Feedback stabilization of a surface tension system modeling the motion of a two-dimensional soap bubble

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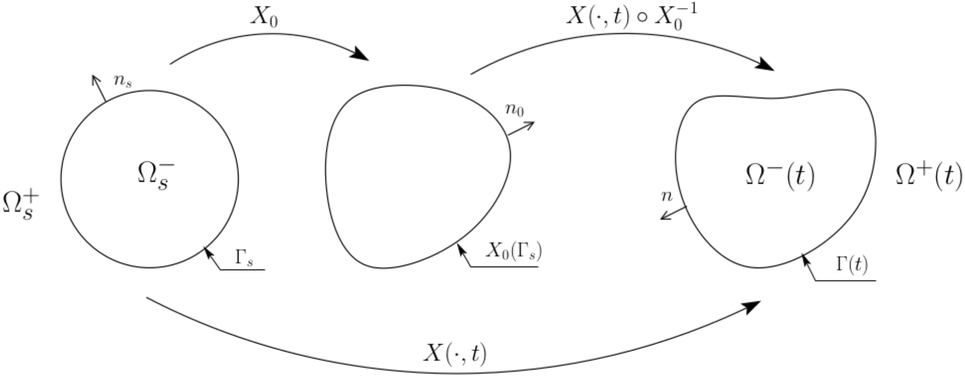
Presentation



Main system

Stokes system in $\Omega^+(t)$ and $\Omega^-(t)$ with (u^+, p^+, u^-, p^-, X) as unknown:

$$
\begin{aligned}
-\operatorname{div}\sigma(u^+, p^+) &= f^+ && \text{in } \Omega^+(t),\ t \in (0,\infty),\\
-\operatorname{div}\sigma(u^-, p^-) &= f^- && \text{in } \Omega^-(t),\ t \in (0,\infty),\\
\operatorname{div} u^+ &= 0 && \text{in } \Omega^+(t),\ t \in (0,\infty),\\
\operatorname{div} u^- &= 0 && \text{in } \Omega^-(t),\ t \in (0,\infty),\\
u^+ &= 0 && \text{on } \partial\Omega \times (0,\infty),\\
u^+ = u^- &= \frac{\partial X}{\partial t}\left(X(\cdot,t)^{-1}, t\right), && \text{on } \Gamma(t),\ t \in (0,\infty),\\
\sigma(u^+, p^+)n^+ + \sigma(u^-, p^-)n^- &= \mu\kappa n^- + g && \text{on } \Gamma(t), t \in (0,\infty),\\
X(\cdot, 0) &= X_0 && \text{on } \Gamma_s,
\end{aligned}
$$

The interface $\Gamma(t) = X(\Gamma_0, t)$ splits Ω into $\Omega^+(t)$ and $\Omega^-(t)$.

A couple system with jump conditions

Cauchy stress tensor: $\sigma(u,p) = 2\nu\varepsilon(u) - p\mathrm{Id} = \nu(\nabla u + \nabla u^T) - p\mathrm{Id}$

$$
\begin{aligned}
-\operatorname{div}\sigma(u^\pm,p^\pm) = f^\pm \quad \text{and} \quad \operatorname{div} u^\pm &= 0 && \text{in } \Omega^\pm(t),\ t\in(0,\infty),\\
u^+ &= 0 && \text{on } \partial\Omega\times(0,\infty),\\
u^\pm = \frac{\partial X}{\partial t}\left(X(\cdot,t)^{-1},t\right) \quad \text{and} \quad -[\sigma(u,p)]\,n &= \mu\kappa n + g && \text{on } \Gamma(t),\ t\in(0,\infty),\\
X(\cdot,0) &= X_0 && \text{on } \Gamma_s.
\end{aligned}
$$

- κ, the mean curvature of $\Gamma(t)$, satisfies: $\kappa n = \Delta_{\Gamma(t)}\mathrm{Id}$ on $\Gamma(t)$.
- Jump of φ across $\Gamma(t)$: $[\varphi] := \varphi^+ - \varphi^-$
- Energy estimate:

$$
\mu\frac{\mathrm{d}}{\mathrm{d}t}|\Gamma(t)| + 2\nu\left(\|\varepsilon(u^+)\|^2_{\mathbb{L}^2(\Omega^+(t))} + \|\varepsilon(u^-)\|^2_{\mathbb{L}^2(\Omega^-(t))}\right) = \langle f^\pm, u^\pm\rangle_{\mathrm{L}^2(\Omega^\pm(t))} + \langle g, u^\pm\rangle_{\mathrm{L}^2(\Gamma(t))}.
$$

References

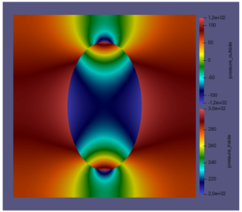
- **Rivkind** et al., 1971–1979: Motion of a drop in a viscous incompressible fluid.
- **Denisova & Solonnikov**, 1989 – 1995, 2021: Wellposedness results, local-in-time.

