À propos de l'électron relativiste de Jean-Marie Souriau

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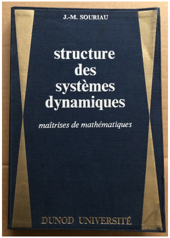
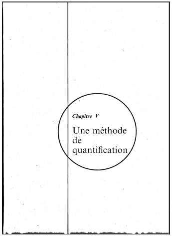
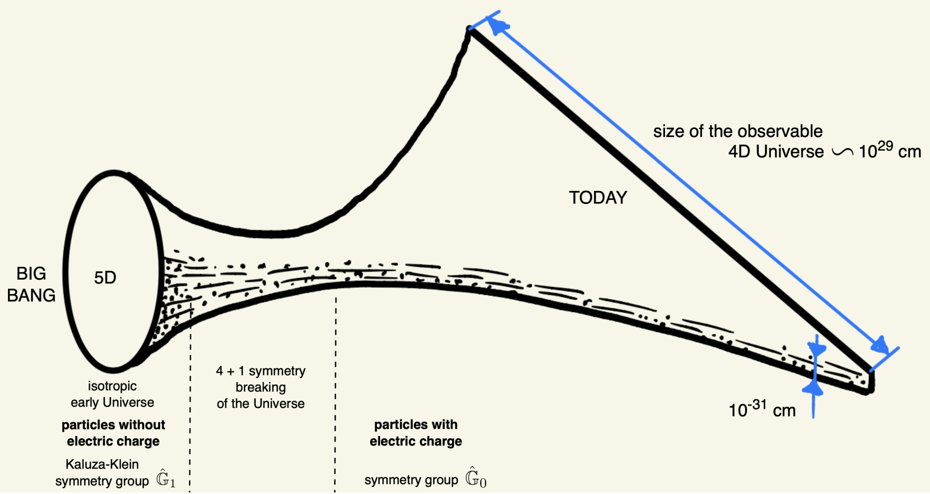


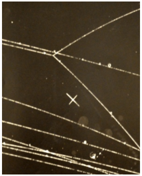


4 + 1 symmetry breaking of the Universe



An elementary particle, a very tiny think

Wave-particle duality



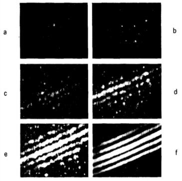


Fig. 1. (a–f) Electron interference fringe patterns filmed from a TV monitor at increasing current densities.

It behaves sometimes as a particle .... sometimes as a wave

Wave equations in natural units $c = \hbar = 1$

Photon		$\Box\,\psi = \partial_t^2 \psi - \Delta\psi = 0$	
Schrödinger electron	$\psi \in \mathbb{C}$	$i\,\partial_t \psi = -\frac{1}{2\,m_0}\,\Delta\psi = \hat{H}\,\psi$	$\leftrightarrow H = \frac{1}{2\,m_0}\ \parallel p \parallel^2$
Pauli electron	$\psi \in \mathbb{C}^2$		
Dirac electron	$\psi \in \mathbb{C}^4$	$i\,\partial_k \psi = \pm\, m_0\,\psi$	$H = \sqrt{\parallel p \parallel^2 + m_0^2}$

History $i\,\partial_t \psi = \hat{H}\,\psi = (\sum_{k=1}^{3} \hat{p}_k \alpha_k + m_0 \beta)\,\psi$ with $\alpha_k, \beta \in \mathbb{C}^{4\times 4}$

Dirac spinors (Souriau, Calcul linéaire)

- The metric $G = \begin{bmatrix} 1_{\mathbb{C}^2} & 0 \\ 0 & -1_{\mathbb{C}^2} \end{bmatrix}$ makes Hermitian the space $\mathbb{C}^4$

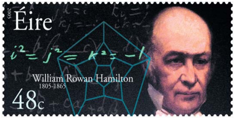
  of **spinors** $\psi = \begin{bmatrix} \psi' \\ \psi'' \end{bmatrix}$, decomposed into **half spinors**

