On the second-order tensorial nature of the state damage variable

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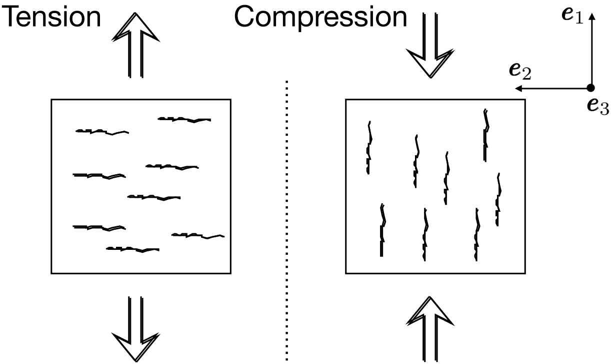
GDR GDM, La Rochelle, 27 juin 2025

LOADING-INDUCED ORTHOTROPIC DAMAGE

(IN CONCRETE, MAZARS ET AL. 1990)



$$\mathbf{D} \approx \begin{pmatrix} D_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{(\boldsymbol{e}_i)}$$

$$\mathbf{D} \approx \begin{pmatrix} 0 & 0 & 0 \\ 0 & D_{22} & 0 \\ 0 & 0 & D_{22} \end{pmatrix}_{(\boldsymbol{e}_i)}$$

Petrography S.P. Shah

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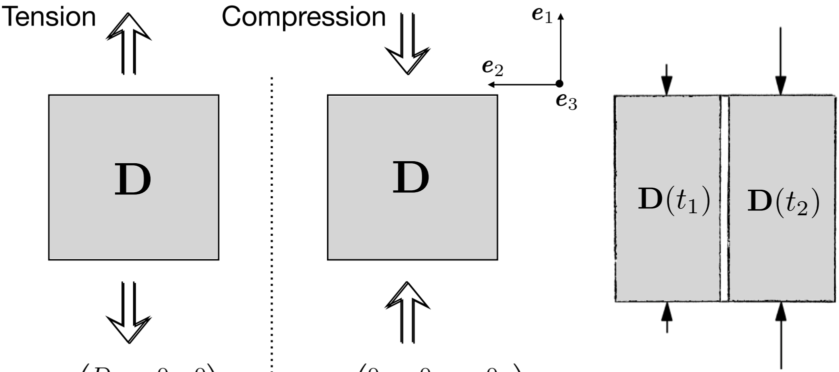

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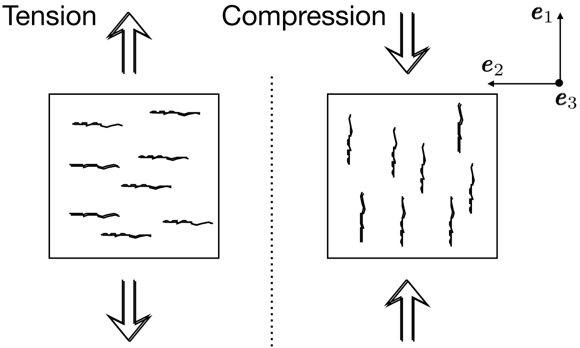
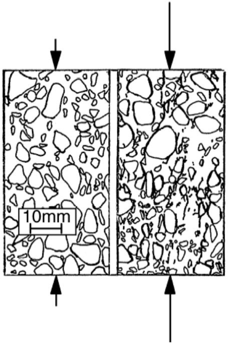
Anisotropic damage evolution law

$$\dot{\mathbf{D}} = \dot{\lambda} \langle \mathbf{N} \rangle^+ \qquad \left(i.e.,\ \dot{D}_{ij} = \dot{\lambda} \left(\langle \mathbf{N} \rangle^+ \right)_{ij} \right)$$

Loading-induced orthotropic damage

(in concrete, Mazars et al. 1990)





Petrography S.P. Shah

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Anisotropic damage evolution law

$$\dot{\mathbf{D}} = \dot{\lambda} \langle \epsilon \rangle^+ \qquad \left(i.e.,\ \dot{D}_{ij} = \dot{\lambda} \left(\langle \boldsymbol{\epsilon} \rangle^+ \right)_{ij} \right)$$

Tensorial nature of damage

The tensorial nature of the damage variable has been discussed in literature (Vakulenko & Kachanov, 1971, Chaboche, 1979, 1984, Leckie & Onat, 1981, Cordebois & Sidoroff, 1982, Lemaitre & Chaboche, 1985, Murakami, 1978, 1988).

The main question was whether or not the damage state of initially isotropic materials could be represented

- by a 4th order tensor $\mathbb{D}$ (Chaboche, 1978, 1979, Leckie & Onat, 1981, Krajcinovic, 1985, Kachanov, 1993),
- or by two 2nd order tensors (Leckie & Onat, 1981, Ladevèze, 1983, 1995, Desmorat et al, 2018, 2021),
- or by a single 2nd order tensor $\mathbf{D}$ (Murakami, 1978, 1988, Cordebois & Sidoroff, 1982, Ramtani et al, 1992, Kachanov, 1993, Halm & Dragon, 1996, 1998, Papa & Talercio, 1996, Voyiadjis and Park, 1997, Steinmann & Carol, 1998, Lemaitre et al, 2000, Menzel et al, 2002, Brunig, 2003, Desmorat et al 2007, 2016, 2018, Loiseau, 2023, Basmaji, 2024),

instead of scalar variable d introduced by Kachanov (1958) and Rabotnov (1969).

