General covariant formulation of the electro-magneto-mechanics coupling

Towards Piezoelectricity and Magnetoelasticity

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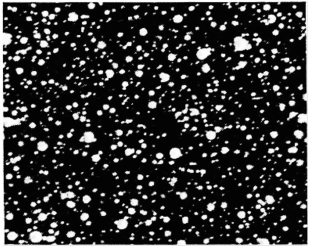
Magnetorheological Elastomers (MRE)

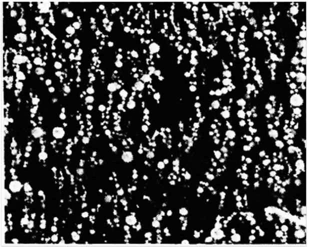
- Depending on the range of magnetostrictive strain:
  - mechanically-hard $\rightarrow 10^{-6} - 10^{-3}$,
  - mechanically-soft $\rightarrow$ up to 10^{-1}.
- $\rightarrow$ Finite strain theory (Polukhov et al. 2021).

Formation of particle chains when curing an MRE in a magnetic field

(Ginder et al. 1999)

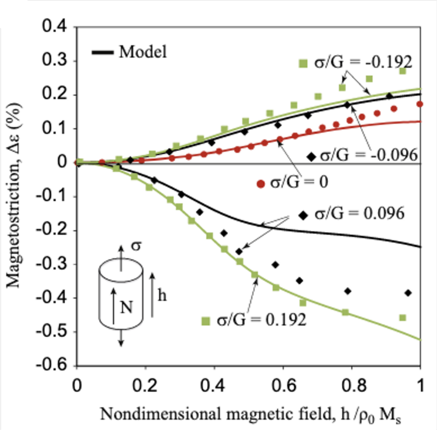


Zero flux density



0.5 T flux density

Effect of uniaxial constant stress on magnetic behavior, for a MRE (Danas et al. 2012)



(G the shear modulus)

Motivation

Obtain 4D conservation equations for non-conducting elastic continuous media.

- General Relativity framework:
  - Aiming at a unifying theory for insulators.
  - General covariance: invariance under local reparametrizations of the Universe (Einstein 1921).
    - **The Lagrangian $\mathscr{L}$ has to be general covariant.**
    - Used to determine suitable arguments for the coupling density L.
  - Definition of the stress-energy tensor in the sense of Hilbert (Hilbert 1915)
    - In presence of matter, generalization of the 3D stress tensor.
    - Symmetric by this definition, but other definitions found in literature (Gotay et al. 1992).
- After obtention of 4D equations, separation of space and time and obtention of 3D conservation.

VARIATIONAL RELATIVITY (SOURIAU 1958)

- Gravitation and inertia (g), electromagnetism ($\boldsymbol{\mathcal{A}}$), matter (Ψ) (hyperelasticity) and their coupling are described by fields defined on a 4-dimensional Universe $\mathscr{M}$.
- Lagrangian $\mathscr{L}$ depending on these fields:

$$\mathscr{L}[\underbrace{g,\ \boldsymbol{\mathcal{A}}, \Psi}_{\text{fields}}] = \int L(g_m, \boldsymbol{\mathcal{A}}_m, \Psi(m), \underbrace{\ldots,\ \ldots,\ \ldots\ \ldots,\ \ldots}_{\text{derivatives at point } m \in \mathscr{M}})\, \mathrm{vol}_g\,,$$

  with Lagrangian density L depending on a finite number of their partial derivatives.
- Sum with a term per phenomenon :

$$\mathscr{L}[g, \Psi, \boldsymbol{\mathcal{A}}] = \mathscr{H}[g] + \mathscr{L}^{\mathrm{EM}}[g, \boldsymbol{\mathcal{A}}] + \mathscr{L}^{\mathrm{Matter}}[g, \Psi, \boldsymbol{\mathcal{A}}]$$

  with the matter Lagrangien $\mathscr{L}^{\mathrm{Matter}}$ itself divided into

$$\mathscr{L}^{\mathrm{Matter}}[g, \Psi, \boldsymbol{\mathcal{A}}] = \mathscr{L}^{\mathrm{HED}}[g, \Psi] + \mathscr{L}^{\mathcal{J}^e}[\Psi, \boldsymbol{\mathcal{A}}] + \mathscr{L}^{\mathrm{EMM}}[g, \Psi, \boldsymbol{\mathcal{A}}]\,.$$