  $\psi' = \begin{bmatrix} z'_1 \\ z'_2 \end{bmatrix}$, $\psi'' = \begin{bmatrix} z''_1 \\ z''_2 \end{bmatrix} \in \mathbb{C}^2$ with the standard metric

  Example : $\psi^\dagger \psi = \bar{\psi}^T G \psi = [\psi'^\dagger \quad -\psi''^\dagger] \begin{bmatrix} \psi' \\ \psi'' \end{bmatrix} = \psi'^\dagger \psi' - \psi''^\dagger \psi" = \mid \psi' \mid^2 - \mid \psi'' \mid^2$

- **Quaternions** $Q \in \mathbb{H} = \mathbb{C}^{2\times 2} \quad \Leftrightarrow \quad Q = y\, 1_{\mathbb{C}^2} + x_1 I + x_2 J + x_3 K, \quad y, x_1, x_2, x_3 \in \mathbb{R}$

  with $I = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}, \qquad J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \qquad K = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}$



- Conjugate $Q^\dagger = \bar{Q} = y\, 1_{\mathbb{C}^2} - x_1 I - x_2 J - x_3 K$ with I, J, K anti-Hermitian

  Pauli matrices $\sigma_1 = i\, I, \quad \sigma_2 = i\, J \quad \sigma_3 = i\, K \quad \Rightarrow \quad \sigma_1, \sigma_2, \sigma_3$ Hermitian

Dirac matrices (Souriau, Calcul linéaire)

- **Quaternionic matrices** $\gamma \in \Gamma = \mathbb{H}^{2\times 2}$
  Canonical decomposition of their space $\Gamma = \Gamma_i \oplus \Gamma_+ \oplus \Gamma_-$

space	elements	dimension	basis
Γ	quaternionic	16	
Γ_i	isotropic	1	$1_{\mathbb{C}^4}$
Γ_+	Hermitian and traceless	5	$\gamma_0, \gamma_1, \gamma_2, \gamma_3, \gamma_5$
Γ_-	anti-Hermitian and traceless	10	$\gamma_i\gamma_5,\ \gamma_i\gamma_j\ (0 \leq i < j \leq 3)$

- **Dirac matrices** $\gamma \in \Gamma_+$

basis : $\gamma_0 = \begin{bmatrix} 1_{\mathbb{C}^2} & 0 \\ 0 & -1_{\mathbb{C}^2} \end{bmatrix}, \quad \gamma_5 = \begin{bmatrix} 0 & -1_{\mathbb{C}^2} \\ 1_{\mathbb{C}^2} & 0 \end{bmatrix}$

$\gamma_1 = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}, \quad \gamma_2 = \begin{bmatrix} 0 & J \\ J & 0 \end{bmatrix}, \quad \gamma_3 = \begin{bmatrix} 0 & K \\ K & 0 \end{bmatrix}$

$\mathbb{R}$-linear and bijective map $\gamma : \mathbb{R}^5 \to \Gamma_+ : X \mapsto \gamma(X) = X^\alpha \gamma_\alpha$

$\mathbb{R}^5$ is endowed with the hyperbolic metric of signature $(+----)$
$X^* X = (X^0)^2 - [(X^1)^2 + (X^2)^2 + (X^3)^2 + (X^5)^2]$
Our hypothesis is that the 5^{th} coordinate y is that of the Kaluza-Klein theory

Properties : $\quad (\gamma(X))^2 = (X^* X)\, 1_{\mathbb{C}^4} \quad \Leftrightarrow \quad \gamma(U)\gamma(V) + \gamma(V)\gamma(U) = 2\,(U^* V)\, 1_{\mathbb{C}^4}$

Spin group (Souriau, Calcul linéaire)