Properties of damage variable

(Fau et al, 2024)

A frame independent damage variable

- is a tensor $\mathbf{D}$,
- can be measured as $\mathbf{D}(\widetilde{\mathbf{E}})$ from the effective (damaged) elasticity tensor $\widetilde{\mathbf{E}}$ (such as $\boldsymbol{\sigma} = \widetilde{\mathbf{E}} : \boldsymbol{\epsilon}^e$),
- can be measured in any basis,

$$\mathbf{Q} \star \left(\mathbf{D}(\widetilde{\mathbf{E}})\right) = \mathbf{D}(\mathbf{Q} \star \widetilde{\mathbf{E}}), \quad \text{for any rotation } \mathbf{Q},$$

where $\mathbf{Q}\,\star$ means the action of a rotation $\mathbf{Q}$.
Mathematically, $\mathbf{D}$ is a covariant of the effective elasticity tensor $\widetilde{\mathbf{E}}$.

Definition ($\mathbf{C}(\widetilde{\mathbf{E}})$ covariant of $\widetilde{\mathbf{E}}$)

$$\mathbf{Q} \star \left(\mathbf{C}(\widetilde{\mathbf{E}})\right) = \mathbf{C}(\mathbf{Q} \star \widetilde{\mathbf{E}}), \quad \text{for any rotation } \mathbf{Q}.$$

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where $\mathbf{Q}\star$ means the action of a rotation $\mathbf{Q}$.

Mathematically, D is a covariant of the effective elasticity tensor $\widetilde{\mathbf{E}}$.

Orthotropy is a symmetry class of effective elasticity tensors (for which $\mathbf{D} \neq 0$). All the vector covariant of orthotropic elasticity tensors vanish (Olive et al, 2021).

To conclude :

The damage variable $\mathbf{D}$ cannot be a vector.

Different approaches allowing to derive ”the” coupling elasticity / damage

- By phenomenology : invent a (good ?) formula

$$\tilde{\mathbf{E}} = \tilde{\mathbf{E}}(\mathbf{D}) = \dots$$

Approach simplified by the effective stress concept (Lemaitre, 1971, Cordebois & Sidoroff, 1982, Lemaitre & Chaboche, 1985)

$$\begin{cases} \tilde{\boldsymbol{\sigma}} = \tilde{\boldsymbol{\sigma}}(\boldsymbol{\sigma}, \mathbf{D}) = \dots \\ \tilde{\boldsymbol{\sigma}} = \mathbf{E} : \boldsymbol{\epsilon}^e \end{cases} \longrightarrow \begin{cases} \boldsymbol{\sigma} = \tilde{\mathbf{E}}(\mathbf{D}) : \boldsymbol{\epsilon}^e \\ \tilde{\mathbf{E}} = \tilde{\mathbf{E}}(\mathbf{D}) = \dots \end{cases}$$

- From micro-mechanics (Vakulenko & Kachanov, 1972, Kachanov, 1994) : solve theoretically a (field) elasticity problem at the micro-scale
$\rightarrow \tilde{\mathbf{E}}(\mathbf{D})$ determined as an average over a Representative Volume Element.

ANISOTROPIC DAMAGE STATE MODELING FROM 2D DISCRETE SIMULATIONS OF FRACTURE

Coworkers : F. Loiseau, C. Oliver-Leblond, T. Verbeke

Discrete virtual testing

AREA ELEMENT DISCRETIZED BY VORONOÏ CELLS

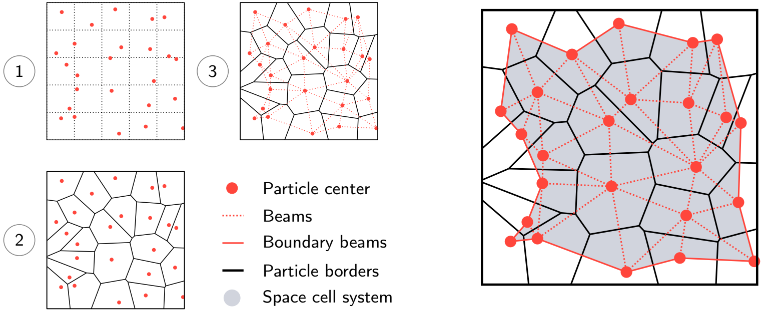
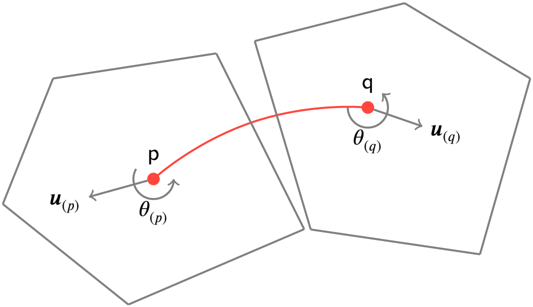


Figure – Representation of the mesh of the beam-particle model.
(1) positioning of the particle centers in the grid.
(2) generation of the particle border by the Voronoi tessellation of the particle centers.
(3) add the beams network based on the Delaunay triangulation of the particle centers.