Theorem: Invariant expression of the coupling Lagrangian

- If the coupling Lagrangian

$$\mathscr{L}^{\text{EMM}}[g, \Psi, \boldsymbol{\mathcal{A}}] = \int L^{\text{EMM}}\left(g_{\mu\nu}, \partial_\rho g_{\mu\nu}, \Psi^I, \partial_\rho \Psi^I, \mathcal{A}_\mu, \partial_\rho \mathcal{A}_\mu\right)$$

is general covariant and gauge invariant then its density recasts as

$$\boxed{L^{\text{EMM}} = F\left(\Psi, \mathbf{K}_r, \boldsymbol{E}_r, \mathbf{B}_r\right) = \rho_r(\Psi, \mathbf{K}_r)\, \mathfrak{f}\left(\Psi, \mathbf{K}_r, \boldsymbol{E}_r, \mathbf{B}_r\right) .}$$

with $\mathbf{K}_r, \boldsymbol{E}_r$ and $\mathbf{B}_r$ the components of g^{-1} and of the Faraday tensor $\boldsymbol{\mathcal{F}}^\sharp = (\mathrm{d}\boldsymbol{\mathcal{A}})^\sharp$ in the matter rest frame $\mathcal{R}^\star_{\text{rest}} = (-\boldsymbol{U}^\flat, \mathrm{d}\, \Psi^I)$ (Chapon, Darondeau, Desmorat, Ecker, Kolev 2025):

$$\left[g^{-1}\right]_{\mathcal{R}^\star_{\text{rest}}} = \begin{pmatrix} -1 & 0 \\ 0 & \mathbf{K}_r \end{pmatrix}, \quad \left[\boldsymbol{\mathcal{F}}^\sharp\right]_{\mathcal{R}^\star_{\text{rest}}} = \begin{pmatrix} 0 & -\frac{1}{c}\boldsymbol{E}_r^\star \\ \frac{1}{c}\boldsymbol{E}_r & -\mathbf{B}_r \end{pmatrix},$$

where $\mathbf{K}_r = T\Psi g^{-1} T\Psi^\star$, $\boldsymbol{E}_r = -cT\Psi g^{-1}\boldsymbol{\mathcal{F}}\boldsymbol{U}$, $\mathbf{B}_r = -T\Psi g^{-1}\boldsymbol{\mathcal{F}} g^{-1} T\Psi^\star$.

Least Action Principle

- The full Lagrangian

$$\mathscr{L}[g,\Psi,\boldsymbol{\mathcal{A}}] = \mathscr{H}[g] + \mathscr{L}^{\mathrm{EM}}[g,\boldsymbol{\mathcal{A}}] + \mathscr{L}^{\mathrm{Matter}}[g,\Psi,\boldsymbol{\mathcal{A}}]\,.$$

verifies the least action principle for each field $[g,\Psi,\boldsymbol{\mathcal{A}}]$:

  - $\delta_g \mathscr{L} = 0 \quad \to$ Einstein equation,
  - $\delta_\Psi \mathscr{L} = 0 \quad \to$ Equilibrium equations,
  - $\delta_{\boldsymbol{\mathcal{A}}} \mathscr{L} = 0 \quad \to$ Maxwell equations with current.

Next: focus on $\delta_g \mathscr{L}$, especially the stress-energy tensor $\mathbf{T} = -2\dfrac{\delta(\mathscr{L} - \mathscr{H})}{\delta g}$

VARIATION WITH RESPECT TO THE METRIC TENSOR g

- Critical points of the Lagrangian with respect to g

$$\delta_g \mathscr{L}[g, \boldsymbol{\mathcal{A}}, \Psi] = 0 \iff 2\frac{\delta \mathscr{H}}{\delta g} + 2\frac{\delta \mathscr{L}^{\mathrm{EM}}}{\delta g} + 2\frac{\delta \mathscr{L}^{\mathrm{Matter}}}{\delta g} = 0$$

  - lead to the Einstein equation

$$2\frac{\delta \mathscr{H}}{\delta g} = \boxed{\frac{1}{\kappa}\left(\mathbf{G}^g\right)^\sharp = \mathbf{T} = \mathbf{T}^{\mathrm{EM}} + \mathbf{T}^{\mathrm{Matter}}} = -2\frac{\delta \mathscr{L}^{\mathrm{EM}}}{\delta g} - 2\frac{\delta \mathscr{L}^{\mathrm{Matter}}}{\delta g}$$

  - with $\left(\mathbf{G}^g\right)^\sharp$ the Einstein tensor, $\mathbf{T} = \mathbf{T}^{\mathrm{EM}} + \mathbf{T}^{\mathrm{Matter}}$ the stress energy tensor:

$$\mathbf{T} = (\mathrm{T}^{\mu\nu}) \to \begin{pmatrix} \text{Energy} & \text{Momentum} \\ \text{Momentum} & \text{Stress} \end{pmatrix}.$$

Splitting of the matter stress energy tensor

- Definition of the 4D stress tensor $\boldsymbol{\Sigma}$ from the expression of $\mathbf{T}^{\text{Matter}}$:

$$
\begin{aligned}
\mathbf{T}^{\text{Matter}} &= \mathbf{T}^{\text{HED}} + \mathbf{T}^{\text{EMM}} \\
&= \rho_r \left(c^2 + \mathfrak{e} + \mathfrak{f}\right) \boldsymbol{U} \otimes \boldsymbol{U} - \underbrace{\left(2\rho_r g^{-1} (T\Psi)^{\star} \mathbf{K}_r^{-1} \frac{\partial(\mathfrak{e} + \mathfrak{f})}{\partial \mathbf{K}_r^{-1}} \mathbf{K}_r^{-1} (T\Psi) g^{-1}\right)}_{\text{4D stress tensor } \boldsymbol{\Sigma}} \\
&\qquad - 2\rho_r \left(\frac{\partial \mathfrak{f}}{\partial \boldsymbol{E}_r} \cdot \frac{\partial \boldsymbol{E}_r}{\partial g} + \frac{\partial \mathfrak{f}}{\partial \mathbf{B}_r} : \frac{\partial \mathbf{B}_r}{\partial g}\right)
\end{aligned}
$$

with the last part being related to electromagnetic invariants:

