Mécanique symplectique et structures de Poisson

Vladimir Salnikov

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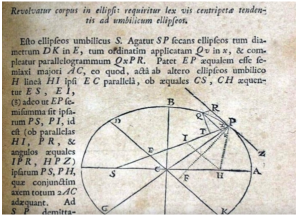
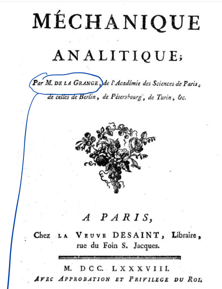
0. Histoire


Revolvatur corpus in ellipsi: requiritur lex vis centripetæ tendentis ad umbilicum ellipseos.

Esto ellipseos umbilicus S. Agatur SP secans ellipseos tum diametrum DK in E, tum ordinatim applicatam Qv in x, & compleatur parallelogrammum QxPR. Patet EP æqualem esse semiaxi majori AC, eo quod, acta ab altero ellipseos umbilico H linea HI ipsi EC parallela, ob æquales CS, CH æquentur ES, EI, (§) adeo ut EP semisumma sit ipsarum PS, PI, id est (ob parallelas HI, PR, & angulos æquales IPR, HPZ) ipsarum PS, PH, quæ conjunctim axem totum 2AC adæquant. Ad SP demitta-

(Newton)

⚠ Charles Michel Marle


MÉCHANIQUE ANALITIQUE;

Par M. DE LA GRANGE, de l'Académie des Sciences de Paris, de celles de Berlin, de Pétersbourg, de Turin, &c.

A PARIS,

Chez LA VEUVE DESAINT, Libraire, rue du Foin S. Jacques.

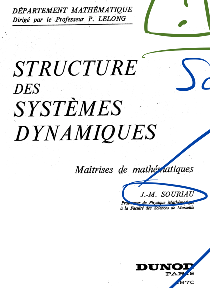
M. DCC. LXXXVIII.

AVEC APPROBATION ET PRIVILEGE DU ROI.

Lagrange

Poisson

⚠ Géry


DÉPARTEMENT MATHÉMATIQUE
Dirigé par le Professeur P. LELONG

STRUCTURE DES SYSTÈMES DYNAMIQUES

Maîtrises de mathématiques

J.-M. SOURIAU
Professeur de Physique Mathématique à la Faculté des Sciences de Marseille

DUNOD
PARIS
1970

Souriau

Arnold

Marsden

⚠ Lecture by Jean-Pierre Bourguignon (google: Souriau symplectic)
https://www.youtube.com/watch?v=93hFolIBoOQ

1. Géométrie symplectique au lycée

$(\star)$ $m_j \vec{\gamma} = \vec{F}_j$, Cas particulier quand F's dérivent d'un potentiel

Un Lagrangien $L = T - V = E_{kin} - U_{pot}$

$$= \sum_{j=1}^{N} \frac{m_j \vec{v}_j^{\,2}}{2} - U(\vec{r}_1 \ldots \vec{r}_N)$$

cas part.

(EL) $$\frac{d}{dt}\frac{\partial L}{\partial \vec{v}_j} - \frac{\partial L}{\partial \vec{r}_j} = 0$$

$$m_j \ddot{\vec{r}}_j + \frac{\partial V}{\partial \vec{r}_j} = 0$$

Jeu de notations : $\quad n := 3N$

$$q := (r_{1x}, r_{1y}, r_{1z}, \ldots, r_{Nx}, r_{Ny}, r_{Nz})$$

$$p := (m_1 v_{1x}, m_1 v_{1y}, m_1 v_{1z}, \ldots, m_N v_{Nx}, m_N v_{Ny}, m_N v_{Nz})$$

$$H := T(p) + V(q)$$

$$(EL) \Rightarrow \begin{cases} \dot{q}^i = \dfrac{p_i}{m_{j(i)}} = \dfrac{\partial H(q,p)}{\partial p_i} \\ \dot{p}_i = -\dfrac{\partial V}{\partial q^i} = -\dfrac{\partial H(q,p)}{\partial q^i} \end{cases} \quad i = 1, \ldots, n$$

<u>Déf</u> Système Hamiltonien

(en coordonnées symplectiques canoniques)

Remarques :