- Unitary matrices $S \in Spin(1,4) \quad \Leftrightarrow \quad S^\dagger S = 1_{\mathbb{C}^4}$
  In "Calcul linéaire", the construction of the spin group is purely algebraic and does not resort to the matrix exponential as in classical textbooks

Theorem (Souriau)

$\diamondsuit \qquad S\,\gamma(V)\,S^\dagger = \gamma(P(V))$ then P is a 5D rotation : $P \in \mathbb{O}(1,4)$

$\mathbb{O}(1,4)$ is a linear representation of $Spin(1,4)$
but, conversely, $S = S(P)$ is defined only up to a factor z root of unity,

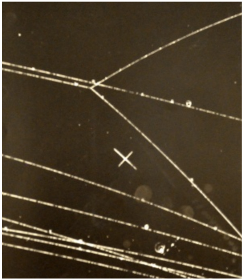
$$\heartsuit \qquad S = \begin{bmatrix} \sqrt{1+\mid H\mid^2}\,1_{\mathbb{C}^2} & \bar{H} \\ H & \sqrt{1+\mid H\mid^2}\,1_{\mathbb{C}^2} \end{bmatrix} \begin{bmatrix} F & 0 \\ 0 & G \end{bmatrix}, \qquad \mid F\mid = \mid G\mid = 1$$

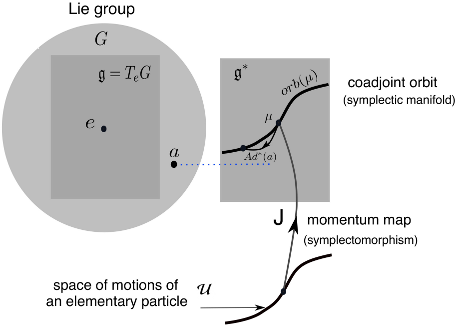
- **4D reduction :** space-time $\mathbb{R}^4$ equipped with the Minkowski metric

$$X^*X = (X^0)^2 - [(X^1)^2 + (X^2)^2 + (X^3)^2]$$

$$S = \underbrace{\begin{bmatrix} \sqrt{1+\mid H\mid^2}\,1_{\mathbb{C}^2} & -H \\ H & \sqrt{1+\mid H\mid^2}\,1_{\mathbb{C}^2} \end{bmatrix}}_{\text{generates a boost}} \underbrace{\begin{bmatrix} F & 0 \\ 0 & F \end{bmatrix}}_{\text{generates a rotation}} \qquad P_S \text{ Lorentz transfo.}$$

The coadjoint orbit method (Souriau, Structure des Systèmes Dynamiques)





classifying the coadjoint orbits,
you classify the elementary particles of physicists

Application of the coadjoint orbit method to the electron

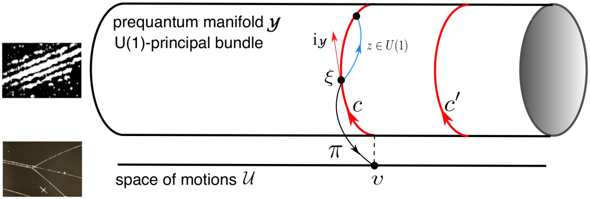
- $\dim(G) - \dim(\mathrm{orb}(\mu)) = 10 - 8 = 2$ invariants :
  the spin $s > 0$ and the rest mass $m_0 > 0$ (or the energy $e = m_0\, c^2 = m_0$)

electron	classical	relativistic	
G	Galilei group	Poincaré group	
linear momentum	$p = m_0 v$	$\Pi = \begin{bmatrix} m \\ p \end{bmatrix} = m_0 U = m_0 \begin{bmatrix} \gamma_u \\ \gamma_u u \end{bmatrix}$	$U^* U = 1$
		with $\gamma_u = 1/\sqrt{1 - \parallel u \parallel^2}$	
angular momentum	$l = x \times m_0 v + s\, n$	$M = m_0\,(X\, U^* - U\, X^*) + s\,\Omega$	$J^* J = -1$
	$\parallel n \parallel = 1$	with $\Omega = \mathcal{J}(U, J)$	$J^* U = 0$

- symplectic form $\boxed{\omega_{\mathcal{U}}(\delta\mu, d\mu) = m_0(\delta X^* dU - dX^* \delta U) - s\, Tr(\delta\Omega\, \Omega\, d\Omega)}$

Other orbits : photon, neutrino, ...