- $\dfrac{\partial \mathfrak{f}}{\partial \boldsymbol{E}_r}$ $\rightarrow$ polarization in the rest frame,
- $\dfrac{\partial \mathfrak{f}}{\partial \mathbf{B}_r}$ $\rightarrow$ magnetization in the rest frame.

Definition of the relativistic stress tensor

- At the Galilean limit $c \to \infty$:

> The conformation $\mathbf{K}_r$ tends to $\mathbf{C}^{-1}$, the inverse of the right Cauchy–Green tensor.

- 4D Relativistic hyper-elasticity law coupled with electromagnetism:

$$\boldsymbol{\Sigma} = 2\rho_r g^{-1} (T\Psi)^{\star} \mathbf{K}_r^{-1} \frac{\partial(\mathfrak{e} + \mathfrak{f})}{\partial \mathbf{K}_r^{-1}} \mathbf{K}_r^{-1} T\Psi\, g^{-1}, \qquad \boldsymbol{\Sigma}\, \boldsymbol{U}^{\flat} = 0$$

- The following terms are not gained from a derivative with respect to the conformation $\mathbf{K}_r$,

$$-2\rho_r \left(\frac{\partial\, \mathfrak{f}}{\partial \boldsymbol{E}_r} \cdot \frac{\partial \boldsymbol{E}_r}{\partial g} + \frac{\partial\, \mathfrak{f}}{\partial \mathbf{B}_r} : \frac{\partial \mathbf{B}_r}{\partial g} \right)$$

In accordance with (Carter 1980), they are not interpreted as part of the mechanical stress tensor $\boldsymbol{\Sigma}$.

3D relativistic second Piola–Kirchhoff stress tensor

- Inside the former 4D relativistic stress tensor, a 3D stress tensor is naturally defined as its components in the rest frame:

$$\boxed{\mathbf{S} = \frac{1}{\rho_r} T\Psi\, \boldsymbol{\Sigma}\, T\Psi^{\star} = 2\,\frac{\partial(\mathfrak{e}+\mathfrak{f})}{\partial \mathbf{K}_r^{-1}}\,.}$$

It is interpreted as the relativistic second Piola–Kirchhoff stress tensor.

- It can be split in two parts:

$$\mathbf{S} = \mathbf{S}^{\mathrm{HED}} + \mathbf{S}^{\mathrm{EMM}}\,, \qquad \mathbf{S}^{\mathrm{HED}} = 2\,\frac{\partial \mathfrak{e}}{\partial \mathbf{K}_r^{-1}}\,, \qquad \mathbf{S}^{\mathrm{EMM}} = 2\,\frac{\partial \mathfrak{f}}{\partial \mathbf{K}_r^{-1}}\,,$$

the last being a manifestation of both piezo-electricity (stress by electric field) and piezo-magnetism (stress by magnetic induction).

Conservation of the stress-energy tensor and Lorentz force

- General covariance of $\mathscr{H}$ (Noether 1918) yields the conservation of the Einstein tensor

$$\operatorname{div}^g \mathbf{G}^g = 0\,.$$

- For $\mathbf{T}$ solution of the Einstein equation $(\mathbf{G}^g)^\sharp = \kappa \mathbf{T}$, it recasts as the conservation of $\mathbf{T}$

$$\boxed{\operatorname{div}^g \mathbf{T} = \operatorname{div}^g \mathbf{T}^{\text{EM}} + \operatorname{div}^g \mathbf{T}^{\text{Matter}} = 0\,.}$$

- First part: the Lorentz force, defined as $\boldsymbol{f}^L = g^{-1}\boldsymbol{\mathcal{F}} \cdot \boldsymbol{\mathcal{J}}^e$, appears in

$$\boxed{\operatorname{div}^g \mathbf{T}^{\text{EM}} = \boldsymbol{f}^L\,.}$$

GENERALIZED LORENTZ FORCE

- From the conservation of the total stress-energy tensor $\mathrm{div}^g\,\mathbf{T} = 0$, we have

$$\boxed{\mathrm{div}^g\,\boldsymbol{\Sigma} - \boldsymbol{f}^{\mathrm{GL}} = \mathrm{div}^g\left(\rho_r\left(c^2 + \mathfrak{e} + \mathfrak{f}\right)\boldsymbol{U}\otimes\boldsymbol{U}\right)\,,}$$

  - where $\boldsymbol{U}$ is the proper time direction of the matter,
  - the Lorentz force, previously $\boldsymbol{f}^L = \mathrm{div}^g\,\mathbf{T}^{\mathrm{EM}}$, is generalized as $\boldsymbol{f}^{\mathrm{GL}}$ a four-vector

$$\boldsymbol{f}^{\mathrm{GL}} = \boldsymbol{f}^L - \mathrm{div}^g\left[2\rho_r\left(\frac{\partial\,\mathfrak{f}}{\partial\boldsymbol{E}_r}\cdot\frac{\partial\boldsymbol{E}_r}{\partial g} + \frac{\partial\,\mathfrak{f}}{\partial\mathbf{B}_r}:\frac{\partial\mathbf{B}_r}{\partial g}\right)\right]\,,$$

    which is possibly affected by the medium deformation.