- $$\frac{dH}{dt} = \sum_i \left(\frac{\partial H}{\partial q^i} \dot{q}^i + \frac{\partial H}{\partial p_i} \dot{p}_i \right) =$$
$$= \sum_i \left(\frac{\partial H}{\partial q^i} \frac{\partial H}{\partial p_i} - \frac{\partial H}{\partial p_i} \frac{\partial H}{\partial q^i} \right) = 0$$

- Dans $\mathbb{R}^{2n}$ (q, p)

$$(**) \quad \begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = X_H = J \nabla H, \quad J = \begin{pmatrix} 0 & Id_n \\ -Id_n & 0 \end{pmatrix}$$

J définit une forme bilinéaire antisymétrique non dégénérée $\Omega(v, w) := v^T J w$

- Ω "≃" forme d'aire (orientée)

$$\Omega(v,w) = \sum_{i=1}^{n} (v_{q_i} w_{p_i} - v_{p_i} w_{q_i}) = \sum_{i=1}^{n} \det(v,w)\big|_{p_i q_i}$$

- Le flot de (**) $\varphi_t : \begin{pmatrix} q(0) \\ p(0) \end{pmatrix} \longmapsto \begin{pmatrix} q(t) \\ p(t) \end{pmatrix}$

est symplectique $(\nabla \varphi_t)^T J (\nabla \varphi_t) = J$.

- Il y a des liens entre

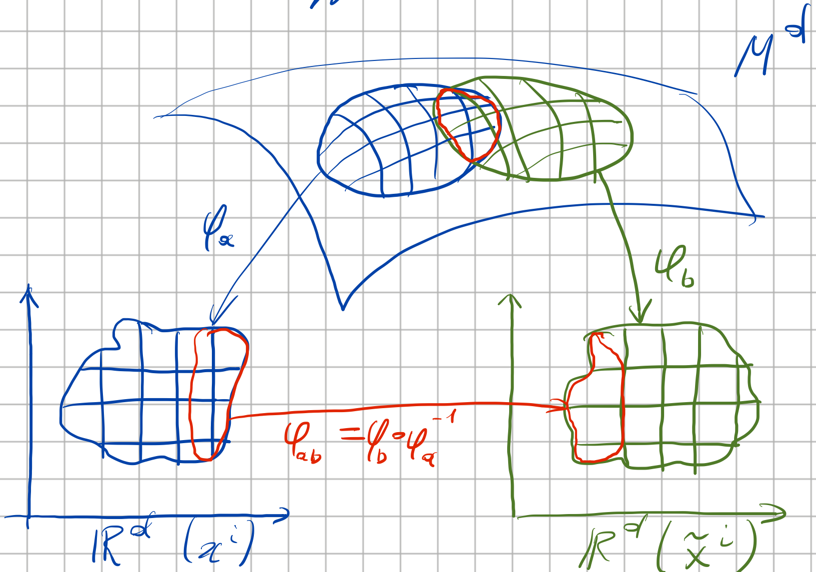
$\rightarrow$ $J^T = J^{-1} = -J$

$\rightarrow$ φ_t symplectique / préserve le volume

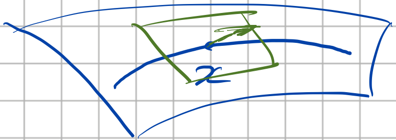
$\rightarrow$ $\dfrac{dH}{dt} = 0$

2. Rappels de la géométrie différentielle

- Variété différentielle (lisse)



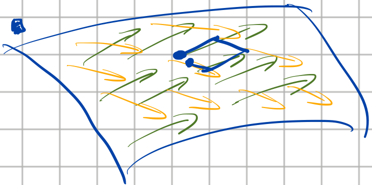
- Fibré tangent $TM = \bigsqcup_{x \in M} T_x M$



- Champ de vecteurs $v \in \Gamma(TM)$

En coordonnées $v = \sum_{i=1}^{d} v^i(x) \frac{\partial}{\partial x^i}$

- $\varphi_t : M \to M$ – flot de v si v est tangent aux trajectoires de φ.