- **Prüss, Simonett** et al., 2009–2016: Existence results with L^p-maximal regularity, phase transitions.
- **Coutand & Shkoller**, 2007, 2013: Euler system, high-order energy estimates.
- **Fischer, Hensel**, 2020: Uniqueness of solutions using *varifold* solutions.

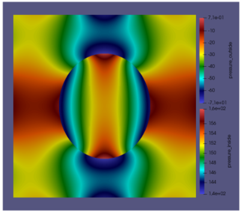
- Simulations: Numerous contributions for addressing the numerical challenges.

Simulations (FEM, fictitious domain approach)

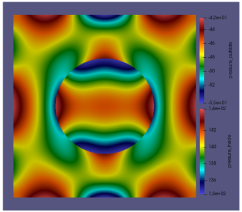
Ellipsoidal deformations, **pressure** field (S. C, J. Comput. Appl. Math, 2019):



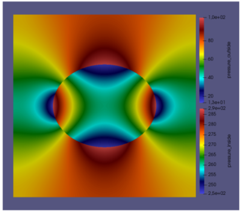
$t = 0.0$



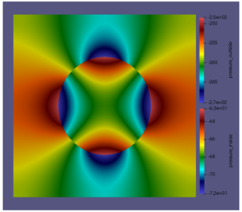
$t = 0.00675$



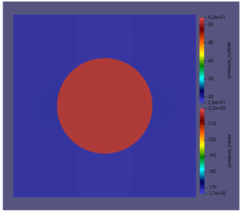
$t = 0.00975$



$t = 0.0135$



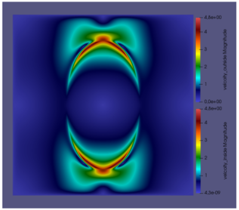
$t = 0.02725$



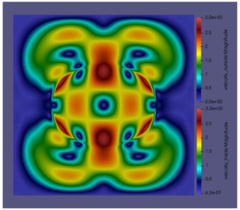
$t = 0.08$

Simulations (FEM, fictitious domain approach)

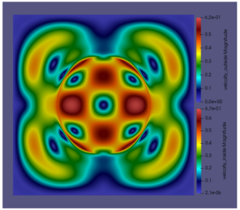
Ellipsoidal deformations, **velocity** field (S. C, J. Comput. Appl. Math, 2019):



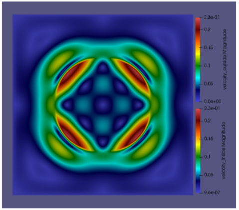
$t = 0.0$



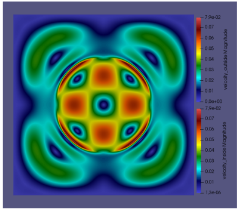
$t = 0.00675$



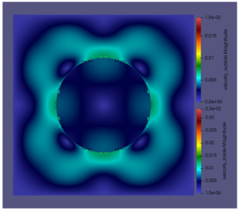
$t = 0.00975$



$t = 0.0135$



$t = 0.02725$



$t = 0.08$

Goal

- For any $\lambda > 0$, find g, a surface tension force, such that

$$\|e^{\lambda t}(X - \mathrm{Id})\|_{\mathrm{L}^\infty(0,\infty;\mathrm{H}^2(\Gamma_s)/\mathbb{R}^2)} \leq C\|X_0 - \mathrm{Id}\|_{\mathrm{H}^2(\Gamma_s)/\mathbb{R}^2}.$$

- Design $g = g_1 + g_2$ in a feedback form:
  - g_1 chosen **explicit** for making the linearized system coercive
  - g_2 of **finite-dimension**, solution of a Riccati equation.
- Instead of Id above, one can choose any smooth diffeomorphism X_c of the circle such that $X_c(\Gamma_s)$ is another circle.
- The stationary states are indeed the mappings X_c, namely $X_c(\Gamma_s)$ is still a circle of the same volume.

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A couple system with jump conditions

Cauchy stress tensor: $\sigma(u,p) = 2\nu\varepsilon(u) - p\mathrm{Id} = \nu(\nabla u + \nabla u^T) - p\mathrm{Id}$

$$\begin{aligned}
-\operatorname{div}\sigma(u^\pm,p^\pm) = f^\pm \quad \text{and} \quad \operatorname{div} u^\pm &= 0 && \text{in } \Omega^\pm(t),\ t\in(0,\infty),\\
u^+ &= 0 && \text{on } \partial\Omega\times(0,\infty),\\
u^\pm = \frac{\partial X}{\partial t}\left(X(\cdot,t)^{-1},t\right) \quad \text{and} \quad -[\sigma(u,p)]\,n &= \mu\kappa n + g && \text{on } \Gamma(t),\ t\in(0,\infty),\\
X(\cdot,0) &= X_0 && \text{on } \Gamma_s.
\end{aligned}$$

- κ, the mean curvature of $\Gamma(t)$, satisfies: $\kappa n = \Delta_{\Gamma(t)}\mathrm{Id}$ on $\Gamma(t)$.
- Jump of φ across $\Gamma(t)$: $[\varphi] := \varphi^+ - \varphi^-$

- System highly coupled with the domains geometry.