Geometric prequantization



The prequantum manifold $\mathcal{Y}$ is endowed with a **contact and symplectic structure** :
α is a 1-form and $\omega_{\mathcal{Y}} = \mathrm{d}\alpha$ such that

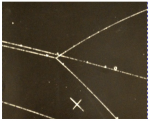
- ♢ $\dim(\ker(\mathrm{d}\alpha)) = 1,$ ♠ $\dim(\ker(\alpha) \cap \ker(\mathrm{d}\alpha)) = 0$
- ♣ $\forall \delta\xi, \quad \omega_{\mathcal{Y}}(\mathrm{i}_{\mathcal{Y}}, \delta\xi) = 0$

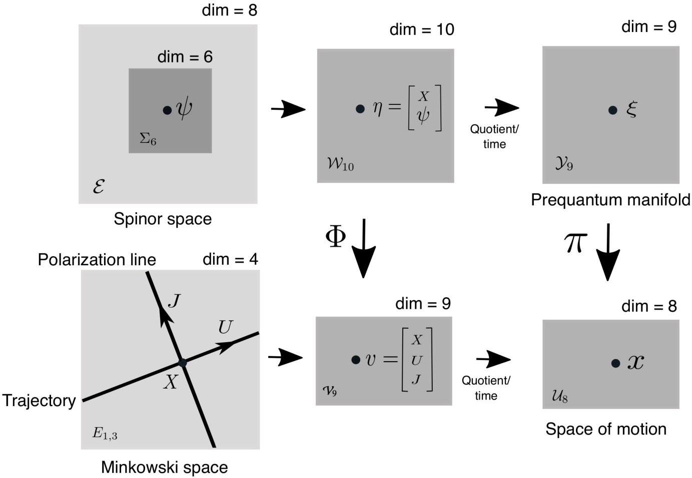
Theorem

Let c and c' be two orbits of U(1) on the prequantum manifold $\mathcal{Y}$.
Then the action integral is conserved $\forall c, c'$ $\quad \int_c \alpha(d\xi) = \int_{c'} \alpha(d\xi)$

♡ **Sommerfeld quantization condition** $\quad \boxed{\int_c \alpha(d\xi) = 2\,\pi\, n, \quad n \in \mathbb{N}}$

Prequantization of the relativistic electron





Prequantization of the relativistic electron

space	basis
$\Gamma = \Gamma_i \oplus \Gamma_+ \oplus \Gamma_-$	
Γ_i	$1_{\mathbb{C}^4}$
Γ_+	$\gamma_0, \gamma_1, \gamma_2, \gamma_3, \gamma_5$
Γ_-	$\gamma_i\gamma_5,\ \gamma_i\gamma_j\ (0 \leq i < j \leq 3)$

- **Step 1 : normalization condition** $\psi^\dagger 1_{\mathbb{C}^4} \psi = \psi^\dagger\psi = \mid \psi' \mid^2 - \mid \psi'' \mid^2 = \epsilon, \quad \epsilon = \pm 1$
  $\Sigma_6 = \left\{ \psi \in \mathbb{C}^4 \quad \text{such that} \quad \psi^\dagger\psi = \epsilon \quad \text{and} \quad \psi^\dagger \gamma_5 \psi = 0 \right\}$ decomposed into
  $\Sigma_+ = \left\{ \psi \in \Sigma_6 \ \text{such that} \ \psi^\dagger\psi = 1 \right\}, \qquad \Sigma_- = \left\{ \psi \in \Sigma_6 \ \text{such that} \ \psi^\dagger\psi = -1 \right\}$
  Remark : J.-M. Souriau considers only Σ_+