- 

Crochet de Lie (commutateur)

$$[v, w]^i = \sum_{j=1}^{d} \left(v^j \frac{\partial}{\partial x^j} w^i - w^j \frac{\partial}{\partial x^j} v^i \right)$$

- Fibré cotangent $T^*M = \bigsqcup_{x\in M} (T_xM)^*$
- Champ de covecteurs $\alpha \in \Gamma(T^*M)$ (AKA 1-forme différentielle)

  En coordonnées $\alpha = \sum_{i=1}^{d} \alpha_i(x)\, dx^i$

  Dualité entre T^*M et TM : $dx^i\left(\frac{\partial}{\partial x^j}\right) = \delta^i_j$

- k-forme différentielle – champ de formes k-linéaires antisymétriques

  en coord's : $\omega = \sum_{1\le i_1 < \dots < i_k \le d} \omega_{i_1 \dots i_k}(x)\, dx^{i_1} \wedge \dots \wedge dx^{i_k}$

- ω – k forme, θ – ℓ forme

  produit extérieur $\omega \wedge \theta$ – $(k+\ell)$-forme

  $\omega \wedge \theta = (-1)^{k\ell}\, \theta \wedge \omega$

ω - k-forme $\quad \omega = \sum_{1 \le i_1 < \dots < i_k \le d} \omega_{i_1 \dots i_k}(x) dx^{i_1} \wedge \dots \wedge dx^{i_k}$

différentielle exterieure (de deRham)

$$d\omega = \sum_{j=1}^{d} \sum_{1 \le i_1 < \dots < i_k \le d} \frac{\partial \omega_{i_1 \dots i_k}}{\partial x^j}(x) dx^j \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k}$$

$(k+1)$-forme

propriétés : $dd\omega \equiv 0 \quad \forall \omega$

$$d(\omega \wedge \theta) = (d\omega) \wedge \theta + (-1)^k \omega \wedge (d\theta)$$

vocabulaire : ω - fermée $\Leftrightarrow d\omega = 0$

ω - exacte $\Leftrightarrow \exists \alpha : d\alpha = \omega$

Cohomologie : exacte $\Longrightarrow$ fermée

$\cancel{\Longleftarrow}$

Symplectique : 2-forme non-dégénérée fermée

- TM et T^*M ensemble

  - ω - k-forme ; $X, v_1 \dots v_k$ - k ch. de vecteurs

    $\omega(v_1, \dots v_k) \in C^\infty(M)$

    Produit intérieur (contraction, substitution)

    $$(\iota_X \omega)(v_1 \dots v_{k-1}) := \omega(X, v_1, \dots v_{k-1})$$

    Propriété : $\iota_X(\omega \wedge \theta) = (\iota_X \omega) \wedge \theta + (-1)^k \omega \wedge \iota_X \theta$

  - $\Phi_t : M \to M$ flot de X, alors

    "l'évolution" de ω est donnée par

    $$\mathcal{L}_X \omega = \iota_X d\omega + d\iota_X \omega$$

    (dérivée de Lie le long X) formule magique de Cartan

3. Mécanique symplectique

$$\boxed{\begin{aligned} \dot{q}^i &= \frac{\partial H(q,p)}{\partial p_i} \\ \dot{p}_i &= -\frac{\partial H(q,p)}{\partial q^i} \end{aligned}}$$

Hamiltonien dans $\mathbb{R}^{2n}$ (coordonnées symplectique canoniques)

$$(\star\star) \quad X_H = J\,\nabla H \qquad J = \begin{pmatrix} 0 & Id_n \\ -Id_n & 0 \end{pmatrix}$$

Exo $(\star\star) \iff i_{X_H}\omega = dH$ pour

$$\omega = \sum_{i=1}^n dq^i \wedge dp_i$$

$$X_H = \sum_{i=1}^n \left(\frac{\partial H}{\partial p_i} \frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i} \frac{\partial}{\partial p_i} \right)$$

Conséquence : ω est préservée par le flot de X_H

$$\mathcal{L}_{X_H}\omega = i_{X_H}(\underbrace{\cancel{d\omega}}_{\text{fermée}}) + d(\underbrace{\cancel{i_{X_H}\omega}}_{ddH}) = 0$$

Déf Soit ω une forme symplectique sur M, un champ de vecteurs X est Hamiltonien si $\exists H : \quad \iota_X \omega = dH$

