On the notion of placement-induced ”lines”

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GDR GDM Lyon, 2026

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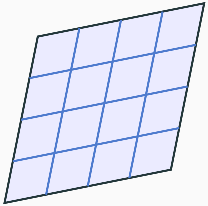
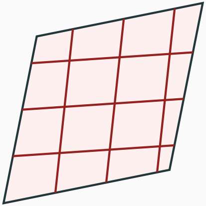
It’s All About Lines

Lines in classical elasticity



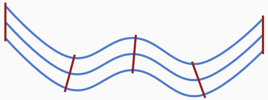
material *A*



material *B*

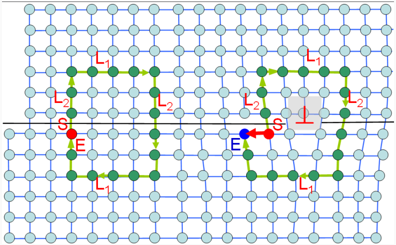
Same outer shape — completely different internal structure

Lines in structural mechanics



- Structures simplified to their **central curve**
- Cross-sections modelled by **lines** connecting them
- Different theories = different modelling of these lines:
  - → **Euler–Bernoulli**: cross-sections perpendicular to axis
  - → **Timoshenko**: cross-sections rotate independently
- Same picture for **plates** and **shells**

Lines in crystallography



- Crystal ≈ **lattice**: a discrete set of points organised in a periodic structure
- Continuous version ≈ **set of lines** following the internal structure
- Deformation of these lines ⟷ **defects**:
  - → **dislocations**: translational irregularity
  - → **disclinations**: rotational irregularity

What Are Lines

Geodesic transport structures

Definition

Let M be a manifold. A **geodesic transport structure** on M consists of:

- an application γ associating to $(p, v) \in TM$ a curve starting at p with tangent v:

$$\begin{aligned} \gamma : \quad \mathrm{T}M &\longrightarrow \mathcal{C}^0([0, \varepsilon], M) \\ (p, v) &\longmapsto \gamma(p, v) \end{aligned}$$

- an application $\overrightarrow{\gamma(p, v)}_s^t$ transporting vectors along γ:

$$\overrightarrow{\gamma(p, v)}_s^t : \mathrm{T}_{\gamma(p,v)(s)}M \longrightarrow \mathrm{T}_{\gamma(p,v)(t)}M$$

satisfying $\gamma(p, v)(0) = p$, $\gamma(p, v)'(0) = v$, $\overrightarrow{\gamma(p, v)}_s^s = \mathrm{Id}$, Chasles' relations, and smoothness.

Euclidean example: $\gamma(p, v)(t) = p + tv$, parallel transport $\overrightarrow{\gamma(p, v)}_s^t(w) = w$.

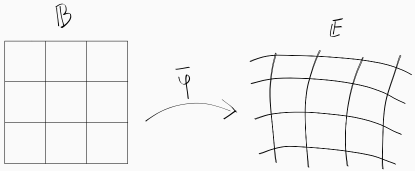
Classical Elasticity

Placement and gradient

- Material body $\mathbb{B}$ placed in Euclidean space $\mathbb{E}$:

$$\overline{\varphi} : \mathbb{B} \longrightarrow \mathbb{E}$$

- $\nabla\overline{\varphi}$ is the **local frame** best approximating the placement
- **Geodesics** = curves whose tangent vectors have **constant coordinates** in the frame $\nabla\overline{\varphi}$



Geodesics in classical elasticity

The grid lines are geodesics: curves $\gamma(p,v)$ whose tangent vectors $\gamma(p,v)'(t)$ have constant coordinates in the frame of $\nabla\overline{\varphi}$. This leads to:

Geodesic PDE

$$\gamma''(p,v) + \nabla_{\gamma(p,v)'(t)}\nabla\overline{\varphi} \cdot (\nabla\overline{\varphi})^{-1} \cdot \gamma(p,v)'(t) = 0$$

These are equivalently:

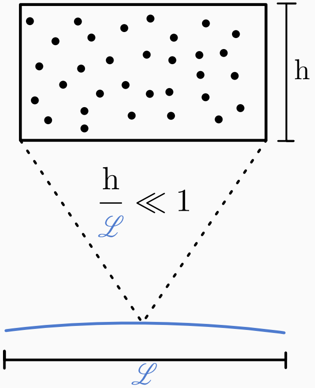
- the geodesics of the **inverse of the left Cauchy–Green tensor**,
- the geodesics of the **Levi-Civita connection** pushed forward by the deformation $\overline{\varphi}$.

$\Rightarrow$ the connection is **torsion-free** and **flat** (since $\nabla\overline{\varphi}$ is a gradient).

