Saint-Venant compatibility condition revisited

Romain Lloria, Rodrigue Desmorat, Boris Kolev

At the Paris-Saclay Mechanics Laboratory (LMPS)

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The problem

Context: Infinitesimal deformations

Compatibility: a displacement vector field ξ generates a symmetric covariant 2-tensor of infinitesimal deformation

$$\varepsilon = \frac{1}{2}\left(\nabla\xi + (\nabla\xi)^t\right).$$

Related questions:

- find a necessary condition of compatibility
- calculate ξ form the data of ε

Cesàro-Volterra's response [1906]

Calculate $\boldsymbol{\xi}$ at $\mathbf{x} \in \Omega \subset \mathbb{R}^3$ by joigning an initial point $\mathbf{x}_0 \in \Omega \subset \mathbb{R}^3$ with a smooth path $\mathcal{C}$. The fundamental lemma of integral calculus gives:

$$\boldsymbol{\xi}(\mathbf{x}) = \boldsymbol{\xi}(\mathbf{x}_0) + \int_{\mathbf{x}_t \in \mathcal{C}} \frac{d}{dt}\boldsymbol{\xi}(\mathbf{x}_t) = \cdots \text{ only } \boldsymbol{\varepsilon}$$

Compatibility condition: $\boldsymbol{\xi}$ does not depend on the path $\mathcal{C}$ and $\mathbf{x}_0$

$\Downarrow$

closed path $\mathbf{x} = \mathbf{x}_0$

$\Downarrow$

the integral is zero

$\Downarrow$ Stokes

$$\boxed{\partial_l \partial_k \varepsilon_{ij} - \partial_l \partial_i \varepsilon_{jk} + \partial_i \partial_j \varepsilon_{kl} - \partial_k \partial_j \varepsilon_{il} = 0}$$

Integrator: taking for $\mathcal{C}$ the straight line $\mathbf{x}_t = t\mathbf{x}$

$$\boxed{\boldsymbol{\xi}(\mathbf{x}) = \boldsymbol{\xi}(0) + \boldsymbol{\Omega}(0)\mathbf{x} + \int_0^1 \boldsymbol{\varepsilon}(t\mathbf{x})\mathbf{x}dt + \int_0^1 (1-t)\left[\nabla\boldsymbol{\varepsilon}(t\mathbf{x}) : (\mathbf{x}\otimes\mathbf{x}) - (\mathbf{x}\otimes\mathbf{x}) : \nabla\boldsymbol{\varepsilon}(t\mathbf{x})\right] dt}$$

where $\boldsymbol{\xi}(0) + \boldsymbol{\Omega}(0)\mathbf{x}$ is an element of $\mathfrak{se}_3 := \left\{\boldsymbol{\xi} + \boldsymbol{\Omega}\mathbf{x} \mid \boldsymbol{\xi} \in \mathbb{R}^3,\ \boldsymbol{\Omega} \in \mathfrak{so}_3(\mathbb{R})\right\}$

Bernstein-Gelfand-Gelfand's response

Differential complex of linear elasticity [Eastwood 2000] which derives from De Rham one

$$\begin{array}{ccccccc}
\boxed{\Omega^0(\mathfrak{se}_3)} & \underset{\mathcal{L}_1}{\overset{\mathrm{d}_1}{\rightleftarrows}} & \boxed{\Omega^1(\mathfrak{se}_3)} & \underset{\mathcal{L}_2}{\overset{\mathrm{d}_2}{\rightleftarrows}} & \boxed{\Omega^2(\mathfrak{se}_3)} & \underset{\mathcal{L}_3}{\overset{\mathrm{d}_3}{\rightleftarrows}} & \boxed{\Omega^3(\mathfrak{se}_3)} \\
\mathcal{R}_0^{-1}\uparrow\downarrow\mathcal{R}_0 & & \mathcal{R}_1^{-1}\uparrow\downarrow\mathcal{R}_1 & & \mathcal{R}_2^{-1}\uparrow\downarrow\mathcal{R}_2 & & \mathcal{R}_3^{-1}\uparrow\downarrow\mathcal{R}_3 \\
\boxed{\boldsymbol{\xi}} & \xrightarrow{\mathcal{D}_1 = \nabla^s} & \boxed{\boldsymbol{\varepsilon}} & \xrightarrow{\mathcal{D}_2 = \textbf{rot} \circ \textbf{rot}^t} & \boxed{\textbf{Inc}} & \xrightarrow{\mathcal{D}_3 = \textbf{div}} & \boxed{\boldsymbol{\nu}}
\end{array}$$

$d_{k+1} d_k \alpha = 0 \implies$ Saint-Venant compatibility condition:

$$\boxed{\text{compatibility} \iff \boldsymbol{\varepsilon} = \mathcal{D}_1 \boldsymbol{\xi} \underset{\mathcal{D}_2}{\implies} \mathcal{D}_2 \boldsymbol{\varepsilon} = \underbrace{\mathcal{D}_2 \mathcal{D}_1}_{=0} \boldsymbol{\xi} = 0 \iff \textbf{Inc} = 0}$$