Propriétés et non-propriétés

- $\dim M = 2n$
- ω est préservée par X
- $Vol = \underbrace{\omega \wedge \omega \wedge \ldots \wedge \omega}_{n}$ aussi
- H est conservée
- Q1: $\mathcal{L}_X \omega = 0 \overset{?}{\Rightarrow}$ Hamiltonien
- Q2: X_H, ω - Hamiltonien $\overset{?}{\Rightarrow}$ canonique

Déf Soit ω une forme symplectique sur M, un champ de vecteur X est Hamiltonien si $\exists H :$ $\iota_X \omega = dH$

Propriétés et non-propriétés

- $\dim M = 2n$ $\Leftarrow$ ω non-dégénérée
- ω est préservée par X $\Leftarrow$ toujours Cartan magique
- $Vol = \underbrace{\omega \wedge \omega \wedge .. \wedge \omega}_{n}$ aussi $\Leftarrow$ etc
- H est conservée $\Leftarrow$ $\mathcal{L}_X H = \iota_X dH = \iota_X \iota_X \omega = 0$
- Q1: $\mathcal{L}_X \omega = 0 \overset{?}{\Rightarrow}$ Hamiltonien
- Q2: X_H, ω - Hamiltonien $\overset{?}{\Rightarrow}$ canonique

Q1 : $\mathcal{L}_X \omega = 0$, $\exists ?\, H$ t.q. $\iota_X \omega = dH$

$$\mathcal{L}_X \omega = d\iota_X \omega + \iota_X d\omega = d\iota_X \omega = 0$$

donc $(\iota_X \omega)$ est fermée.

$\Rightarrow$ OK localement ou dans $\mathbb{R}^{2n}$

⚠ Intégrateurs symplectiques

cf. A. Hamdouni, M. Chhay (chorégraphie des vortexes ponctuels)

Q2 : $\cancel{X}_H, \omega \overset{?}{\Rightarrow}$ canonique

Déf Carte de Darboux :

$$\omega = d\theta = d\left(\sum_{i=1}^{n} p_i \, dq^i\right)$$

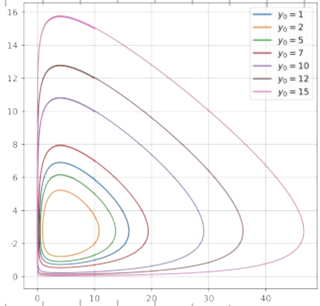
Théorème (Darboux, Poincaré, Moser, Weinstein)

Une variété symplectique peut être recouverte par cartes de Darboux

Démonstration $\in$ Astuce de Moser

Exemple : Système de Lotka-Volterra

$$\begin{cases} \dot{x} = \alpha x - \beta x y \\ \dot{y} = \delta x y - \gamma y \end{cases}$$



Lois de conservation :

$$\delta x - \gamma \ln(x) + \beta y - \alpha \ln(y) = \text{const}$$

$$\omega = \frac{1}{xy} dx \wedge dy$$

Changement de coordonnées :

$$p = \ln(x), \quad q = \ln(y) \implies \omega \text{ canonique}$$

$$H = \delta e^{p} - \gamma p + \beta e^{q} - \alpha q$$

4. Et les Poissons

- La structure de Poisson symplectique

$\mathbb{R}^{2n}$, $\quad \omega = \sum_{i=1}^{n} dq^i \wedge dp_i$

$$X_f = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial p_i} \frac{\partial}{\partial q^i} - \frac{\partial f}{\partial q^i} \frac{\partial}{\partial p_i} \right)$$

$$X_g = \sum_{i=1}^{n} \left(\frac{\partial g}{\partial p_i} \frac{\partial}{\partial q^i} - \frac{\partial g}{\partial q^i} \frac{\partial}{\partial p_i} \right)$$

<u>Exo</u>

$$\iota_{X_g} \iota_{X_f} \omega = \iota_{X_g} df =$$

$$= \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q^i} \frac{\partial f}{\partial p_i} \right) =: \{f, g\}$$

Propriétés $\qquad a, b \in \mathbb{R}, \ f, g, h \in C^\infty(M = \mathbb{R}^{2n})$