- In literature (Eringen et al. 1990), question around the symmetry of stresses.
- Here clear splitting of $\boldsymbol{\Sigma}$ and $\boldsymbol{f}^{\mathrm{GL}}$ and symmetry of $\boldsymbol{\Sigma}$.

Towards the 3D conservation equations

- For now, general relativistic framework: no distinction of space and time.
- To recover mechanics: introduction of a Spacetime.
- After decomposing in space and time parts, obtention of expressions with usual 3D quantities.

Next: Introduction of the observer, and expression of 3D the momentum equilibrium.

Observer

- An observer is a time function $\hat{t}$, whose gradient is everywhere timelike.
- $\boldsymbol{N}$ is a vector field orthogonal to spatial hypersurfaces where $\hat{t}$ is fixed,

$$\boldsymbol{N} = -\frac{\operatorname{grad}\hat{t}}{\sqrt{\|\operatorname{grad}\hat{t}\|^2}} \,.$$

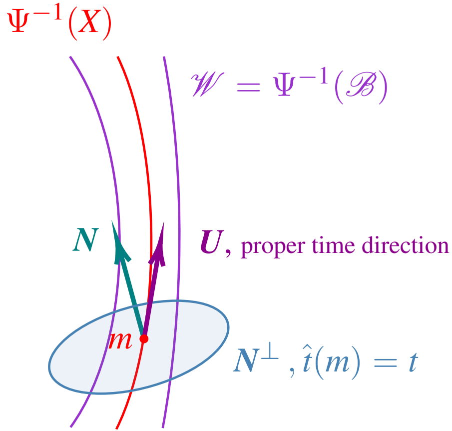
  - Generalized Lorentz factor

$$\gamma = -g(\boldsymbol{U}, \boldsymbol{N}) = \frac{1}{\sqrt{1 - \frac{1}{c^2}\|\boldsymbol{u}\|^2_{g^{3D}}}} \,.$$

  - Orthogonal decomposition of $\boldsymbol{U}$:

$$\boldsymbol{U} = \gamma\left(\boldsymbol{N} + \frac{1}{c}\boldsymbol{u}\right) ,$$

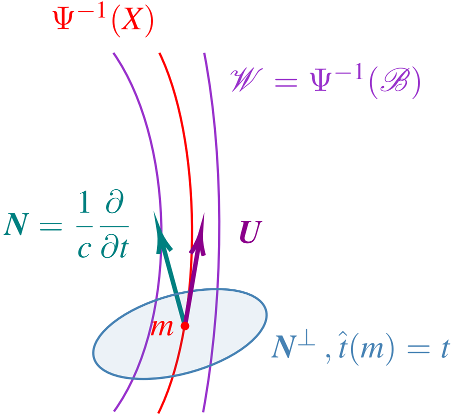
with $\boldsymbol{u}$ the spatial velocity and $\boldsymbol{N} \cdot \boldsymbol{u} = 0$.



The Minkowski Spacetime

Passive matter assumption: neglectible impact of matter and electromagnetism on gravitation.

- Additional choice of a time function: $\hat{t} = t$,
- Diagonal metric tensor $g = \boldsymbol{\eta} = -c^2 \mathrm{d}t \otimes \mathrm{d}t + \mathbf{q}$,
- Decomposition of the vectors:
  - $\boldsymbol{U} = \dfrac{\gamma}{c}\left(\dfrac{\partial}{\partial t} + \boldsymbol{u}\right)$,
  - $\gamma = \dfrac{1}{\sqrt{1 - \frac{\|\boldsymbol{u}\|_{\mathbf{q}}^2}{c^2}}}$,
  - $T\Psi = -\mathbf{F}^{-1}\boldsymbol{u} \otimes \mathrm{d}t + \mathbf{F}^{-1}$,
    with $\mathbf{F} = T\Psi_t^{-1}$ the deformation gradient.