Material as a brittle beams network

DEAP 2D beam-particle model (Delaplace, 2009, Vassaux et al, 2015, 2016)



- The beam-particle specimen is composed of a set of rigid particles.
- The dual graph of the Voronoi tessellation is the Delaunay triangulation. It associates a segment to each pair of neighboring particles.
- Those segments are used as the geometric support for a (brittle) beams network.
- This (brittle) beams network models the cohesion of (quasi-brittle) materials.

Dataset of Discrete Element computations

F. Loiseau (2023)

Dataset of 76 356 effective elasticity tensors available at

https://doi.org/10.57745/LYHM4W

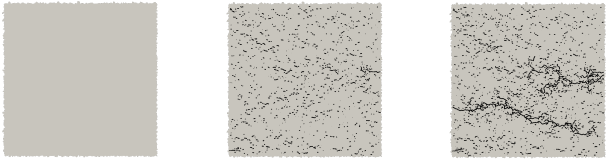


Figure – For an uniaxial tension in vertical direction

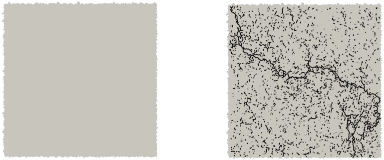
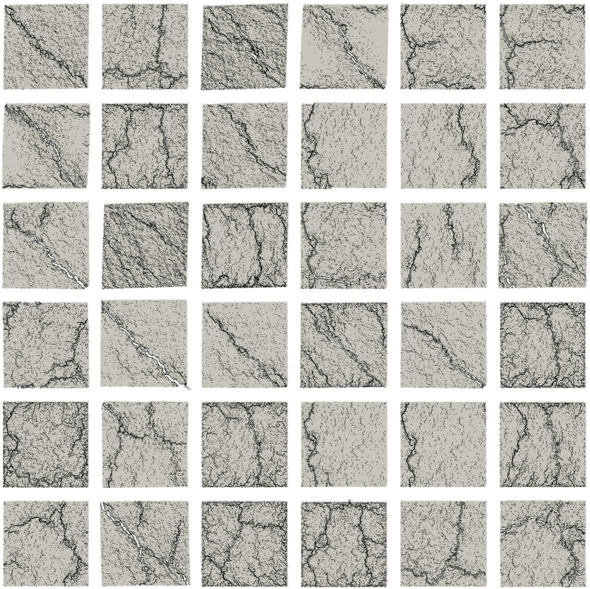


Figure – For an equi-biaxial tension

Illustration of localized cracking patterns in the dataset



Measurements of (damaged) elasticity tensors

76 356 effective elasticity tensors of discrete (possibly cracked) specimens are measured

- from symmetric average strains and stresses computed from the particle displacements and forces (Bagi, 2006),
- using 3 linearly independent strain tensors $\boldsymbol{\varepsilon}_{\text{imp}}^{(i)}$ and associated computed (average) stress tensors $\boldsymbol{\sigma}_{\text{av}}^{(i)}$,

$$[\boldsymbol{\varepsilon}_{\text{imp}}^{(1)}] = \begin{bmatrix} \varepsilon \\ 0 \\ 0 \end{bmatrix}, \quad [\boldsymbol{\varepsilon}_{\text{imp}}^{(2)}] = \begin{bmatrix} 0 \\ \varepsilon \\ 0 \end{bmatrix}, \quad [\boldsymbol{\varepsilon}_{\text{imp}}^{(3)}] = \begin{bmatrix} 0 \\ 0 \\ \sqrt{2}\varepsilon \end{bmatrix},$$

where ε is sufficiently small (chosen such that the loading remains elastic).

The elasticity tensor of a specimen is obtained in Kelvin (matrix) notation as the symmetrized 3×3 matrix product