- Algébriques

$\{f, g\} = -\{g, f\}$; $\{af + bg, h\} = a\{f, h\} + b\{g, h\}$

Leibniz : $\{fg, h\} = \{f, h\}g + f\{g, h\}$

Jacobi : $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$

- Mécaniques :

$\{q^i, q^j\} = 0$; $\{p_i, p_j\} = 0$; $\{q^i, p_j\} = \delta^i_j$

$\{q^i, H\} = \dfrac{\partial H}{\partial p_i} = \dot{q}^i$; $\{p_i, H\} = -\dfrac{\partial H}{\partial q^i} = \dot{p}_i$

$\dfrac{d}{dt} f(q, p) = \{f, H\}$

Déf $\{\cdot,\cdot\} : C^\infty(M)\times C^\infty(M) \to C^\infty(M)$ est une structure de Poisson si $\forall a,b\in\mathbb{R}$ $f,g,h \in C^\infty(M \overset{(x^1..x^d)}{=} \cancel{\mathbb{R}^{2n}})$

$$\{f,g\} = -\{g,f\} \;;\; \{af+bg,h\} = a\{f,h\} + b\{g,h\}$$

Leibniz : $\{fg,h\} = \{f,h\}g + f\{g,h\}$

Jacobi : $\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0$

Déf système Hamiltonien sur M $(x^1..x^d)$

$$\cancel{\{q^i,q^j\} = 0 \;;\; \{p_i,p_j\} = 0 \;;\; \{q^i,p_j\} = \delta^i_j}$$

$$\cancel{\{q^i,H\} = \frac{\partial H}{\partial p_i} = \dot q^i \;;\; \{p_i,H\} = -\frac{\partial H}{\partial q^i} = \dot p_i}$$

$$\frac{d}{dt} f(\overset{x}{\cancel{q,p}}) = \{f,H\}$$

Description de $\{\cdot,\cdot\}$ avec les bivecteurs:

$$\Pi := \sum_{1\le i<j\le n} \pi^{ij}(x)\frac{\partial}{\partial x^i}\wedge\frac{\partial}{\partial x^j}, \quad \pi^{ij}=\{x^i, x^j\}$$

Théorème (Weinstein splitting)

Une variété de Poisson (M,Π) autour de chaque point x_0, admet une carte avec les coordonnées $(q^1 \ldots q^k, p_1 \ldots p_k, c_1 \ldots c_m)$ t.q.

$$\Pi = \sum_{i=1}^{k}\frac{\partial}{\partial q^i}\wedge\frac{\partial}{\partial p_i} + \sum_{1\le i<j\le m}\varphi_{ij}(c)\frac{\partial}{\partial c_i}\wedge\frac{\partial}{\partial c_j},$$

et $\varphi_{ij}(0)=0$.

Démonstration par récurrence sur rang$(\Pi(x_0))$ avec l'astuce de Moser.

Lasagne de Poisson

c

Casimir

$\omega(p,q,c=c_{reg\,1})$

$\omega(p,q,c=c_{reg\,2})$

$\vdots$

$\omega(p,q,c=c_{sing})$

q

p

Application 1:

Systems & Control Letters 14 (1990) 341–346
North-Holland

Stabilization of rigid body dynamics by the Energy–Casimir method

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Abstract: We show how the Energy–Casimir method can be used to prove stabilizability of the angular momentum equations of the rigid body about its intermediate axis of inertia, by a single torque applied about the major or minor axis. We also show how this system has associated with it, a Lie–Poisson bracket which is invariant under SO(3) for small feedback, but is invariant under SO(2, 1) for feedback large enough to achieve stability.

Keywords: Stabilization; feedback; rigid body; Energy–Casimir; Hamiltonian.

equations may be asymptotically stabilized to revolute motion about a principal axis.

In Brockett [12], it is shown by finding a Lyapunov function that the null solution of the angular velocity equations may be stabilized by two control torques. In Aeyels [2], the same result is demonstrated by center manifold theory. In Aeyels [1], it is shown that the null solution of the angular velocity equations may be 'robustly' stabilized (though not asymptotically stabilized) by a single torque about the major or minor axis. This result is in fact sharp since Aeyels and Szafranski [3] show that the equations cannot be asymptotically stabilized by a single torque about a principal axis.

