial L_0}{\partial\mathbf{F}_\phi}\right)-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L_0}{\partial(\partial_t\phi)}\right)\right\}\cdot\mathbf{F}_\phi$$
$$+\operatorname{div}\left(\mathbf{F}_\phi^\star\cdot\frac{\partial L_0}{\partial\mathbf{F}_\phi}-L_0\,\mathbf{I}_3^\star\right)+\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L_0}{\partial(\partial_t\phi)}\cdot\mathbf{F}_\phi\right)=0.$$

Making use of Euler–Lagrange equation

$$\frac{\partial L_0}{\partial\mathbf{x}_0}+\operatorname{div}\left(\mathbf{F}_\phi^\star\cdot\frac{\partial L_0}{\partial\mathbf{F}_\phi}-L_0\,\mathbf{I}_3^\star\right)+\mathrm{d}t\left(\frac{\partial L_0}{\partial(\partial_t\phi)}\cdot\mathbf{F}_\phi\right)=0,$$

Introducing linear momentum density $\boldsymbol{p}$ and first Piola–Kirchhoff tensor

$$\hat{\mathbf{P}} = -\frac{\partial L_0}{\partial \mathbf{F}_\phi}$$

we end up with the balance equation

$$\frac{\partial L_0}{\partial \mathbf{x}_0} - \operatorname{div}\left(\mathbf{F}_\phi^\star \cdot \hat{\mathbf{P}} + L_0\, \mathbf{I}_3^\star\right) + \frac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{p} \cdot \mathbf{F}_\phi\right) = 0, \qquad \text{on } \Omega_0$$

which is the configurational forces balance of Continuum Mechanics considered as fundamental by Gurtin (1995)

The second order tensor

$$\hat{\mathbf{B}} := -L_0\, \mathbf{I}_3^\star - \mathbf{F}_\phi^\star \cdot \hat{\mathbf{P}} = -L_0\, \mathbf{I}_3^\star - \hat{\mathbf{M}} \qquad \text{on } \Omega_0$$

is the 3D Eshelby stress tensor, a mixed tensor field $\hat{\mathbf{B}} \colon T^\star\Omega_0 \;\to\; T^\star\Omega_0$, and $\hat{\mathbf{M}} = \mathbf{F}_\phi^\star \cdot \hat{\mathbf{P}} = \mathbf{C}\,\mathbf{S}$ is the Mandel stress tensor.

In particular case of homogeneous Hyperelasticity, the Lagrangian density is

$$L_0(\phi, \boldsymbol{V}, \mathbf{F}_\phi) = K(\boldsymbol{V}) - W(\mathbf{F}_\phi) + \boldsymbol{f} \cdot \phi, \qquad K = \frac{1}{2}\rho_0 \langle \boldsymbol{V}, \boldsymbol{V} \rangle,$$

with K the kinematic energy density and W the hyperelastic energy density.

In homogeneous Hyperelastodynamics

the configurational forces balance takes then the common form

$$\operatorname{div} \hat{\mathbf{B}} + \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{p} \cdot \mathbf{F}_\phi) = 0, \quad \text{where} \quad \begin{cases} \hat{\mathbf{B}} := (W - K)\, \mathbf{I}_3^\star - \mathbf{F}_\phi^\star \cdot \hat{\mathbf{P}}, \\ \hat{\mathbf{P}} = \dfrac{\partial W}{\partial \mathbf{F}_\phi}, \\ \boldsymbol{p} = \rho_0 \boldsymbol{V}^\flat, \end{cases}$$

where $\boldsymbol{V}^\flat = \mathbf{q}\boldsymbol{V}$ is the covariant version of the Lagrangian velocity.