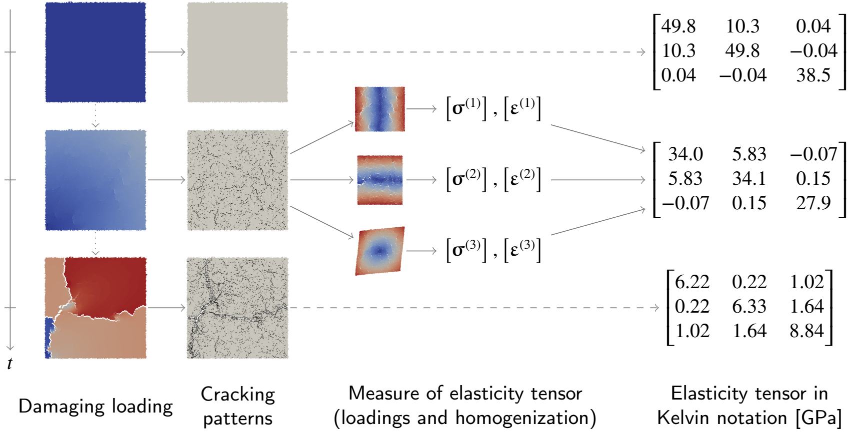
$$[\mathbf{E}] = \left\{ \left[\left[\boldsymbol{\sigma}_{\text{av}}^{(1)}\right], \left[\boldsymbol{\sigma}_{\text{av}}^{(2)}\right], \left[\boldsymbol{\sigma}_{\text{av}}^{(3)}\right] \right] \cdot \left[\left[\varepsilon_{\text{imp}}^{(1)}\right], \left[\varepsilon_{\text{imp}}^{(2)}\right], \left[\varepsilon_{\text{imp}}^{(3)}\right] \right]^{-1} \right\}^{S}.$$

$$[\varepsilon_{\text{imp}}^{(i)}] = \begin{bmatrix} \varepsilon_{xx}^{(i)} \\ \varepsilon_{yy}^{(i)} \\ \sqrt{2}\varepsilon_{xy}^{(i)} \end{bmatrix}, \quad [\boldsymbol{\sigma}_{\text{av}}^{(i)}] = \begin{bmatrix} (\boldsymbol{\sigma}_{\text{av}}^{(i)})_{xx} \\ (\boldsymbol{\sigma}_{\text{av}}^{(i)})_{yy} \\ \sqrt{2}(\boldsymbol{\sigma}_{\text{av}}^{(i)})_{xy} \end{bmatrix}$$

In Kelvin notation (here in 2D) :

$$[\mathbf{E}] = [\mathbf{E}]^{T} = \begin{bmatrix} E_{1111} & E_{1122} & E_{1112} \\ E_{2211} & E_{2222} & E_{2212} \\ E_{1211} & E_{1222} & E_{1212} \end{bmatrix}$$

Effective elasticity tensors in a bi-tension loading



ISOTROPIC (SCALAR) DAMAGE ?

Damage state represented by scalar variable d

Macroscopic Area Element = AE

Virgin AE

$$d = 0$$

Damaged AE

d

scalar

$$\boldsymbol{\sigma} = \widetilde{\mathbf{E}} : \boldsymbol{\epsilon} = \mathbf{E}(1-d) : \boldsymbol{\epsilon}, \qquad d = 1 - \frac{\|\widetilde{\mathbf{E}}\|}{\|\mathbf{E}\|}.$$

Multiaxial non-proportional loading case

Periodic boundary conditions, shear → (plane) tension loading

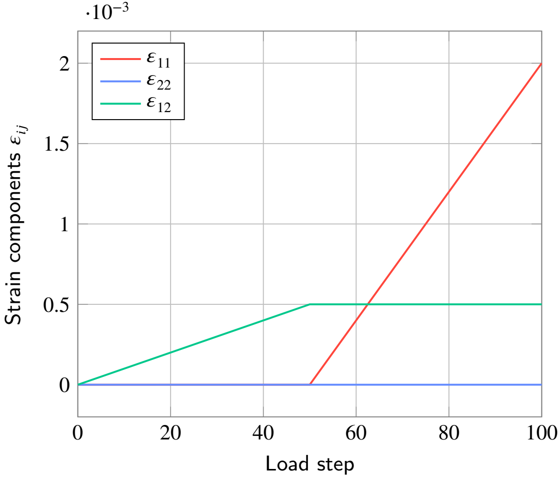
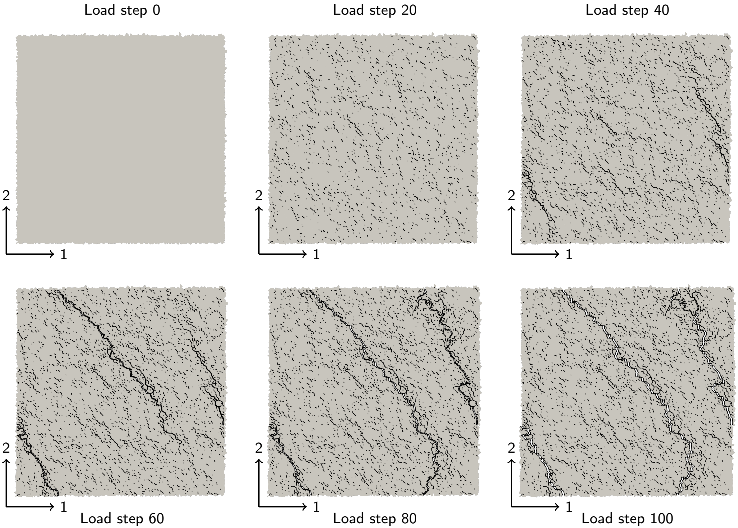


Figure – Applied macroscopic strain (on the Area Element)