Another area in which there has been much fruitful activity in recent years is the analysis of the stability of coupled rigid and flexible bodies. Some new techniques for this problem were introduced in Baillieul and Levi [7] and Krishnaprasad and Marsden [16] both based on geometric formu-

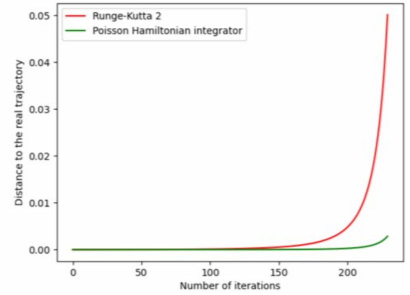
Application 2 :

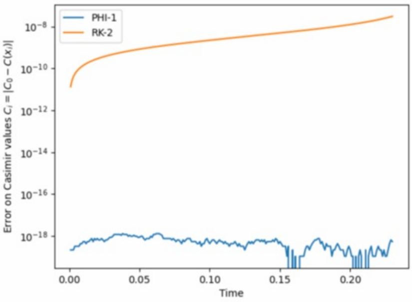
$$\begin{aligned} \dot{x}_1 &= x_1(x_2+x_3) \\ \dot{x}_2 &= x_2(-x_1+x_3) \\ \dot{x}_3 &= -x_3(x_1+x_2). \end{aligned}$$

Π quadratique

H linéaire

Lotka-Volterra en $\mathbb{R}^3$ singularité e C





cf Pol Vanhaecke

et les intégrateurs de Poisson de Cosserat

Intégrabilité ("Application" 3)

Liouville–Arnold theorem (*complete integrability*):
Let n smooth functions F_i on a $2n$-dimensional manifold M be in involution: $\{F_i, F_j\} = 0$. Consider the level set of functions F_i $M_f = \{(\mathbf{q}, \mathbf{p}) : F_i(\mathbf{q}, \mathbf{p}) = f_i, i = 1, \ldots, n\}$.
Let the functions F_i be independent on M_f. Then:

- *M_f is invariant under the phase flow with the hamiltonian function $H = F_1(\mathbf{q}, \mathbf{p})$ ($\dot{p}_i = -\frac{\partial H}{\partial q_i}$, $\dot{q}_i = \frac{\partial H}{\partial p_i}$).*
- *If M_f is compact, then each connected component of it is diffeomorphic to an n-dimensional torus* $\mathbb{T}^n = \{(\varphi_1, \ldots, \varphi_n) \mod 2\pi\}$
- *The phase flow with the hamiltonian H defines on M_f a quasi-periodic motion: $\dot{\varphi} = \omega(\mathbf{f})$ (action-angle variables)*
- *Canonical hamiltonian equations can be integrated in quadratures.*

K
A
M

In this case the system is called *completely integrable* or *integrable in Liouville–Arnold sense*

Vers les EDPs

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Kadomtsev–Petviashvili equation

From Wikipedia, the free encyclopedia

In mathematics and physics, the **Kadomtsev–Petviashvili equation** (often abbreviated as **KP equation**) is a partial differential equation to describe nonlinear wave motion. Named after Boris Borisovich Kadomtsev and Vladimir Iosifovich Petviashvili, the KP equation is usually written as:

$$\partial_x(\partial_t u + u\partial_x u + \epsilon^2 \partial_{xxx} u) + \lambda \partial_{yy} u = 0$$

where $\lambda = \pm 1$. The above form shows that the KP equation is a generalization to two spatial dimensions, *x* and *y*, of the one-dimensional Korteweg–de Vries (KdV) equation. To be physically meaningful, the wave propagation direction has to be not-too-far from the *x* direction, i.e. with only slow variations of solutions in the *y* direction.

Like the KdV equation, the KP equation is completely integrable.[1][2][3][4][5] It can also be solved using the inverse scattering transform much like the nonlinear Schrödinger equation.[6]

Crossing swells, consisting of near-cnoidal wave trains. Photo taken from Phares des Baleines (Whale Lighthouse) at the western point of Île de Ré (Isle of Rhé), France, in the Atlantic Ocean. The interaction of such near-solitons in shallow water may be modeled through the Kadomtsev–Petviashvili equation.

Contents [hide]

Merci
pour
votre
patience

⚠ Quiberon ⚠