Extension of diffeomorphisms

- We need to extend $X(\cdot, t) : \Gamma_s \to \Gamma(t)$ to the disc Ω_s^-.
- Existence of harmonic extensions relies on the Radó-Kneser-Choquet theorem (for diffeomrohpisms of the disc).
- Douady-Earle extension in 2D: Explicit formula.

- Extension of these theorems to the case of diffeomorphisms from the circle to another Jordan curve often requires that the latter enclosed a convex set.
- More recently, **Alessandrini & Nesi**, 2009-2017: Harmonic extension from the disc Ω_s^- to the domain enclosed by $X(\Gamma_s, t) = \Gamma(t)$, provided smoothness of X, that X is orientation-preserving, and that $\mathrm{det} \nabla X > 0$.

$\to \tilde{X}$ diffeomorphism from $\overline{\Omega_s^-}$ onto $\overline{\Omega(t)^-}$, such that $\tilde{X}_{|\Gamma_s} = X$.

Rewriting the main system in cylindrical domains

Using the **Piola** transform:

$$\begin{aligned}
-\operatorname{div}\left(\left(\sigma(u^{\pm},p^{\pm})\circ\tilde{X}\right)\operatorname{cof}(\nabla\tilde{X})\right) = \tilde{f}^{\pm} \quad \text{and} \quad \operatorname{div}\left(\operatorname{cof}(\nabla\tilde{X})^T\tilde{u}\right) &= 0 && \text{in } \Omega_s^{\pm}\times(0,\infty),\\
\tilde{u}^{+} &= 0 && \text{on } \partial\Omega\times(0,\infty),\\
\tilde{u}^{\pm} = \frac{\partial X}{\partial t} \quad \text{and} \quad -\left[(\sigma(u,p)\circ X)\right]\operatorname{cof}\nabla\tilde{X}n_s = \mu|\operatorname{cof}\nabla\tilde{X}n_s|(\Delta_{\Gamma(t)}\mathrm{Id})\circ X + \tilde{g} & && \text{on } \Gamma_s\times(0,\infty),\\
X(\cdot,0) &= X_0 && \text{on } \Gamma_s,
\end{aligned}$$

with

$$(\Delta_{\Gamma(t)}\mathrm{Id})_i\circ X\circ X_s = (\det\mathfrak{g}(t))^{-1/2}\frac{\partial}{\partial\xi_j}\left((\det\mathfrak{g}(t))^{1/2}\frac{\partial(X\circ X_s)_i}{\partial\xi_k}(\mathfrak{g}(t)^{-1})_{kj}\right),$$

$$(\det\mathfrak{g}(t))^{1/2} = |\operatorname{cof}\nabla\tilde{X}n_s|(\det\mathfrak{g}_s)^{1/2},$$

$$|\operatorname{cof}\nabla\tilde{X}n_s|(\Delta_{\Gamma(t)}\mathrm{Id})\circ X \approx \operatorname{div}_{\Gamma_s}\left((n_s\otimes n_s)\nabla_{\Gamma_s}X\right) \qquad (\neq \Delta_{\Gamma_s}X)$$

Linearization

Change of variables:

$$
\begin{aligned}
&\hat{u}^{\pm} := e^{\lambda t}(u^{\pm} \circ \tilde{X}), \quad \hat{p}^{\pm}(y,t) := e^{\lambda t}(p^{\pm} \circ \tilde{X} - p_s^{\pm}), && \text{in } \Omega_s^{\pm} \times (0,\infty),\\
&Z = e^{\lambda t}(X - \mathrm{Id}), && \text{on } \Gamma_s \times (0,\infty)\\
&\hat{f}^{\pm} := e^{\lambda t}(\det \nabla \tilde{X})(f^{\pm} \circ \tilde{X}), && \text{in } \Omega_s^{\pm} \times (0,\infty),\\
&\hat{g} := e^{\lambda t}\left|\mathrm{cof}(\nabla \tilde{X}) n_s\right| (g \circ X), && \text{on } \Gamma_s \times (0,\infty).
\end{aligned}
$$

The linearized system is then:

$$
\begin{aligned}
-\operatorname{div} \sigma(\hat{u}^{\pm}, \hat{p}^{\pm}) = F^{\pm} \quad \text{and} \quad \operatorname{div} \hat{u}^{\pm} &= 0 && \text{in } \Omega_s^{\pm} \times (0,T),\\
\hat{u}^{+} &= 0 && \text{on } \partial\Omega \times (0,T),\\
\hat{u}^{+} = \hat{u}^{-} = \frac{\partial Z}{\partial t} - \lambda Z \quad \text{and} \quad -[\sigma(\hat{u},\hat{p})]\, n_s &= \mu \operatorname{div}_{\Gamma_s} \nabla_{\Gamma_s}^{n_s} Z + G && \text{on } \Gamma_s \times (0,T),\\
Z(\cdot, 0) &= Z_0 && \text{on } \Gamma_s.
\end{aligned}
$$