Micro-cracking patterns at different load steps

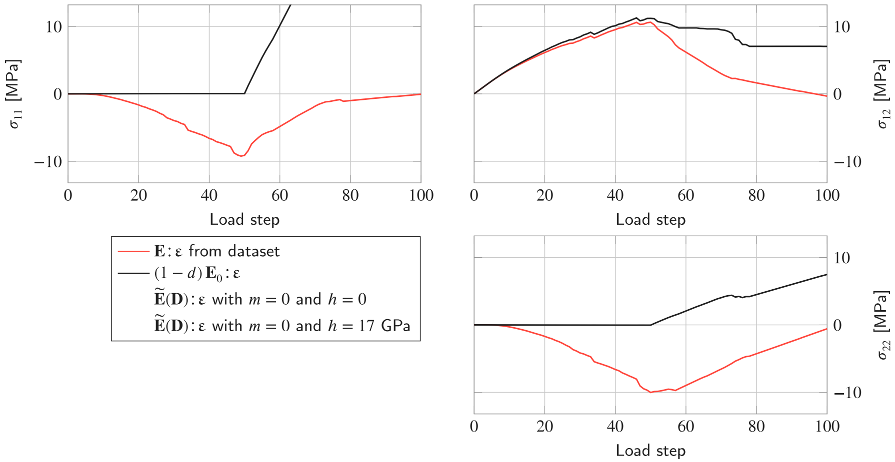
Loading-induced anisotropic damage



(Area Elements are deformed, displacements are scaled with a factor of 10)

Macroscopic stress components by isotropic damage

Non-proportional shear→tension loading, Comparison to DE data



The isotropic damage variable d is computed as $d = 1 - \|\widetilde{\mathbf{E}}\|/\|\mathbf{E}\|$, in (Loiseau et al, 2023).

Damage state by (which ?) tensorial variable $\mathbf{D}$

Macroscopic Area Element = AE

Virgin AE

$$\mathbf{D} = 0$$

Damaged AE

$$\mathbf{D}$$

tensor

Harmonic decomposition Covariant construction of 2D elasticity tensors

Definition

A covariant of an elasticity tensor $\mathbf{E}$ is a tensor $\mathbf{C} = \mathbf{C}(\mathbf{E})$ which satisfies

$$\mathbf{C}(\mathbf{Q} \star \mathbf{E}) = \mathbf{Q} \star \mathbf{C}(\mathbf{E}), \quad \forall \mathbf{Q} \in \mathrm{O}(2).$$

Harmonic decomposition of 2D elasticity tensors

In 2D, we have the harmonic decomposition of effective elasticity tensors [a]

$$\widetilde{\mathbf{E}} = \left\{ \tilde{\mu} \,, \tilde{\kappa} \,, \widetilde{\mathbf{di}}' \,, \widetilde{\mathbf{H}} \right\} \in \mathbb{H}^0 \times \mathbb{H}^0 \times \mathbb{H}^2 \times \mathbb{H}^4, \qquad \operatorname{tr} \widetilde{\mathbf{di}}' = 0, \quad \operatorname{tr}_{ij} \widetilde{\mathbf{H}} = 0$$

Let $\mathbf{J} = \mathbf{I} - \frac{1}{2} \mathbf{1} \otimes \mathbf{1}$ be the deviatoric projector and $\widetilde{\mathbf{di}}' * \widetilde{\mathbf{di}}' = (\widetilde{\mathbf{di}}' \otimes \widetilde{\mathbf{di}}')^{s\prime}$.

a. $\mathbf{a}' = \mathbf{a} - \frac{1}{2} \operatorname{tr} \mathbf{a} \, \mathbf{1}, \quad I_{ijkl} = \frac{1}{2} (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \quad J_{ijkl} = I_{ijkl} - \frac{1}{2} \delta_{ij}\delta_{kl}.$

Dilatation and Voigt tensors are 2nd order covariants of the elasticity tensor [1]

$$\widetilde{\mathbf{di}} = \operatorname{tr}_{34} \widetilde{\mathbf{E}} = \widetilde{\mathbf{E}} : \mathbf{1}, \qquad \widetilde{\mathbf{vo}} = \operatorname{tr}_{13} \widetilde{\mathbf{E}},$$

and the following scalars are invariants of $\widetilde{\mathbf{E}}$,

$$\tilde{\mu} = \frac{1}{8} \left(2 \operatorname{tr} \widetilde{\mathbf{vo}} - \operatorname{tr} \widetilde{\mathbf{di}} \right), \qquad \tilde{\kappa} = \frac{1}{4} \operatorname{tr} \widetilde{\mathbf{di}},$$

$$\tilde{I}_2 = \widetilde{\mathbf{di}}' : \widetilde{\mathbf{di}}', \qquad \tilde{K}_3 = \widetilde{\mathbf{di}} : \widetilde{\mathbf{H}} : \widetilde{\mathbf{di}},$$

1. $\mathbf{C}(\mathbf{Q} \star \widetilde{\mathbf{E}}) = \mathbf{Q} \star \mathbf{C}(\widetilde{\mathbf{E}}), \quad \forall \mathbf{Q} \in \mathrm{O}(2).$