Allowing for Dislocations

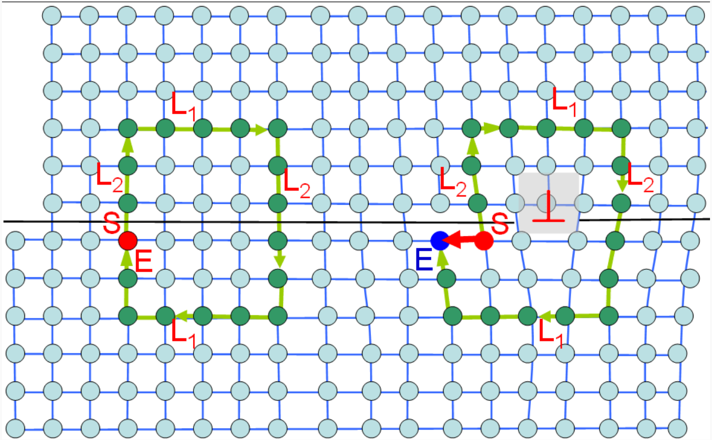
From smooth to generic

- The **actual** placement is discrete and highly irregular
- $\overline{\varphi}$ is only a **smooth approximation** of reality
- The best linear approximation of the real placement is a **generic matrix field** P
- Smooth approx. of linear approx. $\neq$ linear approx. of smooth approx.



Continuous approximation of a micro-structured material

Burgers circuit and dislocations



$$\text{Burgers vector} = \oint_{\partial\Sigma} P\,\mathrm{d}\ell \;\longleftrightarrow\; \operatorname{Curl} P\big|_{\Sigma} \quad (\text{non-zero} \Leftrightarrow \text{dislocation})$$

The two operators do not commute

Let $\mathscr{C}^\infty$ = smooth approx., $\mathscr{L}$ = linear approx.

$$\begin{tikzcd}
& \text{reality} \arrow[dl, "\mathscr{L}"'] \arrow[dr, "\mathscr{C}^\infty"] & \\
\text{lin. approx.} \arrow[dr, "\mathscr{C}^\infty"'] & & \overline{\varphi} \arrow[dl, "\mathscr{L}"] \\
& P \quad \nabla\overline{\varphi} &
\end{tikzcd}$$

- $\nabla\overline{\varphi}$ is **curl-free** (gradient)
- P can have **non-zero curl**
- Non-zero curl $\Leftrightarrow$ **dislocations**

Geodesics with P — Micromorphic model

Geodesic PDE with dislocations

$$\gamma''(p,v) + \nabla_{\gamma(p,v)'(t)} P \cdot P^{-1} \cdot \gamma(p,v)'(t) = 0$$

- P = **deformation of the microscopic neighbourhood** of material points
- Together $(\overline{\varphi}, P)$ define a single map $\varphi : T\mathbb{B} \longrightarrow T\mathbb{E}$
- The geodesics are those of the **push-forward by** φ of the Euclidean Levi-Civita connection
- Non-zero torsion $\longleftrightarrow$ **dislocations**

E. & F. Cosserat, *Théorie des Corps Déformables*, 1909 A.C. Eringen, *Nonlinear theory of simple micro-elastic solids*, 1964
R.D. Mindlin, *Micro-structure in linear elasticity*, 1964 R.A. Toupin, *Elastic materials with couple-stresses*, 1962

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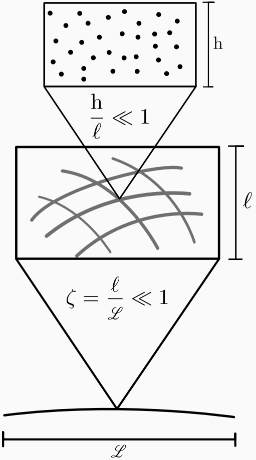

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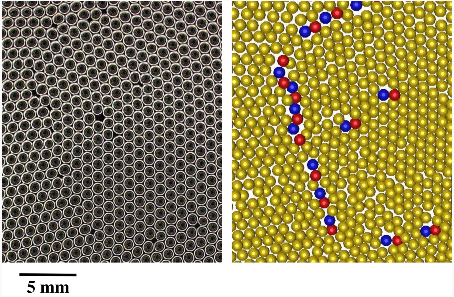
Disclinations



Photography of soap bubbles (left) and the computer reconstruction (right).
red = 5-fold $\left(-\frac{\pi}{3}\text{ rad}\right)$ disclinations,
blue = 7-fold $\left(+\frac{\pi}{3}\text{ rad}\right)$ disclinations,
yellow = standard 6-fold coordinated (0 rad).