THE STRESS TENSOR IN THE MINKOWSKI SPACETIME

- By placing ourselves in a Spacetime at rest with $\boldsymbol{N} = \boldsymbol{U}$ ($\boldsymbol{u} = 0$), only the spacial part of the stress tensor remains:

$$\boldsymbol{\Sigma} = \begin{pmatrix} 0 & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix}, \qquad \boldsymbol{\sigma} = 2\rho_r \mathbf{F} \frac{\partial(\mathfrak{e} + \mathfrak{f})}{\partial \mathbf{K}_r^{-1}} \mathbf{F}^\star = \boldsymbol{\sigma}^{\text{HED}} + \boldsymbol{\sigma}^{\text{EMM}}$$

- The Maxwell stress tensor emerges from the Electromagnetic Lagrangian $\mathscr{L}^{\text{EM}}$:

$$\mathbf{T}^{\text{EM}} = \begin{pmatrix} \varepsilon^{\text{EM}} & \frac{1}{c}(\mathbf{S}^{\text{EM}})^\star \\ \frac{1}{c}\mathbf{S}^{\text{EM}} & -\boldsymbol{\sigma}^{\text{EM}} \end{pmatrix}, \qquad \boldsymbol{\sigma}^{\text{EM}} = \varepsilon_0 \boldsymbol{E} \otimes \boldsymbol{E} + \frac{1}{\mu_0} \boldsymbol{B} \otimes \boldsymbol{B} - \varepsilon^{\text{EM}} \mathbf{q}^{-1}$$

it represents the stress exerted by the electromagnetic field on matter, and is linked to the Lorentz force $\boldsymbol{f}^{\text{L}}$.

Equilibrium in the Minkowski Spacetime

- Still placing ourselves in a Spacetime at rest with $\boldsymbol{N} = \boldsymbol{U}$ ($\boldsymbol{u} = 0$), the spacial part of the conservation $\operatorname{div}^g \mathbf{T}$ is:

$$\boxed{\sigma_{ij,j} = f_i^{\mathrm{GL}}}$$

  - With $\boldsymbol{f}^{\mathrm{GL}}$ the generalized Lorentz force

$$\boldsymbol{f}^{\mathrm{GL}} = \boldsymbol{f}^{L} - \operatorname{div}^g \left[2\rho_r \left(\frac{\partial \mathfrak{f}}{\partial \boldsymbol{E}_r} \cdot \frac{\partial \boldsymbol{E}_r}{\partial g} + \frac{\partial \mathfrak{f}}{\partial \mathbf{B}_r} : \frac{\partial \mathbf{B}_r}{\partial g} \right) \right] ,$$

  - And the classical Lorentz force deriving from

$$\boldsymbol{f}^{L} = \operatorname{div}^g \mathbf{T}^{\mathrm{EM}} = \operatorname{div}^{\mathbf{q}} \boldsymbol{\sigma}^{\mathrm{EM}} .$$

  - The stress tensor still is

$$\boldsymbol{\sigma} = 2\rho_r \mathbf{F} \frac{\partial(\mathfrak{e} + \mathfrak{f})}{\partial \mathbf{K}_r^{-1}} \mathbf{F}^{\star} = \boldsymbol{\sigma}^{\mathrm{HED}} + \boldsymbol{\sigma}^{\mathrm{EMM}} .$$

The relativistic invariants in the Minkowski Spacetime

- Starting from the Faraday tensor $\mathcal{F} = \mathrm{d}\mathcal{A}$

$$\mathcal{F} = \mathrm{d}t \wedge \mathbf{q}\boldsymbol{E} - i_{\boldsymbol{B}} \ \mathrm{vol}_{\mathbf{q}} \ .$$

- The relativistic invariants have for expression

$$\boldsymbol{E}_r = \gamma \mathbf{F}^{-1} \left(-\frac{1}{c^2} (\boldsymbol{E} \cdot \boldsymbol{u}) \boldsymbol{u} + \boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B} \right) \approx \mathbf{F}^{-1} \left(\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B} \right) \ ,$$

$$\mathbf{B}_r = \mathbf{F}^{-1} \left(\frac{1}{c^2} (-\boldsymbol{E} \otimes \boldsymbol{u} + \boldsymbol{u} \otimes \boldsymbol{E}) + \hat{\mathbf{B}} \right) \mathbf{F}^{-\star} \approx \mathbf{F}^{-1} \hat{\mathbf{B}} \mathbf{F}^{-\star} \, ,$$

$$\mathbf{K}_r = \mathbf{F}^{-1} \left(-\frac{1}{c^2} \boldsymbol{u} \otimes \boldsymbol{u} + \mathbf{q}^{-1} \right) \mathbf{F}^{-\star} \approx \mathbf{F}^{-1} \mathbf{q}^{-1} \mathbf{F}^{-\star} = \mathbf{C}^{-1} \, ,$$

when $\boldsymbol{u}^2 << c^2$, and where $\hat{\mathbf{B}}$ is defined as $\hat{\mathbf{B}} = \mathbf{q}^{-1} i_{\boldsymbol{B}} \, \mathrm{vol}_{\mathbf{q}} \, \mathbf{q}^{-1}$, and $\mathbf{F}$ is the deformation gradient.

Conclusion

- Last year, about the coupling Lagrangian
  - Formulation of electro-magnetism and hyperelasticity, and their coupling, in Variational Relativity.
  - Calculation of relativistic invariants using the matter rest frame.
  - Expression of general covariant coupling Lagrangians that depends on first partial derivative of fields.
- New steps
  - Use of the Least action principle: $\delta\mathscr{L} = 0$.
  - Obtention of Maxwell equations, Einstein equation, definition of the stress-energy tensor.
  - Definition of the 4D relativistic stress tensor and its 3D counterpart.