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1. $\mathbf{C}(\mathbf{Q} \star \widetilde{\mathbf{E}}) = \mathbf{Q} \star \mathbf{C}(\widetilde{\mathbf{E}}), \quad \forall \mathbf{Q} \in \mathrm{O}(2).$

In 2D, we have the explicit harmonic decomposition

$$\widetilde{\mathbf{E}} = \underbrace{2\tilde{\mu}\,\mathbf{J} + \tilde{\kappa}\,\mathbf{1}\otimes\mathbf{1}}_{\mathbf{Iso}(\widetilde{\mathbf{E}})} + \underbrace{\frac{1}{2}\left(\mathbf{1}\otimes\widetilde{\mathbf{di}}' + \widetilde{\mathbf{di}}'\otimes\mathbf{1}\right)}_{\mathbf{Dil}(\widetilde{\mathbf{E}})} + \widetilde{\mathbf{H}},$$

where $\mathbf{J} = \mathbf{I} - \frac{1}{2}\mathbf{1}\otimes\mathbf{1}$ be the deviatoric projector and $\widetilde{\mathbf{di}}' * \widetilde{\mathbf{di}}' = (\widetilde{\mathbf{di}}'\otimes\widetilde{\mathbf{di}}')^{s\prime}$.

Dilatation and Voigt tensors are 2nd order covariants of the elasticity tensor [2]

$$\widetilde{\mathbf{di}} = \mathrm{tr}_{34}\,\widetilde{\mathbf{E}} = \widetilde{\mathbf{E}} : \mathbf{1}, \qquad \widetilde{\mathbf{vo}} = \mathrm{tr}_{13}\,\widetilde{\mathbf{E}},$$

and the following scalars are invariants of $\widetilde{\mathbf{E}}$,

$$\tilde{\mu} = \frac{1}{8}\left(2\,\mathrm{tr}\,\widetilde{\mathbf{vo}} - \mathrm{tr}\,\widetilde{\mathbf{di}}\right), \qquad \tilde{\kappa} = \frac{1}{4}\,\mathrm{tr}\,\widetilde{\mathbf{di}},$$

$$\tilde{I}_2 = \widetilde{\mathbf{di}}' : \widetilde{\mathbf{di}}', \qquad \tilde{K}_3 = \widetilde{\mathbf{di}} : \widetilde{\mathbf{H}} : \widetilde{\mathbf{di}},$$

2. $\mathbf{C}(\mathbf{Q}\star\widetilde{\mathbf{E}}) = \mathbf{Q}\star\mathbf{C}(\widetilde{\mathbf{E}}), \quad \forall\mathbf{Q}\in\mathrm{O}(2).$

Covariant reconstruction of 2D elasticity tensors for orthotropy symmetry class

In 2D, we have the reconstruction formula for orthotropic elasticity tensors (Oliver-Leblond et al, 2021) based on harmonic decomposition

$$\widetilde{\mathbf{E}} = \underbrace{2\tilde{\mu}\,\mathbf{J} + \tilde{\kappa}\,\mathbf{1}\otimes\mathbf{1}}_{\mathbf{Iso}(\widetilde{\mathbf{E}})} + \underbrace{\frac{1}{2}\left(\mathbf{1}\otimes\widetilde{\mathbf{di}}' + \widetilde{\mathbf{di}}'\otimes\mathbf{1}\right)}_{\mathbf{Dil}(\widetilde{\mathbf{E}})} + \widetilde{\mathbf{H}}, \qquad \widetilde{\mathbf{H}} = \frac{2\tilde{K}_3}{\tilde{I}_2^2}\,\widetilde{\mathbf{di}}' * \widetilde{\mathbf{di}}',$$

where $\mathbf{J} = \mathbf{I} - \frac{1}{2}\mathbf{1}\otimes\mathbf{1}$ is the deviatoric projector and $\widetilde{\mathbf{di}}' * \widetilde{\mathbf{di}}' = (\widetilde{\mathbf{di}}'\otimes\widetilde{\mathbf{di}}')^{s\prime}$.

The (orthotropic) dilatation and Voigt tensors are 2nd order covariants of $\widetilde{\mathbf{E}}$ [3]

$$\widetilde{\mathbf{di}} = \mathrm{tr}_{34}\,\widetilde{\mathbf{E}} = \widetilde{\mathbf{E}} : \mathbf{1}, \qquad \widetilde{\mathbf{vo}} = \mathrm{tr}_{13}\,\widetilde{\mathbf{E}},$$

and the following scalars are invariants of $\widetilde{\mathbf{E}}$,

$$\tilde{\mu} = \frac{1}{8}\left(2\,\mathrm{tr}\,\widetilde{\mathbf{vo}} - \mathrm{tr}\,\widetilde{\mathbf{di}}\right), \qquad \tilde{\kappa} = \frac{1}{4}\,\mathrm{tr}\,\widetilde{\mathbf{di}},$$

$$\tilde{I}_2 = \widetilde{\mathbf{di}}' : \widetilde{\mathbf{di}}', \qquad \tilde{K}_3 = \widetilde{\mathbf{di}} : \widetilde{\mathbf{H}} : \widetilde{\mathbf{di}},$$