- **Step 2 :** $\gamma^k = G^{km}\gamma_m$ where G is the Minkowski metric
  $\boxed{I^k = \psi^\dagger \gamma^k \psi}$ $(k = 0, \dots, 3)$, components of a 4-vector I

  U **timelike, futur-directed** such that $U^* U = 1$ (as the 4-velocity of the electron)
  Let ψ be an **eigenvector** of $\gamma(U) = \gamma_k U^k$, of **eigenvalue** $\epsilon_U = \pm U^* U = \pm 1$
  U futur-directed $\Rightarrow\ 0 < \psi^\dagger \gamma(U)\, \psi = \epsilon_U\, \psi^\dagger\psi = \epsilon_U \epsilon \ \Rightarrow\ \epsilon_U = \epsilon$

- **Step 3 :** $\boxed{J^k = i\, \psi^\dagger \gamma^k \gamma_5\, \psi}$, $(k = 0, \dots, 3)$, components of a 4-vector J

Prequantization of the relativistic electron

Theorem

If U is timelike futur-directed such that $U^* U = 1$, there is equivalence between $\diamondsuit$ and $\heartsuit$:

$\diamondsuit$ $\exists \psi \in \Sigma_6$, eigenvector of $\gamma(U)$ s.t. I, J are defined by
$I^k = \psi^\dagger \gamma^k \psi$ (1), $J^k = i\, \psi^\dagger \gamma^k \gamma_5\, \psi$ (2) with $k = 0, \ldots, 3$

$\heartsuit$ The 4-vectors I and J verify $I = U$, $U^* U = 1$, $J^* J = -1$, $J^* U = 0$

Proof : $\diamondsuit \Rightarrow \heartsuit$: $\psi, U = \begin{bmatrix} \gamma_u \\ \gamma_u u \end{bmatrix}$ being given, boil down to a simple case :

Apply the Lorentz boost P of velocity u s.t. $\tilde{U} = P^{-1} U = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

then calculate $\tilde{J} = P^{-1} J$, $\tilde{\psi} = S(P^{-1})\psi$

(1) $\forall dX \in \mathbb{R}^4$, $I^* dX = \psi^\dagger \gamma(dX)\, \psi$, next use the **relativistic covariance** :

$\forall d\tilde{X} = P^{-1} dX$, $\tilde{\psi}^\dagger \gamma(d\tilde{X})\, \tilde{\psi} = (S^\dagger \psi)^\dagger (S^\dagger \gamma(P^{-1} dX)\, S)\, (S^\dagger \psi) = \psi^\dagger \gamma(dX)\, \psi$

thus $\gamma(\tilde{U})\, \tilde{\psi} = \gamma_0\, \tilde{\psi} = \epsilon\, \tilde{\psi}$

If $\epsilon = 1$, $\tilde{\psi} = \begin{bmatrix} \psi' \\ 0 \end{bmatrix}$, if $\epsilon = -1$, $\tilde{\psi} = \begin{bmatrix} 0 \\ \psi'' \end{bmatrix}$ $\Rightarrow \left\{ \begin{array}{l} \tilde{\psi}^\dagger \gamma_5\, \tilde{\psi} = 0 \\ \tilde{J} = \begin{bmatrix} 0 \\ n \end{bmatrix}, \ \| n \| = 1 \end{array} \right.$

and we verify easily $\heartsuit$ for $\tilde{U}, \tilde{I}, \tilde{J}$, then for U, I, J.