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Addition of a coupling Lagrangian

- New term added:

$$\mathscr{L}^{\text{EMM}}[g, \Psi, \boldsymbol{\mathcal{A}}] = \int L^{\text{EMM}}\left(g_{\mu\nu}, \partial_\rho g_{\mu\nu}, \Psi^I, \partial_\rho \Psi^I, \mathcal{A}_\mu, \partial_\rho \mathcal{A}_\mu\right) \mathrm{vol}_g \,.$$

- This coupling Lagrangian has to be
  - general covariant, meaning for any diffeomorphism φ:

$$\mathscr{L}^{\text{EMM}}[\varphi^* g, \varphi^* \Psi, \varphi^* \boldsymbol{\mathcal{A}}] = \mathscr{L}^{\text{EMM}}[g, \Psi, \boldsymbol{\mathcal{A}}]\,,$$

  - as well as strongly gauge invariant, so that for any function χ one wants:

$$\mathscr{L}^{\text{EMM}}[g, \Psi, \boldsymbol{\mathcal{A}} + \mathrm{d}\chi] = \mathscr{L}^{\text{EMM}}[g, \Psi, \boldsymbol{\mathcal{A}}]\,.$$

Criteria for determination of invariants, used as arguments of the density L^{EMM}

Expression of the full Lagrangian

- The full Lagrangian is now

$$\mathscr{L}[g,\boldsymbol{\mathcal{A}},\Psi] = \int \frac{1}{2\kappa} R^g \,\mathrm{vol}_g - \int \frac{1}{4\mu_0} ||\boldsymbol{\mathcal{F}}||_g^2 \,\mathrm{vol}_g - \int \boldsymbol{\mathcal{A}} \cdot \boldsymbol{\mathcal{J}}^e \,\mathrm{vol}_g$$

$$+ \int \rho_r \left(\underbrace{c^2 + \mathfrak{e}\left(\Psi, \mathbf{K}_r\right)}_{\text{hyperelastodynamics}} + \underbrace{\mathfrak{f}\left(\Psi, \mathbf{K}_r, \boldsymbol{E}_r, \mathbf{B}_r\right)}_{\text{piezo-electricity, piezo-magnetism}} \right) \mathrm{vol}_g \,.$$

- Each Lagrangian was already function of relativistic invariants:
  - $R^g = R^g(g)$ is by definition a relativistic invariant,
  - $\|F\|_g^2 = \dfrac{2}{c^2} \boldsymbol{E}_r \cdot \mathbf{K}_r^{-1} \cdot \boldsymbol{E}_r + \mathrm{tr}(\mathbf{B}_r \mathbf{K}_r^{-1} \mathbf{B}_r \mathbf{K}_r^{-1})$,
  - $\rho_r = \rho_r(\Psi, \mathbf{K}_r)$,
  - $\boldsymbol{\mathcal{A}} \cdot \boldsymbol{\mathcal{J}}^e = e_r \, \boldsymbol{\mathcal{A}} \cdot \boldsymbol{U}$, first component of $\boldsymbol{\mathcal{A}}$ in $\mathcal{R}^\star_{\text{rest}}$.

Matter rest coframe - Construction

- The unit vector $\boldsymbol{U}$ represents the direction of time for matter.
- A basis of its orthogonal space, made of the vectors $\operatorname{grad}\Psi^I$, can be added to form a frame:

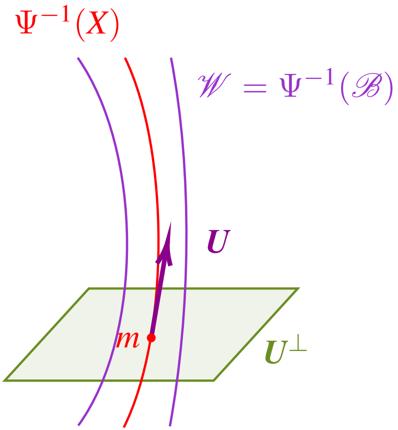
$$\mathcal{R}_{\text{rest}} = \left(\boldsymbol{U}, \operatorname{grad}\Psi^1, \operatorname{grad}\Psi^2, \operatorname{grad}\Psi^3\right)$$

  - Function of g and the matter field Ψ.
  - It is general covariant of g and Ψ (Chapon et al. 2024):

$$\mathcal{R}_{\text{rest}}(\varphi^* g, \varphi^* \Psi) = \varphi^* \mathcal{R}_{\text{rest}}(g, \Psi)$$

  - For technical reasons we will use its dual, the coframe $\mathcal{R}^\star_{\text{rest}}$:

$$\mathcal{R}^\star_{\text{rest}} = \left(-\boldsymbol{U}^\flat, \mathrm{d}\Psi^1, \mathrm{d}\Psi^2, \mathrm{d}\Psi^3\right) .$$