3. $\mathbf{C}(\mathbf{Q}\star\widetilde{\mathbf{E}}) = \mathbf{Q}\star\mathbf{C}(\widetilde{\mathbf{E}}), \quad \forall\mathbf{Q}\in\mathrm{O}(2).$

SECOND ORDER DAMAGE TENSOR

Definition of a tensorial damage variable

We assume that the elasticity tensor $\widetilde{\mathbf{E}}$ of a quasi-brittle material evolves during loading, due to damage $\mathbf{D}$, and that it has the initial isotropic value

$$\widetilde{\mathbf{E}}(\mathbf{D}=0)=\mathbf{E}=2\mu\,\mathbf{J}+\kappa\,\mathbf{1}\otimes\mathbf{1}, \qquad \mathbf{J}=\mathbf{I}-\frac{1}{2}\mathbf{1}\otimes\mathbf{1}.$$

The initial dilatation tensor is then isotropic (*i.e.*, spherical),

$$\mathbf{di}=\mathbf{E}:\mathbf{1}=2\kappa\,\mathbf{1}.$$

Definition (2nd order damage tensor, from harmonic decomposition)

$$\mathbf{D}=\mathbf{1}-\frac{\widetilde{\mathbf{di}}}{2\kappa}, \qquad \widetilde{\mathbf{di}}=\widetilde{\mathbf{E}}:\mathbf{1}.$$

- $\mathbf{D}$ can be mechanically measured from (effective) elasticity.
- $\mathbf{D}$ represents the state of micro-cracking of the quasi-brittle material.
- $\mathbf{D}$ is zero when the effective tensor $\widetilde{\mathbf{E}}=\mathbf{E}_0$ is the initial elasticity tensor.

It remains to determine the general coupling with damage $\widetilde{\mathbf{E}}=\widetilde{\mathbf{E}}(\mathbf{D})$.

BIJECTION BETWEEN $\mathbf{D}$ AND DILATATION TENSOR $\widetilde{\mathbf{di}}$

Since $\mathbf{E}$, therefore $\mathbf{di} = \mathbf{E} : \mathbf{1}$, are isotropic tensors

$$(\mathbf{di} - \widetilde{\mathbf{di}}) \cdot \mathbf{di}^{-1} = \mathbf{di}^{-1} \cdot (\mathbf{di} - \widetilde{\mathbf{di}}).$$

Remark

By initial isotropy we have the equalities

$$\mathbf{D} = \frac{1}{2\kappa} \operatorname{tr}_{12}(\mathbf{E} - \widetilde{\mathbf{E}}) = \mathbf{1} - \frac{\widetilde{\mathbf{di}}}{2\kappa}$$

$\Longrightarrow$ bijection between the damage variable $\mathbf{D}$ and the dilatation tensor $\widetilde{\mathbf{di}}$,

$$\widetilde{\mathbf{di}} = 2\kappa\,(\mathbf{1} - \mathbf{D})\,, \qquad \widetilde{\kappa} = \frac{1}{4} \operatorname{tr} \widetilde{\mathbf{di}} = \kappa \left(1 - \frac{1}{2} \operatorname{tr} \mathbf{D}\right).$$

Distance of effective elasticity tensors to isotropy, to orthotropy

A measured (effective) elasticity tensor will generically be biclinic in 2D (triclinic in 3D).

Can the damage of initially isotropic quasi-brittle materials be represented in 2D by the single second-order damage tensor $\mathbf{D}$?

Two underlying questions

1. The initial (virgin) material is assumed isotropic.
   Is $\mathbf{E}$ measured truly isotropic ?
2. $\mathbf{D}$ being (at least) orthotropic , $\widetilde{\mathbf{E}} = \widetilde{\mathbf{E}}(\mathbf{D})$ shall be (at least) orthotropic.
   Is $\widetilde{\mathbf{E}}$ measured far to be orthotropic ?

Those issues can be mitigated by calculating the distance to the expected elasticity symmetry stratum [4].

4. the set of all tensors which have the same symmetry class (Auffray et al, 2014).

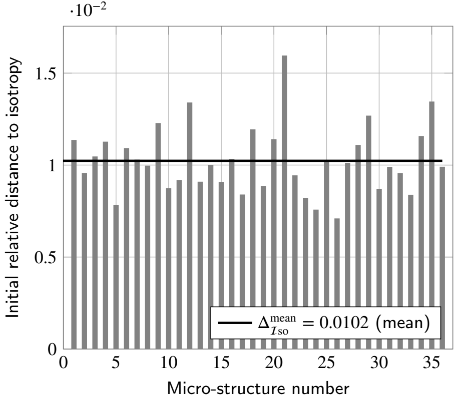
Dataset of 76 356 effective elasticity tensors

(PhD F. Loiseau, 2023)
Dataset available at https://doi.org/10.57745/LYHM4W.