Remark : The method is **constructive** but there is **no uniqueness of** ψ

Prequantization of the relativistic electron

Observe that $\omega_\Sigma(d\psi, \delta\psi) = -\Im(d\psi^\dagger \delta\psi)$ is a 2-form

Theorem

If $\psi \in \Sigma_6$, we have $\boxed{\frac{1}{4}\, Tr(d\Omega\, \Omega\, \delta\Omega) = -\Im(d\psi^\dagger \delta\psi)}$ with $\Omega = \mathcal{J}(U, J)$

Proof : observing that the identity has the **relativistic covariance** we only prove it for a simple value of ψ

Calculate $\Omega = \begin{bmatrix} 0 & V^T \\ V & j(W) \end{bmatrix}$ with

$$\left\{ \begin{array}{l} V = -2\,\Im(\psi''^\dagger \boldsymbol{\sigma}\, \psi') \times (\psi'^\dagger \boldsymbol{\sigma}\, \psi' + \psi''^\dagger \boldsymbol{\sigma}\, \psi'') \\ W = -(\mid \psi' \mid^2 + \mid \psi'' \mid^2)\,(\psi'^\dagger \boldsymbol{\sigma}\, \psi' + \psi''^\dagger \boldsymbol{\sigma}\, \psi'') - 4\,\Im(\psi'^\dagger \psi'')\,\Im(\psi''^\dagger \boldsymbol{\sigma}\, \psi') \end{array} \right.$$

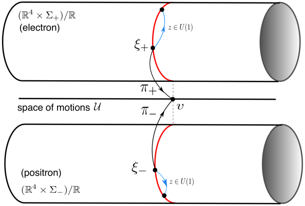
in the simple case

$$\psi = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \ U = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \ J = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix} \ \Rightarrow \ \frac{1}{4}\, Tr(d\Omega\, \Omega\, \delta\Omega) = -\Im(\overline{dz_2'}\, \delta z_2' - \overline{dz_2''}\, \delta z_2'')$$

On the other hand, $-\Im(d\psi^\dagger \delta\psi) = -\Im(\overline{dz_1'}\, \delta z_1' + \overline{dz_2'}\, \delta z_2' - \overline{dz_1''}\, \delta z_1'' - \overline{dz_2''}\, \delta z_2'')$

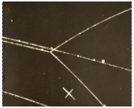
$$\psi \in \Sigma_6 \ \Rightarrow \ -\Im(d\psi^\dagger \delta\psi) = -\Im(\overline{dz_2'}\, \delta z_2' - \overline{dz_2''}\, \delta z_2'')$$

Prequantum manifolds of the electron and the positron



	electron			positron		
spin	$\psi \in \Sigma_+$	$U \in \mathbb{R}^4$	$J \in \mathbb{R}^4$	$\psi \in \Sigma_-$	$U \in \mathbb{R}^4$	$J \in \mathbb{R}^4$
$\downarrow$	$\begin{bmatrix}1\\0\\0\\0\end{bmatrix}$	$\begin{bmatrix}1\\0\\0\\0\end{bmatrix}$	$\begin{bmatrix}0\\0\\0\\-1\end{bmatrix}$	$\begin{bmatrix}0\\0\\1\\0\end{bmatrix}$	$\begin{bmatrix}1\\0\\0\\0\end{bmatrix}$	$\begin{bmatrix}0\\0\\0\\-1\end{bmatrix}$
$\uparrow$	$\begin{bmatrix}0\\1\\0\\0\end{bmatrix}$	$\begin{bmatrix}1\\0\\0\\0\end{bmatrix}$	$\begin{bmatrix}0\\0\\0\\+1\end{bmatrix}$	$\begin{bmatrix}0\\0\\0\\1\end{bmatrix}$	$\begin{bmatrix}1\\0\\0\\0\end{bmatrix}$	$\begin{bmatrix}0\\0\\0\\+1\end{bmatrix}$