Relativistic invariants

- Secondary variables are covariant functions of the primary (field) variables $(g, \boldsymbol{\mathcal{A}}, \Psi)$. For example for the conformation:

$$\mathbf{K}_r(\varphi^*\Psi, \varphi^* g) = \varphi^* \mathbf{K}_r(\Psi, g)\,.$$

- The components of secondary variables in the matter rest frame are relativistic invariants: function of primary variables $(g, \Psi, \boldsymbol{\mathcal{A}})$, and any diffeomorphism φ acts as

$$\boxed{\mathcal{I}\,(\varphi^* g, \varphi^*\Psi, \varphi^*\boldsymbol{\mathcal{A}}) = \varphi^*\,\mathcal{I}\,(g, \Psi, \boldsymbol{\mathcal{A}}) = \mathcal{I}\,(g, \Psi, \boldsymbol{\mathcal{A}}) \circ \varphi\,.}$$

- Here $\mathcal{I}$ is the components of either g^{-1} or $\boldsymbol{\mathcal{F}}$ in $\mathcal{R}^\star_{\text{rest}}$: $\mathbf{K}_r$, $\boldsymbol{E}_r$, or $\mathbf{B}_r$

$$\left[g^{-1}\right]_{\mathcal{R}^\star_{\text{rest}}} = \begin{pmatrix} -1 & 0 \\ 0 & \mathbf{K}_r \end{pmatrix}, \quad \left[\boldsymbol{\mathcal{F}}^\sharp\right]_{\mathcal{R}^\star_{\text{rest}}} = \begin{pmatrix} 0 & -\frac{1}{c}\boldsymbol{E}_r^\star \\ \frac{1}{c}\boldsymbol{E}_r & -\mathbf{B}_r \end{pmatrix}.$$

Is there a contribution from $\mathscr{L}^{\mathcal{J}^e}$?

- Actually the critical points are

$$\delta_g \mathscr{L}[g, \boldsymbol{\mathcal{A}}, \Psi] = 0 \iff 2\frac{\delta \mathscr{H}}{\delta g} + 2\frac{\delta \mathscr{L}^{\mathrm{EM}}}{\delta g} + 2\frac{\delta \mathscr{L}^{\mathrm{HED}}}{\delta g} + 2\frac{\delta \mathscr{L}^{\mathcal{J}^e}}{\delta g} + 2\frac{\delta \mathscr{L}^{\mathrm{EMM}}}{\delta g} = 0$$

- But $\mathscr{L}^{\mathcal{J}^e}$ does not depend on $g \to$ no contribution to $\mathbf{T}$:

$$\mathscr{L}^{\mathcal{J}^e}[\Psi, \boldsymbol{\mathcal{A}}] = \int -\boldsymbol{\mathcal{A}} \cdot \boldsymbol{\mathcal{J}}^e \,\mathrm{vol}_g = \int -\boldsymbol{\mathcal{A}} \wedge (\Psi^* \mu^e)\,.$$

Expression of the stress-energy tensor

- Starting from

$$\mathbf{T} = \mathbf{T}^{\text{EM}} + \mathbf{T}^{\text{HED}} + \mathbf{T}^{\text{EMM}} = -2\frac{\delta \mathscr{L}^{\text{EM}}}{\delta g} - 2\frac{\delta \mathscr{L}^{\text{HED}}}{\delta g} - 2\frac{\delta \mathscr{L}^{\text{EMM}}}{\delta g} \,,$$

we get for the three terms:

  - $\mathbf{T}^{\text{EM}} = \dfrac{1}{\mu_0}\boldsymbol{\mathcal{F}}^{\sharp}\boldsymbol{\mathcal{F}} g^{-1} + \dfrac{1}{4\mu_0}\|\boldsymbol{\mathcal{F}}\|_g^2 g^{-1} \,,$
  - $\mathbf{T}^{\text{HED}} = \rho_r \left(c^2 + \mathfrak{e}\right) \boldsymbol{U} \otimes \boldsymbol{U} + 2\rho_r g^{-1}(T\Psi)^{\star} \dfrac{\partial e}{\partial \mathbf{K}_r}(T\Psi) g^{-1} \,,$
  - $\mathbf{T}^{\text{EMM}} = \rho_r \mathfrak{f}\, \boldsymbol{U} \otimes \boldsymbol{U} + 2\rho_r\, g^{-1}(T\Psi)^{\star} \dfrac{\partial \mathfrak{f}}{\partial \mathbf{K}_r} T\Psi g^{-1} - 2\rho_r \left(\dfrac{\partial \mathfrak{f}}{\partial \boldsymbol{E}_r} \cdot \dfrac{\partial \boldsymbol{E}_r}{\partial g} + \dfrac{\partial \mathfrak{f}}{\partial \mathbf{B}_r} : \dfrac{\partial \mathbf{B}_r}{\partial g} \right) .$

$\rightarrow$ By definition they all are symmetrical (contrary to (Eringen et al. 1990)).