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The linearized system

For $\lambda > 0$, find $Z \in \mathrm{L}^2(0,T;\mathrm{H}^{5/2}(\Gamma_s)) \cap \mathrm{H}^1(0,T;\mathrm{H}^{3/2}(\Gamma_s))$ such that

$$\begin{aligned}
-\operatorname{div}\sigma(u^\pm,p^\pm) = F^\pm \quad \text{and} \quad \operatorname{div} u^\pm &= 0 && \text{in } \Omega_s^\pm \times (0,T),\\
u^+ &= 0 && \text{on } \partial\Omega \times (0,T),\\
u^+ = u^- = \frac{\partial Z}{\partial t} - \lambda Z \quad \text{and} \quad -[\sigma(u,p)]\, n_s &= \mu \operatorname{div}_{\Gamma_s} \nabla_{\Gamma_s}^{n_s} Z + G && \text{on } \Gamma_s \times (0,T),\\
Z(\cdot,0) &= Z_0 && \text{on } \Gamma_s,
\end{aligned}$$

where the data are assumed to satisfy

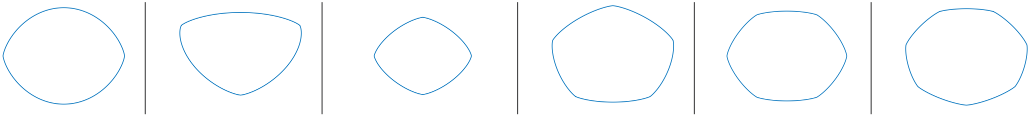
$$F^\pm \in \mathrm{L}^2(0,T;\mathbf{L}^2(\Omega_s^\pm)), \quad G \in \mathrm{L}^2(0,T;\mathbf{H}^{1/2}(\Gamma_s)), \quad Z_0 \in= \mathrm{H}^2(\Gamma_s)/\mathbb{R}^2.$$

Problem: $\nabla_{\Gamma_s}^{n_s} = (n_s \otimes n_s)\nabla_{\Gamma_s}$ is not coercive...

The kernel of $\nabla^{n_s}_{\Gamma_s} = (n_s \otimes n_s)\nabla_{\Gamma_s}$

There are non-trivial mappings $X = Z + \mathrm{Id}$ that are:

- smooth
- orientation-preserving
- volume-preserving
- transforming the circle into a Jordan curve
- that can be chosen arbitrarily close to the identity
- and such that $\nabla^{n_s}_{\Gamma_s} Z = 0$.



First feedback operator - regularization

- In order to get back coercivity, and unique continuation, we add an additional term.
  In stead of considering $\mathrm{div}_{\Gamma_s} \circ \nabla_{\Gamma_s}^{n_s}$, we consider
  $$\mathrm{div}_{\Gamma_s} \nabla_{\Gamma_s}^{n_s} + \mathrm{div}_{\Gamma_s} \nabla_{\Gamma_s}^{\tau_s} = \mathrm{div}_{\Gamma_s} \nabla_{\Gamma_s} = \Delta_{\Gamma_s}$$
  where $\nabla_{\Gamma_s}^{\tau_s} Z := (\tau_s \otimes \tau_s) \nabla_{\Gamma_s} Z$.
- **Rigidity**: If $\Delta_{\Gamma_s} Z = 0$, then Z is a constant.
  The corresponding displacement $X = Z + \mathrm{Id}$ is then a translation.
- **Poincaré/Gårding**-type inequalities:
  $$\begin{aligned} \|Z\|_{\mathrm{H}^1(\Gamma_s)/\mathbb{R}^2} & \leq C \|\nabla_{\Gamma_s} Z\|_{\mathbb{L}^2(\Gamma_s)}, \\ \|Z\|_{\mathrm{H}^2(\Gamma_s)/\mathbb{R}^2} & \leq C \|\Delta_{\Gamma_s} Z\|_{\mathrm{L}^2(\Gamma_s)}. \end{aligned}$$

A Poincaré-Styeklov operator

Let be $G \in \mathbf{H}^{-1/2}(\Gamma_s)$. The Stokes system with jump conditions:

$$\begin{cases} -\operatorname{div}\sigma(u^\pm,p^\pm) = 0 \quad \text{and} \quad \operatorname{div} u^\pm = 0 & \text{in } \Omega_s^\pm, \\ u^+ = 0 & \text{on } \partial\Omega, \\ [u] = 0 \quad \text{and} \quad -[\sigma(u,p)]\, n_s = G & \text{on } \Gamma_s. \end{cases}$$

Goal: Define $\mathcal{P}_{\Gamma_s} : G \mapsto u^\pm_{|\Gamma_s}$.