OUTLINE

Aparté

[Trimarco and Maugin (1992)] recover the configurational forces balance using Noether stress–energy tensor (see also Maugin, 2002)

The deformation as feld variable (a submersion, $I, i = 1, 2, 3$)

$$\phi\colon \mathbb{R} \times \Omega_0 \to \mathscr{E}, \qquad ({x_0}^0 = t, {x_0}^I) \mapsto (x^i = \phi^i(t, {x_0}^I)), \qquad ,$$

4D Noether–Eshelby tensor (spacetime Eshelby tensor (Eshelby, 1975)),

$$\hat{\mathbb{B}} := \hat{\mathbb{T}}_\phi = (T\phi)^\star \cdot \frac{\partial L_0}{\partial T\phi} - L_0\, \mathbf{I}_4^\star \qquad \left(\mathbb{B}_\mu{}^\nu = (T\phi)^k{}_\mu \left(\frac{\partial L_0}{\partial T\phi}\right)_k{}^\nu - L_0\, \delta_\mu{}^\nu \right)$$

$$\mathbb{B}_t{}^t = \partial_t\phi^k \frac{\partial L_0}{\partial \partial_t \phi^k} - L_0 = E_0, \qquad \mathbb{B}_t{}^I = \partial_t\phi^k \left(\frac{\partial L_0}{\partial \mathbf{F}_\phi}\right)_k{}^I = -V^k P_k{}^I,$$

$$\mathbb{B}_I{}^t = (\mathbf{F}_\phi)^k{}_I \left(\frac{\partial L_0}{\partial \partial_t \phi^k}\right) = p_k (\mathbf{F}_\phi)^k{}_I, \quad \mathbb{B}_I{}^J = (\mathbf{F}_\phi)^k{}_I \left(\frac{\partial L_0}{\partial \mathbf{F}_\phi}\right)_k{}^J - L_0\, \delta_I{}^J = -(\mathbf{F}_\phi^\star)_I{}^k P_k{}^J - L_0\, \delta_I{}^J = B_I{}^J$$

The relations

$$\left(\operatorname{div}\hat{\mathbb{B}}\right)_t = -\frac{\partial L_0}{\partial t} = 0 \quad \text{and} \quad \left(\operatorname{div}\hat{\mathbb{B}}\right)_I = -\frac{\partial L_0}{\partial {x_0}^I}$$

give back both the energy balance and the configurational forces balance.

General spacetime matrix form (spacetime 4D Eshelby tensor)

$$\hat{\mathbb{B}} = \begin{bmatrix} \dfrac{\partial L_0}{\partial(\partial_t \phi)} \cdot \partial_t \phi - L_0 = ({\mathbb{B}_t}^t) & \partial_t \phi . \dfrac{\partial L_0}{\partial \mathbf{F}_\phi} = ({\mathbb{B}_t}^J) \\ \mathbf{F}_\phi^\star \cdot \dfrac{\partial L_0}{\partial(\partial_t \phi)} = ({\mathbb{B}_I}^t) & \mathbf{F}_\phi^\star \cdot \dfrac{\partial L_0}{\partial \mathbf{F}_\phi} - L_0\, \mathbf{I}_3^\star = ({\mathbb{B}_I}^J) \end{bmatrix} = \begin{bmatrix} E_0 & -\boldsymbol{V} \cdot \hat{\mathbf{P}} \\ \boldsymbol{p} \cdot \mathbf{F}_\phi & \hat{\mathbf{B}} \end{bmatrix},$$

and

$$\frac{\partial L_0}{\partial(t, \mathbf{x}_0)} + \operatorname{div}^{4D} \hat{\mathbb{B}} = 0$$

First line : balance of energy

Second line : balance of configurational forces

CONCLUSION

- A reference configuration in Continuum Mechanics is
  - a reference embedding $p_0 : \mathscr{B} \to \mathscr{E}$
  - a choice of location $\Omega_0 = p_0(\mathscr{B})$ where are defined the constitutive tensors
- Balance of configurational forces by variational equation $\delta_{p_0}\mathscr{L}_0$

$$\delta_{p_0}\mathscr{L}_0 = D_{p_0}\mathscr{L}_0 \cdot \delta p_0 = \text{(balance of configurational forces)} \cdot \delta p_0 = 0$$

- Mix (pullback) of balance of linear momentum and of the existence of an hyperelastic energy density
- Definition very general as soon as $\mathscr{L}_0 = \mathscr{L}_0[\dots, p_0]$
  (extension to Variational Relativity in DGK, 2026[1])

1. R. Desmorat, A. Gravouil, B. Kolev, Classical and relativistic balance of configurational forces (en hommage à Jean Lemaitre), European Journal of Mechanics - A/Solids, 118, 2026