The relativistic invariants in the Minkowski Spacetime

- Starting from the Faraday tensor

$$\mathcal{F} = \mathrm{d}t \wedge \mathbf{q}\boldsymbol{E} - i_{\boldsymbol{B}} \ \mathrm{vol}_{\mathbf{q}} \ .$$

- The relativistic invariants have for expression

$$\mathbf{K}_r = \mathbf{F}^{-1}\left(-\frac{1}{c^2}\boldsymbol{u}\otimes\boldsymbol{u} + \mathbf{q}^{-1}\right)\mathbf{F}^{-\star} \approx \mathbf{F}^{-1}\mathbf{q}^{-1}\mathbf{F}^{-\star} = \mathbf{C}^{-1},$$

$$\boldsymbol{E}_r = \gamma\mathbf{F}^{-1}\left(-\frac{1}{c^2}(\boldsymbol{E}\cdot\boldsymbol{u})\boldsymbol{u} + \boldsymbol{E} + \boldsymbol{u}\times\boldsymbol{B}\right) \approx \mathbf{F}^{-1}\left(\boldsymbol{E} + \boldsymbol{u}\times\boldsymbol{B}\right) \ ,$$

$$\mathbf{B}_r = \mathbf{F}^{-1}\left(\frac{1}{c^2}(-\boldsymbol{E}\otimes\boldsymbol{u} + \boldsymbol{u}\otimes\boldsymbol{E}) + \hat{\mathbf{B}}\right)\mathbf{F}^{-\star} \approx \mathbf{F}^{-1}\hat{\mathbf{B}}\mathbf{F}^{-\star},$$

when $\boldsymbol{u}^2 << c^2$, and where $\hat{\mathbf{B}}$ is defined as $\hat{\mathbf{B}} = \mathbf{q}^{-1} i_{\boldsymbol{B}} \,\mathrm{vol}_{\mathbf{q}}\, \mathbf{q}^{-1}$, and $\mathbf{F}$ is the deformation gradient.

Hyperelastic stress-energy tensor $\mathbf{T}^{\text{HED}}$ in the Minkowski Spacetime

- The 4D stress tensor falls back to

$$\mathbf{T}^{\text{HED}} = \left((T^{\text{HED}})^{\mu\nu} \right) = \begin{pmatrix} \varepsilon^{\text{HED}} & \frac{1}{c}\,\boldsymbol{p}^{\star} \\ \frac{1}{c}\,\boldsymbol{p} & \boldsymbol{\sigma} \end{pmatrix},$$

with

  - Hyperelastic energy density $\varepsilon^{\text{HED}} = \dfrac{1}{c^2}\boldsymbol{u}^{\flat} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{u}^{\flat}$,
  - Momentum vector $\boldsymbol{p} = -\boldsymbol{\sigma} \cdot \boldsymbol{u}^{\flat}$,
  - Mechanic stress tensor $\boldsymbol{\sigma} = -\dfrac{2\rho_r}{\gamma}\mathbf{q}^{-1}\mathbf{F}^{-\star}\,\dfrac{\partial \mathfrak{e} + \mathfrak{f}}{\partial \mathbf{K}_r}\mathbf{F}^{-1}\mathbf{q}^{-1}$.

$\rightarrow$ When $\boldsymbol{u} = 0$, only the stress tensor remains.

Recovering the Maxwell stress tensor

- The 4D stress-energy tensor falls back to

$$\mathbf{T}^{\mathrm{EM}} = \frac{1}{\mu_0}\boldsymbol{\eta}^{-1}\,\boldsymbol{\mathcal{F}}\boldsymbol{\eta}^{-1}\,\boldsymbol{\mathcal{F}}\boldsymbol{\eta}^{-1} + \frac{1}{4\mu_0}\|\boldsymbol{\mathcal{F}}\|^2_{\boldsymbol{\eta}}\,\boldsymbol{\eta}^{-1} = \left(\left(\mathrm{T}^{\mathrm{EM}}\right)^{\mu\nu}\right) = -\begin{pmatrix} \varepsilon^{\mathrm{EM}} & \frac{1}{c}\boldsymbol{S}^{\star} \\ \frac{1}{c}\boldsymbol{S} & -\boldsymbol{\sigma}^{\mathrm{EM}} \end{pmatrix},$$

  with
  - Electromagnetic energy density $\varepsilon^{\mathrm{EM}} = \dfrac{1}{2}(\varepsilon_0||\boldsymbol{E}||^2_{\mathbf{q}} + \dfrac{1}{\mu_0}||\boldsymbol{B}||^2_{\mathbf{q}})$,
  - Poynting vector $\boldsymbol{S} = \dfrac{1}{\mu_0}\boldsymbol{E}\times\boldsymbol{B}$,
  - Maxwell electromagnetic stress tensor $\boldsymbol{\sigma}^{\mathrm{EM}} = \varepsilon_0\,\boldsymbol{E}\otimes\boldsymbol{E} + \dfrac{1}{\mu_0}\boldsymbol{B}\otimes\boldsymbol{B} - \varepsilon^{\mathrm{EM}}\mathbf{q}^{-1}$,

    where ε_0 is the vacuum permittivity, $\varepsilon_0\mu_0c^2 = 1$, and $\mathbf{q} = \mathrm{diag}(1,1,1)$ the Euclidean metric tensor.