Method: A weak solution $(u^\pm, p^\pm, \phi = u_{|\Gamma_s})$ is a saddle-point of

$$\begin{aligned} \mathcal{L}(u^+,p^+,u^-,p^-,\lambda^+,\lambda^-,\phi) \quad = \quad & 2\nu\|\varepsilon(u^+)\|^2_{\mathbb{L}^2(\Omega_s^+)} + 2\nu\|\varepsilon(u^-)\|^2_{\mathbb{L}^2(\Omega_s^-)} \\ & -\langle p^+, \operatorname{div} u^+\rangle_{\mathrm{L}^2(\Omega_s^+)} - \langle p^-, \operatorname{div} u^-\rangle_{\mathrm{L}^2(\Omega_s^-)} \\ & -\langle \lambda^+, u^+ - \phi\rangle_{\mathrm{H}^{-1/2}(\Gamma_s):\mathrm{H}^{1/2}(\Gamma_s)} - \langle \lambda^-, u^- - \phi\rangle_{\mathrm{H}^{-1/2}(\Gamma_s):\mathrm{H}^{1/2}(\Gamma_s)} \\ & -\langle G, \phi\rangle_{\mathrm{H}^{-1/2}(\Gamma_s):\mathrm{H}^{1/2}(\Gamma_s)}. \end{aligned}$$

At the optimality: $\lambda^\pm = \sigma(u^\pm,p^\pm)n^\pm, \quad \lambda^+ + \lambda^- + G = 0, \quad$ and $\phi = u_{|\Gamma_s}$.

Definition of $\mathcal{P}_{\Gamma_s}$

A saddle-point of $\mathcal{L}$ satisfies a variational formulation in the form

Denote $\mathfrak{u} = (u^+, p^+, u^-, p^-, \lambda^+, \lambda^-, \phi)$, $\quad \mathfrak{v} = (v^+, q^+, v^-, q^-, \mu^+, \mu^-, \varphi)$

Find $\mathfrak{u} \in \mathfrak{V}$ such that $\quad \mathcal{M}(\mathfrak{u}; \mathfrak{v}) = \mathcal{G}(\mathfrak{v}) \quad \forall \mathfrak{v} \in \mathfrak{V}$,

$$
\text{where} \quad \begin{aligned}
\mathcal{M}(\mathfrak{u}; \mathfrak{v}) \quad := \quad & 2\nu\langle \varepsilon(u^+), \varepsilon(v^+)\rangle_{\mathbb{L}^2(\Omega_s^+)} + 2\nu\langle \varepsilon(u^-) : \varepsilon(v^-)\rangle_{\mathbb{L}^2(\Omega_s^-)} \\
& -\langle p^+, \operatorname{div} v^+\rangle_{\mathrm{L}^2(\Omega_s^+)} - \langle q^+, \operatorname{div} u^+\rangle_{\mathrm{L}^2(\Omega_s^+)} - \langle p^-, \operatorname{div} v^-\rangle_{\mathrm{L}^2(\Omega_s^-)} - \langle q^-, \operatorname{div} u^-\rangle_{\mathrm{L}^2(\Omega_s^-)} \\
& -\langle \lambda^+, v^+ - \varphi\rangle_{\mathbf{W}';\mathbf{W}} - \langle \mu^+, u^+ - \Phi\rangle_{\mathbf{W}';\mathbf{W}} - \langle \lambda^-, v^- - \varphi\rangle_{\mathbf{W}';\mathbf{W}} - \langle \mu^-, u^- - \Phi\rangle_{\mathbf{W}';\mathbf{W}}, \\
\mathcal{G}(\mathfrak{v}) \quad := \quad & \langle G, \varphi\rangle_{\mathbf{W}';\mathbf{W}} .
\end{aligned}
$$

Ladyzhenskaya-Babuška-Brezzi inf-sup condition:

$$
\inf_{\mathfrak{u}\in\mathfrak{V}\setminus\{0\}} \sup_{\mathfrak{v}\in\mathfrak{V}\setminus\{0\}} \frac{\mathcal{M}(\mathfrak{u};\mathfrak{v})}{\|\mathfrak{u}\|_{\mathfrak{V}}\|\mathfrak{v}\|_{\mathfrak{V}}} \quad \geq \quad C.
$$

$\rightarrow \qquad \mathcal{P}_{\Gamma_s} : \mathbf{H}^{-1/2}(\Gamma_s) \ni G \mapsto u^{\pm}_{|\Gamma_s} \in \mathbf{H}^{1/2}(\Gamma_s)$ is well-defined.

Operator formulation

- The operator $\mathcal{P}_{\Gamma_s}$ is symmetric and positive.
- The linear system

$$
\begin{aligned}
-\operatorname{div}\sigma(u^{\pm},p^{\pm}) = 0 \quad \text{and} \quad \operatorname{div} u^{\pm} &= 0 && \text{in } \Omega_s^{\pm}\times(0,T),\\
u^{+} &= 0 && \text{on } \partial\Omega\times(0,T),\\
u^{+} = u^{-} = \frac{\partial Z}{\partial t} - \lambda Z \quad \text{and} \quad -\left[\sigma(u,p)\right] n_s &= \mu\Delta_{\Gamma_s} Z + G && \text{on } \Gamma_s\times(0,T),\\
Z(\cdot,0) &= Z_0 && \text{on } \Gamma_s,
\end{aligned}
$$

is rewritten as

$$
\begin{cases}
\dfrac{\partial Z}{\partial t} - \lambda Z = \mathcal{P}_{\Gamma_s}(\mu\Delta_{\Gamma_s} Z) + \mathcal{P}_{\Gamma_s} G & \text{in } \Gamma_s\times(0,\infty),\\
Z(0) = Z_0 \qquad \text{on } \Gamma_s.
\end{cases}
$$