M. J. Bowick & L. Giomi, *Two-dimensional matter: order, curvature and defects*, 2009

- **Rotational** counterpart of dislocations
- **Excess or lack of angle** in the lattice
- Parallelogram loop around defect: transport vectors $\Rightarrow$ final angle $\neq$ initial
- **Frank vector** $\longleftrightarrow$ Curl ∇P
- In micromorphic model: Curl $\nabla P \equiv 0$ identically

Introducing N — the second diagram

N is the smooth approx. of the **linear approx. of the rough first-order term**:

$$
\begin{tikzcd}
 & & \text{reality} \arrow[dl, "\mathscr{L}"'] \arrow[dr, "\mathscr{C}^\infty"] & \\
 & \text{lin. approx.} \arrow[dl, "\mathscr{L}"'] \arrow[dr, "\mathscr{C}^\infty"] & & \overline{\varphi} \arrow[d, "\mathscr{L}"'] \\
\text{quad. approx.} \arrow[dr, "\mathscr{C}^\infty"] & & P \arrow[d, "\mathscr{L}"'] & \nabla\overline{\varphi} \\
 & N & \nabla P &
\end{tikzcd}
$$

N is a **third-order tensor field**, usually close to ∇P, but potentially with non-zero Lie-bracket of columns.

Geodesics with N — Non-holonomic model

Geodesic PDE with disclinations

$$\gamma''(p,v) + N_{\gamma(p,v)'(t)} \cdot P^{-1} \cdot \gamma(p,v)'(t) = 0$$

- $(\overline{\varphi}, P, N)$ combine into $\mathbf{F} : \mathrm{TT}\mathbb{B} \longrightarrow \mathrm{TT}\mathbb{E}$ (second-order tangent spaces)
- Geodesics = push-forward by **F** of the Euclidean Levi-Civita connection
- **Curvature tensor** non-trivial $\longleftrightarrow$ disclinations
- N arbitrary $\Rightarrow$ **Ehresmann connections** $\Rightarrow$ generic curves
- Our work: N affine in its index $\Rightarrow$ tractable PDEs

The Two-Scale Model

The complete kinematics

The final model — the **two-scale non-holonomic model** — has three kinematic degrees of freedom:

1. **Macroscopic gradient** $\nabla\overline{\varphi} : \mathbb{B} \longrightarrow GL(\mathbb{R}^3)$
   classical elasticity
2. **Microscopic field** $P : \mathbb{B} \longrightarrow GL(\mathbb{R}^3)$
   Cosserat and micromorphic models
3. **Third-order tensor** $N : \mathbb{B} \times \mathbb{R}^3 \longrightarrow \mathbb{R}^{3\times 3}$
   new

Since all degrees of freedom are kinematical, the **principle of virtual power** applies and yields the **Euler–Lagrange PDEs** governing the motion.

Applications: two scales everywhere

Crystals (macro + micro)

- **Stage 1**: $\nabla\overline{\varphi}$ only $\Rightarrow$ classical elasticity
- **Stage 2**: add $P \Rightarrow$ micromorphic model (dislocations)
- **Stage 3**: add $N \Rightarrow$ dislocations + disclinations + non-metricity

Beams (central curve + 2D cross-sections)

- **Stage 1** (small def.): Euler–Bernoulli beam
- **Stage 2** (small def.): Timoshenko beam
- **Stage 3** (small def.): new beam model — not found in the literature
- Same pattern for large deformations

Plates and shells (2D macro + 1D thickness)

- Same three-stage hierarchy

These match in **kinematics** and in **governing Euler–Lagrange PDEs**.

Conclusion

There are many other contexts where **lines are important**:

past and current works:

- **Generic kD two-scale model**: MC, G. Casale & L. Le Marrec, *Two-Scale Geometric Modelling for Defective Media*, 2024
- **3D materials, Crystals and defects**: submitted preprint, a bibliographic comparison between Cauchy, Cosserat, Eringen, Le & Stumpf, Kröner and the presented model.
- **Beams**: MC, G. Casale & L. Le Marrec, *A New Framework for Unidimensional Structures Based on Generalised Continua*, 2025; with other ongoing collaborations
- **Space-time**: general relativity uses such geodesics that do not come from such a $\mathbf{F} : \mathrm{TT}\mathbb{B} \rightarrow \mathrm{TT}\mathbb{E}$. It is an ongoing work

some other potential applications (not planned):

- **Plates and shells**: there, curvature could play a significant role
- **Time**: processes with two characteristic speeds could be interesting
- **Fluids**: vorticity lines have similar properties but do not necessarily have an associated Lagrangian placement
- **Astronomy**: galaxies exhibit different spatial and temporal scales