Symplectic and contact structure



dim = 9 $\qquad\qquad\qquad\qquad$ dim = 10

$$\bullet\; v = \begin{bmatrix} X \\ U \\ J \end{bmatrix} \quad \mathcal{V}_9 \qquad \xleftarrow{\quad \Phi \quad} \qquad \bullet\; \eta = \begin{bmatrix} X \\ \psi \end{bmatrix} \quad \mathcal{W}_{10}$$

$$\begin{aligned} &\omega_{\mathcal{V}}(\delta\mu, d\mu) = -s\, Tr(\delta\Omega\, \Omega\, d\Omega) \\ &+ m_0(\delta X^* dU - dX^* \delta U) \end{aligned} \quad \Rightarrow \quad \begin{aligned} &\omega_{\mathcal{W}}(\delta\eta, d\eta) = 4\, s\, \Im(d\psi^\dagger \delta\psi) \\ &-2\, m_0\, \Re[\{\delta\psi^\dagger \gamma(dX) - d\psi^\dagger \gamma(\delta X)\}\, \psi] \end{aligned}$$

On this basis, we define the **contact structure**

$$\boxed{\alpha(d\eta) = -2\, i\, s\, \psi^\dagger d\psi - m_0\, \psi^\dagger \gamma(dX)\, \psi} \qquad \text{verifying} \quad \omega_{\mathcal{Y}} = \mathrm{d}\alpha$$

We apply the **Sommerfeld quantization condition** :

$$\int_c \alpha(d\xi) = 2\, i\, s \int_c \psi^\dagger d\psi = 4\, \pi\, s = 2\, \pi\, n \quad \Rightarrow \quad s = \frac{n}{2}, \quad n \in \mathbb{N}$$

For the electron, $\boxed{s = \frac{1}{2}}$

Geometric quantization

- A **state vector** is a field $\xi \mapsto \Psi(\xi)$ on the prequantum manifold $\mathcal{Y}$ with compact support, constant on the U(1)-orbits, then we may apply the measure $d\mu$ of $\mathcal{U}$

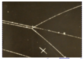
Construction of the wave equation

- **Planck condition** : $\boxed{\mathcal{L}_{\text{grad}\, u_j} \Psi = i\, u_j\, \Psi}$ for observables u_j in involution
- If we take as observables minus the components Π^j of the linear 4-momentum Π the Planck condition gives $\frac{\partial \Psi}{\partial X} = -i\, m_0\, U^*\, \Psi$, of solutions : $\Psi(\xi) = e^{-i\, m_0\, U^* X}\, \psi(U)$
- If $U \mapsto \psi(U)$ is a smooth function on the manifold $\mathcal{M} = \{U \in \mathbb{R}^4$ such that $U^* U = 1$ and U future-directed$\}$

  we define $\tilde{\Psi}(X) = \int_{\mathcal{M}} e^{-i\, m_0\, U^* X}\, \psi\, d\mu$
  where ψ is an eigenvector of $\gamma(U)$ with eigenvalue ϵ

  It verifies the **Dirac equation** $\boxed{i\, \gamma^k \partial_k \tilde{\Psi} - \epsilon\, m_0\, \tilde{\Psi} = 0}$
- Let us define the **probability current** $\tilde{U}$ of components $\tilde{U}^j = \tilde{\Psi}^\dagger \gamma^j \tilde{\Psi}$
- **Particular case** : for the free electron, $d\mu$ is a Dirac measure : $\tilde{\Psi} = e^{-i\, m_0\, U^* X}\, \psi$ then

$$\boxed{\tilde{U} = U}$$



CPT symmetries

The transformation law of a spinor $\psi \in \mathcal{E}$ and $\boldsymbol{\gamma^k} \in End(\mathcal{E})$ are respectively
$\psi = S^{-1}\tilde{\psi}$ and $\gamma^k = S^{-1}\tilde{\gamma}^k\, S$

Through $(\gamma(X))^2 = (X^* X)\, 1_{\mathbb{C}^4}$, there is a correspondance
between γ_k and the coordinate X^k, up to the sign