Operator formulation and fundamental properties

It is more convenient to consider the tangential gradient of this equation:

$$\begin{aligned}
&\frac{\partial \nabla_{\Gamma_s} Z}{\partial t} - \lambda \nabla_{\Gamma_s} Z = \nabla_{\Gamma_s} \mathcal{P}_{\Gamma_s}(\mu \,\mathrm{div}_{\Gamma_s} \nabla_{\Gamma_s} Z) + \nabla_{\Gamma_s} \mathcal{P}_{\Gamma_s} G \qquad \text{in } \Gamma_s \times (0,\infty), \\
&\nabla_{\Gamma_s} Z(0) = \nabla_{\Gamma_s} Z_0 \qquad \text{on } \Gamma_s.
\end{aligned}$$

The operator $\mathcal{A} = \mu \nabla_{\Gamma_s} \circ \mathcal{P}_{\Gamma_s} \circ \mathrm{div}_{\Gamma_s}$ below is self-adjoint, dissipative, and infinitesimal generator of an analytic semigroup of contraction on $\nabla_{\Gamma_s} \mathrm{H}^{5/2}(\Gamma_s)$:

$$\begin{array}{rcll}
\mathcal{A}: & \nabla_{\Gamma_s} \mathrm{H}^{5/2}(\Gamma_s) =: D(\mathcal{A}) & \rightarrow & \nabla_{\Gamma_s} \mathrm{H}^{3/2}(\Gamma_s) \\
& \nabla_{\Gamma_s} Z & \mapsto & \mu \nabla_{\Gamma_s} \mathcal{P}_{\Gamma_s}(\mathrm{div}_{\Gamma_s} \nabla_{\Gamma_s} Z)
\end{array}$$

Moreover, its resolvent is **compact**.

Plan

Approximate controllability

For some control function $G \in \mathcal{G}_T(\Gamma_s)$:

$$\left\{\begin{array}{rcl} -\operatorname{div}\sigma(u^\pm,p^\pm) = 0 \quad \text{and} \quad \operatorname{div} u^\pm = 0 & & \text{in } \Omega_s^\pm \times (0,T), \\ u^+ = 0 & & \text{on } \partial\Omega \times (0,T), \\ u^\pm = \dfrac{\partial Z}{\partial t} \quad \text{and} \quad -[\sigma(u,p)]\,n = \mu\Delta_{\Gamma_s} Z + G & & \text{on } \Gamma_s \times (0,T), \\ Z(\cdot,0) = 0 & & \text{on } \Gamma_s. \end{array}\right. \qquad (*)$$

The reachable set $R(T)$ is dense in $\mathrm{L}^2(\Gamma_s)/\mathbb{R}^2$.

$$R(T) := \left\{ Z(\cdot,T) \text{ such that } Z \text{ is solution of } (*) \mid G \in \mathcal{G}_T(\Gamma_s) \right\}.$$

The set $R(T)^\perp$ is reduced to constants.

Let be $Z_T \in R(T)^\perp$, and let us show that necessarily Z_T is constant.

Unique continuation argument

How do we prove approximate controllabiltiy?
Introduce the adjoint system with unknowns $(\phi^+, \psi^+, \phi^-, \psi^-, \zeta)$:

$$
\begin{array}{rlr}
-\operatorname{div}\sigma(\phi^\pm, \psi^\pm) = 0 \quad \text{and} \quad \operatorname{div}\phi^\pm = 0 & & \text{in } \Omega_s^\pm \times (0,T), \\
\phi^+ = 0 & & \text{on } \partial\Omega \times (0,T), \\
\phi^\pm = -\dfrac{\partial \zeta}{\partial t} \quad \text{and} \quad -[\sigma(\phi,\psi)]\, n_s = \mu \Delta_{\Gamma_s}\zeta & & \text{on } \Gamma_s \times (0,T), \\
\zeta(T) = Z_T & & \text{on } \Gamma_s,
\end{array}
$$

Taking the scalar product of the first equation by $u^\pm$ and integrating gives:

$$
\mu \left\langle \nabla_{\Gamma_s} Z_T, \nabla_{\Gamma_s} Z(T) \right\rangle_{\mathbb{L}^2(\Gamma_s)} = \int_0^T \left\langle G, \frac{\partial \zeta}{\partial t} \right\rangle_{\mathrm{L}^2(\Gamma_s)} \mathrm{d}t.
$$

$$
\Rightarrow \dot{\zeta} \equiv 0 \quad \Rightarrow \phi^\pm \equiv 0 \quad \Rightarrow \quad \mu\Delta_{\Gamma_s}\zeta = [\psi] n_s = 0 \quad \Rightarrow \quad \zeta \equiv \text{constant.}
$$