By composition the Poincaré integrator gives [Snorre et al. 2020]

$$\boxed{\boldsymbol{\xi}} \underset{\mathcal{K}_1}{\overset{\nabla^s}{\rightleftarrows}} \boxed{\boldsymbol{\varepsilon}} \underset{\mathcal{K}_2}{\overset{\textbf{rot} \circ \textbf{rot}^t}{\rightleftarrows}} \boxed{\textbf{Inc}} \underset{\mathcal{K}_3}{\overset{\textbf{div}}{\rightleftarrows}} \boxed{\boldsymbol{\nu}}$$

In particular, the integrator $\mathcal{K}_1$ is the **same as the Cesàro-Volterra one**.

Generalization of De Rham complex

Dubois-Violette [2002] generalizes De Rham's theory (any symmetries)

$\Downarrow$

linear elasticity complex naturally adapted to symmetries

How to do it explicitly?

De Rham complex on $\mathbb{R}^3$: width 1

$$\underset{\boxed{1}}{\Omega_1^1(\mathbb{R}^3)} \xrightarrow{\textbf{grad}} \underset{\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}}{\Omega_1^2(\mathbb{R}^3)} \xrightarrow{\textbf{rot}} \underset{\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}}{\Omega_1^3(\mathbb{R}^3)} \xrightarrow{\textbf{div}} \underset{\bullet}{\{0\}}$$

$$\begin{cases} \textbf{rot} \circ \textbf{grad} = 0 \\ \textbf{div} \circ \textbf{rot} = 0 \end{cases}$$

Elasticity complex on $\mathbb{R}^3$: width 2

$$\underset{\boxed{1}}{\Omega_2^1(\mathbb{R}^3)} \xrightarrow{d_1} \underset{\begin{array}{|c|c|}\hline 1 & 2\\\hline\end{array}}{\Omega_2^2(\mathbb{R}^3)} \xrightarrow{d_2} \underset{\begin{array}{|c|c|}\hline 1 & 2\\\hline 3 & \\\hline\end{array}}{\Omega_2^3(\mathbb{R}^3)} \xrightarrow{d_3} \underset{\begin{array}{|c|c|}\hline 1 & 2\\\hline 3 & 4\\\hline\end{array}}{\Omega_2^4(\mathbb{R}^3)}$$

$$d_3 d_2 d_1 = 0$$

$$D_1 = d_1, \quad D_2 = d_3 d_2$$

Contribution 1: New complex of linear elasticity

$X, Y, Z, T \in \mathrm{Vect}(T\Omega)$ and ∇ flat derivative

$$D_1\xi = \nabla^s\xi := \varepsilon$$

1	2

$$D_2\varepsilon = (\nabla_T\nabla_Z\varepsilon)(X,Y) - (\nabla_T\nabla_X\varepsilon)(Y,Z) + (\nabla_X\nabla_Y\varepsilon)(Z,T) - (\nabla_Z\nabla_Y\varepsilon)(X,T)$$

$$:= \mathbf{W}(X,Y,Z,T)$$

1	2
3	4

Proposition

$$D_2D_1 = 0 \implies \textit{Saint-Venant compatibility condition}$$

Proof: If the deformation ε is generated by a displacement ξ we have

$$\varepsilon = D_1\xi \xrightarrow{D_2} D_2\varepsilon = D_2D_1\xi = 0 \quad \text{i.e.} \quad \mathbf{W} = 0.$$

Interest: no more ad hoc bijections, but a complex naturally adapted to the symmetries of the elasticity tensors!

Contribution 2: Integrator

Question: How to calculate the displacement field ξ from the deformation ε?

$$\text{Dubois} - \text{Violette} \Longrightarrow \text{homotopy formula :}$$
$$\underbrace{D_1 K_1 \varepsilon}_{\text{integrator}} + \underbrace{K_2 D_2 \varepsilon}_{\text{obstruction}} = \varepsilon$$

How to do it explicitly?

- **(1) obstruction:** we define

$$K_2\mathbf{W}(\mathbf{x}) := \int_0^1 dt \int_0^t s\mathbf{W}(s\mathbf{x}) : (\mathbf{x} \otimes \mathbf{x})\, ds.$$

- **(2) intégrator:** primitive of $D_1 K_1 \varepsilon = \varepsilon - K_2\mathbf{W} \Longrightarrow$ Cesàro-Volterra

$$K_1\varepsilon(\mathbf{x}) = \xi(0) + \boldsymbol{\Omega}(0)\mathbf{x} + \int_0^1 \varepsilon(t\mathbf{x})\,\mathbf{x}\, dt + \int_0^1 (1-t)\left[\nabla\varepsilon(t\mathbf{x}) : (\mathbf{x}\otimes\mathbf{x}) - (\mathbf{x}\otimes\mathbf{x}) : \nabla\varepsilon(t\mathbf{x})\right] dt$$