Correspondance rule

When a symmetry acts on spinors by $S = \gamma_j$ in such way that

$$\gamma^k = \begin{cases} \tilde{\gamma}^k & \text{if } k = j \\ -\tilde{\gamma}^k & \text{otherwise} \end{cases}$$

it acts on the 5D space by

$$X^k = \begin{cases} -\tilde{X}^k & \text{if } k = j \\ \tilde{X}^k & \text{otherwise} \end{cases}$$

CPT symmetries

symmetry		C charge conjugation	P parity	T time reversal
action on spinors	ψ	γ_5	$\gamma_1\gamma_2\gamma_3$	γ_0
action on coordinates	$t =$	$\tilde{t}$	$\tilde{t}$	$-\tilde{t}$
	$x =$	$\tilde{x}$	$-\tilde{x}$	$\tilde{x}$
	$y =$	$-\tilde{y}$	$\tilde{y}$	$\tilde{y}$

- **Our hypothesis is that the 5^{th} coordinate y is that of the Kaluza-Klein theory** in which the electric charge q is the linear momentum with respect to y, then for the corresponding symmetry C, $\tilde{q} = m_0 \frac{d\tilde{y}}{d\tilde{t}} = -m_0 \frac{dy}{dt} = -q$

- For the electron $\tilde{\Psi} \in \Sigma_+$. If it is at rest, the action of C on the spinors gives

$$\Psi = (\gamma_5)^{-1}\,\tilde{\Psi} = \begin{bmatrix} 0 & 1_{\mathbb{C}^2} \\ -1_{\mathbb{C}^2} & 0 \end{bmatrix} \begin{bmatrix} \psi' \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -\psi' \end{bmatrix} \in \Sigma_-$$

  and the wave function $\tilde{\Psi}$ of the electron is transformed into that Ψ of the positron. Because of the relativistic covariance, it is true even if these particles are not at rest.

- We verify that the Dirac equation of the electron $[i\,\gamma^k \partial_k - m_0\, 1_{\mathbb{C}^4}]\,\tilde{\Psi} = 0$ is transformed by C into that of the positron $[i\,\gamma^k \partial_k + m_0\, 1_{\mathbb{C}^4}]\,\Psi = 0$

Conclusions and perspectives

- Souriau's geometric quantization can be considered as the deepening, the improvement of the **wave-particle duality**.

- In Souriau's theory of the electron, there are four key formulae :
  $I^k = \psi^\dagger \gamma^k \psi, \qquad J^k = i\,\psi^\dagger \gamma^k \gamma_5\,\psi, \qquad \frac{1}{4}\,Tr(d\Omega\,\Omega\,\delta\Omega) = -\Im(d\psi^\dagger \delta\psi)$
  and $\qquad \alpha(d\eta) = -2\,i\,s\,\psi^\dagger\,d\psi - m_0\,\psi^\dagger \gamma(dX)\,\psi$

- Dirac claims that his equation models both the **electron** and the **positron**. In our work, they may be associated respectively to the manifolds Σ_+ and Σ_-. Unlike Dirac, these two particles have the same spin and rest mass (or energy, not of opposite sign).

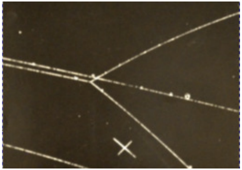
- Our opinion is that this ambiguity in Dirac theory on the sign of the energy results from the absence of the charge in the formalism, ambiguity which might be resolved by introducing the **Kaluza-Klein** 5^{th} dimension of the Universe that allows to identify naturally the charge as an extra invariant of the coadjoint orbit as proposed in :

de Saxcé, G., Which symmetry group for elementary particles with an electric charge today and in the past ? *Journal of Geometry and Physics* 213 (2025) 105491

- For more details, you can read

de Saxcé, G., On Jean-Marie Souriau's geometric quantization of the relativistic electron *arXiv :2606.01121v1* (2026)

Thank you for your attention !



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