Back to the operator formulation

Considering the tangential gradient of this equation:

$$\begin{aligned}
&\frac{\partial \nabla_{\Gamma_s} Z}{\partial t} - \lambda \nabla_{\Gamma_s} Z = \nabla_{\Gamma_s}\mathcal{P}_{\Gamma_s}(\mu \operatorname{div}_{\Gamma_s} \nabla_{\Gamma_s} Z) + \nabla_{\Gamma_s}\mathcal{P}_{\Gamma_s} G \qquad \text{in } \Gamma_s \times (0,\infty),\\
&\nabla_{\Gamma_s} Z(0) = \nabla_{\Gamma_s} Z_0 \qquad \text{on } \Gamma_s.
\end{aligned}$$

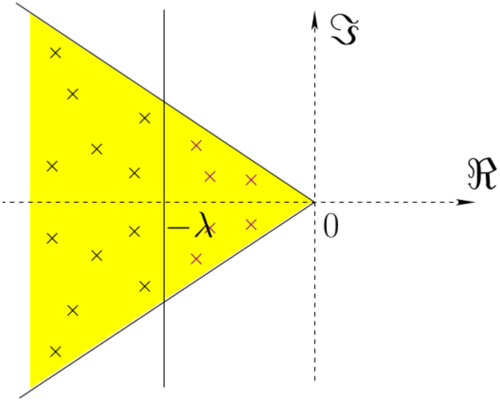
The operator $\mathcal{A} = \mu \nabla_{\Gamma_s} \circ \mathcal{P}_{\Gamma_s} \circ \operatorname{div}_{\Gamma_s}$ below is self-adjoint, dissipative, and infinitesimal generator of an analytic semigroup of contraction on $\nabla_{\Gamma_s}\mathrm{H}^{5/2}(\Gamma_s)$:

$$\begin{aligned}
\mathcal{A}: \quad \nabla_{\Gamma_s}\mathrm{H}^{5/2}(\Gamma_s) =: D(\mathcal{A}) \quad &\to \quad \nabla_{\Gamma_s}\mathrm{H}^{3/2}(\Gamma_s)\\
\nabla_{\Gamma_s} Z \quad &\mapsto \quad \mu \nabla_{\Gamma_s}\mathcal{P}_{\Gamma_s}(\operatorname{div}_{\Gamma_s} \nabla_{\Gamma_s} Z)
\end{aligned}$$

Moreover, its resolvent is **compact**.

The unstable modes are of finite number

The spectrum of $\mathcal{A}$ is sectorial:



Finite-dimensional feedback operator

We define the following finite-dimensional feedback operator

$$\mathcal{K}_\lambda := -\mathcal{P}_{\Gamma_s}(\nabla_{\Gamma_s})^* \Pi_\lambda P_\lambda \nabla_{\Gamma_s} \in \mathcal{L}\left(\mathbf{H}^{5/2}(\Gamma_s), \Xi\right)$$

where $\Pi_\lambda \in \mathcal{L}\left(\mathbb{H}_u^{(\lambda)}, (\mathbb{H}_u^{(\lambda)})^*\right)$ satisfies the Riccati equation

$$\Pi_\lambda = \Pi_\lambda{}^* \succeq 0, \quad \Pi_\lambda \mathcal{A}_\lambda + \mathcal{A}_\lambda^* \Pi_\lambda - \Pi_\lambda \mathcal{B}_\lambda \mathcal{B}_\lambda^* \Pi_\lambda + \mathrm{I} = 0,$$

where

$$\mathcal{A}_\lambda = P_\lambda \mathcal{A} P_\lambda \in \mathcal{L}(\mathbb{H}_u^{(\lambda)}, \mathbb{H}_u^{(\lambda)})$$
$$\mathcal{B}_\lambda = P_\lambda \mathcal{P}_{\Gamma_s} \in \mathcal{L}(\mathbf{H}^{1/2}(\Gamma_s), \mathbb{H}_u^{(\lambda)}),$$

where P_λ is the orthogonal projection on $\mathbb{H}_u^{(\lambda)}$.

Optimal control problem in infinite-time horizon

Where does the feedback operator $\mathcal{K}_\lambda$ come from?

$$\inf_G \left\{ \mathcal{J}(Z,G) \mid \ Z \text{ satisfies } (*) \right\},$$

with

$$\mathcal{J}(Z,G) = \frac{1}{2}\int_0^\infty \|P_\lambda \nabla_{\Gamma_s} Z\|^2_{\mathbb{H}_u^{(\lambda)}} \mathrm{d}t + \frac{1}{2}\int_0^\infty \|G\|^2 \mathrm{d}t.$$

The 1st-order optimality conditions lead to

$$G = -\mathcal{P}^*_{\Gamma_s}(\nabla_{\Gamma_s})^* \Pi_\lambda P_\lambda \nabla_{\Gamma_s} Z = \mathcal{P}_{\Gamma_s} \operatorname{div}_{\Gamma_s} \left(\Pi_\lambda P_\lambda \nabla_{\Gamma_s} Z\right),$$

where $\Pi_\lambda = {\Pi_\lambda}^* \succeq 0$ satisfies the Riccati equation of the previous slide.