Contribution 3: Reinterpretation of $\mathbf{rot}\,\mathbf{rot}^t\,\varepsilon$

In the literature, incompatibility tensor: $\mathbf{Inc} = \mathbf{rot}\,\mathbf{rot}^t\,\varepsilon$

But rotationnal of a 2-tensor is poorly defined

$$\Downarrow$$

reinterpretation of $\mathbf{Inc} = \mathbf{rot}\,\mathbf{rot}^t\,\varepsilon$ by harmonic decomposition of the Saint-Venant tensor $\mathbf{W}$ into irreducible components

$$\mathbf{Inc} = -\operatorname{tr}_{12}\mathbf{W} + \frac{1}{2}\operatorname{tr}\left[\operatorname{tr}_{12}\mathbf{W}\right]\mathbf{q} \qquad \text{et} \qquad \mathbf{W} = -\mathbf{Inc} \owedge \mathbf{q} + \frac{1}{2}\operatorname{tr}\left[\mathbf{Inc}\right]\mathbf{q} \owedge \mathbf{q},$$

where $\owedge$ is the Kulkarni-Nomizu product modified to have the symmetries of $\mathbf{W}$ and $\mathbf{q}$ the Euclidean metric

Proof of $\mathbf{Inc} = -\operatorname{tr}_{12}\mathbf{W} + \frac{1}{2}\operatorname{tr}\left[\operatorname{tr}_{12}\mathbf{W}\right]\mathbf{q}$ in coordinate:

$$\mathbf{Inc} = \mathbf{rot}\,\mathbf{rot}^t\,\varepsilon \text{ and } W_{ijkl} = \partial_l\partial_k\varepsilon_{ij} - \partial_l\partial_i\varepsilon_{jk} + \partial_i\partial_j\varepsilon_{kl} - \partial_k\partial_j\varepsilon_{il}$$

Reconstruction of **W** from **Inc**

$X, Y, Z, T \in \mathrm{Vect}(T\Omega)$

Definition (Kulkarni-Nomizu product)

h and **q**, symmetric covariant 2-tensors field

$$(\mathbf{h} \owedge \mathbf{q})(X, Y, Z, T) := \mathbf{h}(X, Y)\mathbf{q}(Z, T) + \mathbf{h}(Z, T)\mathbf{q}(X, Y) - \mathbf{h}(Y, Z)\mathbf{q}(X, T) - \mathbf{h}(X, T)\mathbf{q}(Y, Z)$$

In component:

$$(\mathbf{h} \owedge \mathbf{q})_{ijkl} := h_{ij}q_{kl} + h_{kl}q_{ij} - h_{jk}q_{il} - h_{il}q_{jk}$$

$$\underset{\begin{array}{|c|c|}\hline X & Y \\ \hline\end{array}\ \begin{array}{|c|c|}\hline Z & T \\ \hline\end{array}}{\mathbf{h}(X, Y)\mathbf{q}(Z, T)} \quad \underset{\longrightarrow}{\longmapsto} \quad \underset{\text{pairs permutation}}{+\mathbf{h}(Z, T)\mathbf{q}(X, Y)} \quad \underset{\longrightarrow}{\longmapsto} \quad \underset{\begin{array}{|c|}\hline X \\ \hline Z \\ \hline\end{array}}{-\mathbf{h}(Y, Z)\mathbf{k}(X, T) - \mathbf{h}(X, T)\mathbf{k}(Y, Z)}$$

We obtain the symmetry

$$\begin{array}{|c|c|}\hline X & Y \\ \hline Z & T \\ \hline\end{array}$$

Reconstruction of **W** from **Inc**

(1) We show that the following application is a linear isomorphism:

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array} \longrightarrow \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array}, \qquad \mathbf{h} \longmapsto \mathbf{h} \owedge \mathbf{q}.$$

- linear
- injective: $\mathrm{Ker} = \{0\}$
- surjective: equality of dimensions

(2) We show the formula:

$$\text{bijectivity} \implies \exists!\, \mathbf{h},\ \mathbf{W} = \mathbf{h} \owedge \mathbf{q} \overset{\mathrm{tr}}{\implies} \mathbf{h} = \mathrm{tr}_{12}\, \mathbf{W} - \frac{1}{4}\, \mathrm{tr}\left[\mathrm{tr}_{12}\, \mathbf{W}\right] \mathbf{q}$$

Conclusion

- reformulation de l'élasticité par complexe naturellement adapté aux symétries
- reformulation de la condition de compatibilité de Saint-Venant $\mathbf{W} = 0$
- on a retrouvé l'intégrateur de Cesàro-Volterra
- terme d'obstruction qui mesure le degré d'incompatibilité d'une transformation
- réinterprétation de $\mathbf{rot}\,\mathbf{rot}^{t}\,\varepsilon$ par décomposition harmonique de $\mathbf{W}$