A non-homogeneous linear system

Assume $Z_0 \in \mathrm{H}^2(\Gamma_s)/\mathbb{R}^2$, $F^{\pm} \in \mathrm{L}^2(0,\infty;\mathrm{L}^2(\Omega_s^{\pm}))$ and $G \in \mathrm{L}^2(0,\infty;\mathrm{H}^{1/2}(\Gamma_s))$.
There exists a unique solution $Z \in \mathrm{L}^2(0,\infty;\mathrm{H}^{5/2}(\Gamma_s)) \cap \mathrm{H}^1(0,\infty;\mathrm{H}^{3/2}(\Gamma_s))$ to the following system

$$\begin{array}{rlr}
-\operatorname{div}\sigma(u^{\pm},p^{\pm}) = F^{\pm} \quad \text{and} \quad \operatorname{div} u^{\pm} = 0 & & \text{in } \Omega_s^{\pm} \times (0,\infty), \\
u^{+} = 0 & & \text{on } \partial\Omega \times (0,\infty), \\
u^{+} = u^{-} = \dfrac{\partial Z}{\partial t} - \lambda Z \quad \text{and} \quad -[\sigma(u,p)]\, n = \mu \Delta_{\Gamma_s} Z + \mathcal{K}_{\lambda} Z + G & & \text{on } \Gamma_s \times (0,\infty), \\
Z(\cdot,0) = Z_0 & & \text{on } \Gamma_s,
\end{array}$$

and it satisfies

$$\begin{aligned}
& \|u^{+}\|_{\mathcal{U}_\infty(\Omega_s^{+})} + \|p^{+}\|_{\mathcal{Q}_\infty(\Omega_s^{+})} + \|u^{-}\|_{\mathcal{U}_\infty(\Omega_s^{-})} + \|p^{-}\|_{\mathcal{Q}_\infty(\Omega_s^{-})} + \|Z\|_{\mathcal{Z}_\infty(\Gamma_s)} \\
& \leq C \left(\|Z_0\|_{\mathcal{Z}_0(\Gamma_s)} + \|F^{+}\|_{\mathcal{F}_\infty(\Omega_s^{+})} + \|F^{-}\|_{\mathcal{F}_\infty(\Omega_s^{-})} + \|G\|_{\mathcal{G}_\infty(\Gamma_s)} \right),
\end{aligned}$$

Plan

Back to the original system - Summary

- We introduced 2 linear feedback operators:
  - One explicit: $G_1 = \mathrm{div}_{\Gamma_s}\left((\tau_s \otimes \tau_s)\nabla_{\Gamma_s} Z\right)$ with $Z = e^{\lambda t}(X - \mathrm{Id})$.
  - One of finite-dimension: $G_2 = \mathcal{K}_\lambda Z$.
- The resulting operator $G = G_1 + G_2$ stabilizes the linearized system around a given stationary state (a displacement corresponding to another circle).
- This feedback operator also stabilizes the nonlinear system rewritten in cylindrical domain
  (via a fixed-point argument, with **smallness** assumptions).
- For coming back to the original variables in time-dependent domains:

$$g = \left(r(\det \mathfrak{g})^{-1/2} \left(\mathrm{div}_{\Gamma_s}\left((\tau_s \otimes \tau_s)\nabla_{\Gamma_s}(X - \mathrm{Id})\right) + \mathcal{K}_\lambda (X - \mathrm{Id})\right)\right) \circ X^{-1} \qquad \text{on } \Gamma(t).$$

Possible extension to 3D/Navier-Stokes

- NSE: Not clear how an operator formulation could be derived.
- NSE: Approxicmate controllability involves zeros of Bessel functions (2D) or zeros of spherical harmonics (3D) $\rightarrow$ Rummler 2010.
- 3D: The linearized system involving $\nabla_{\Gamma_s}^{n_s}$ was obtained with simplifications that are specific to dimension 2.
- 3D: The mean curvature is only extrinsic $\rightarrow$ stationary states?
  However, Alexandrov 1956 (see 1962 for an English translation).
- 3D: The question of extension of diffeomorphism defined on the sphere is much more delicate.
  - Sufficient condition: The set of diffeomorphisms of this boundary preserving the orientation is connected.
  - Smale 1959, *Diffeomorphisms of the 2-sphere*: Requires C^∞ regularity.

Extension to Navier-Stokes (2D)

Linearized Navier-Stokes:

$$\begin{aligned} \frac{\partial u}{\partial t} - \mathrm{div}(\sigma(u,p)) &= f && \text{in } \Omega^{\pm}, \\ \mathrm{div}\, u &= 0 && \text{in } \Omega^{\pm}, \\ -\left[\sigma(u,p)\right] n_s &= \mu\, \Delta_{\Gamma_s} Z + G && \text{on } \Gamma_s. \end{aligned}$$

Theorem

Assume that the radius $r_s > 0$ of the sphere of reference Γ satisfies

$$r_s \notin \left\{ \sqrt{\frac{\nu^-}{\lambda}} r_0;\ \lambda \in \mathrm{Sp}(A^-),\ r_0 \in J_1^{-1}(\{0\}) \right\},$$

where A^- denotes the Stokes operator in Ω^-. Then system the linear system is approximately controllable.

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Thank you Springer

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- Approximate controllability of a 2D linear system related to the motion of two fluids with surface tension. PAMM 2023.