- 36 virtual specimens with different micro-structures (but with the same macroscopic properties) are submitted to 21 mechanical loadings, uniaxial or multiaxial.
- Each mechanical loading is discretized into 100 loading steps.
- The dataset contains $36 \times 21 = 756$ evolutions of elasticity tensors, each containing 101 elasticity tensors (total of 76 356 elasticity tensors).
- Some elasticity tensors appear multiple times in the dataset (when the specimen is not yet damaged).

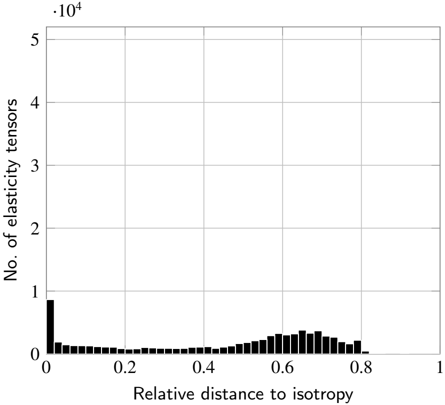
Isotropy of the initial elasticity tensor $\mathbf{E}$

Is elasticity isotropic for the 36 uncracked elasticity tensors in the dataset ?



All relative distances range between $\Delta^{min}_{iso} = 0.007$ and $\Delta^{max}_{iso} = 0.016$.
This is sufficient to consider that the elasticity tensors are initially isotropic.

Tensorial nature of the damage variable
Is damage D a scalar variable? By relatives distances

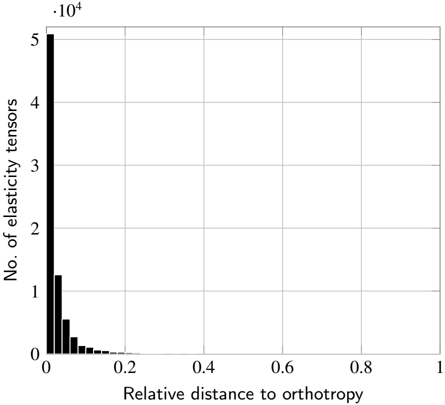


The distribution of relative distances to isotropy Δ_{iso} shows that a large part of the 76 356 effective elasticity tensors is far from being isotropic.

This means that a scalar (isotropic) damage variable is insufficient to represent the loss of stiffness (due to micro-cracking).

Tensorial nature of the damage variable

Is damage $\mathbf{D}$ a 2nd order tensor ? By relatives distances



The distribution of relative distances to orthotropy shows that most of the 76 356 effective elasticity tensors are close to be orthotropic.

This implies that at most two second-order tensors are required to represent the state of micro-cracking of bi-dimensional elasticity.

PARTIAL CONCLUSION

Partial conclusion

Definition (2nd order damage tensor)

$$\mathbf{D} = \mathbf{1} - \frac{\widetilde{\mathbf{di}}}{2\kappa}, \qquad \widetilde{\mathbf{di}} = \widetilde{\mathbf{E}} : \mathbf{1}.$$

Reconstruction (approximate, partial so far) by damage tensor $\mathbf{D}$

$$\widetilde{\mathbf{E}} = 2\tilde{\mu}\,\mathbf{J} + \underbrace{\kappa\big(1 - \frac{1}{2}\operatorname{tr}\mathbf{D}\big)\,\mathbf{1}\otimes\mathbf{1} + \kappa\left(\mathbf{1}\otimes(\mathbf{1}-\mathbf{D})' + (\mathbf{1}-\mathbf{D})'\otimes\mathbf{1}\right)}_{\text{Exact from } \mathbf{D}} + \widetilde{\mathbf{H}},$$

$$\tilde{\mu} = \frac{1}{4}\operatorname{tr}\widetilde{\mathbf{vo}} - \frac{1}{4}\kappa\,(2 - \operatorname{tr}\mathbf{D})\,, \qquad \widetilde{\mathbf{H}} = \frac{8\kappa^2\tilde{K}_3}{\tilde{I}_2^2}\,\mathbf{D}' * \mathbf{D}'.$$

Partial conclusion

Definition (2nd order damage tensor)

$$\mathbf{D} = \mathbf{1} - \frac{\widetilde{\mathbf{di}}}{2\kappa}, \qquad \widetilde{\mathbf{di}} = \widetilde{\mathbf{E}} : \mathbf{1}.$$

Reconstruction (approximate, partial so far) by damage tensor $\mathbf{D}$

$$\widetilde{\mathbf{E}} = 2\tilde{\mu}\,\mathbf{J} + \underbrace{\kappa\left(1 - \frac{1}{2}\operatorname{tr}\mathbf{D}\right)\mathbf{1}\otimes\mathbf{1} + \kappa\left(\mathbf{1}\otimes(\mathbf{1}-\mathbf{D})' + (\mathbf{1}-\mathbf{D})'\otimes\mathbf{1}\right)}_{\text{Exact from } \mathbf{D}} + \widetilde{\mathbf{H}},$$

$$\tilde{\mu} = \frac{1}{2}(2\mu+\kappa)(1-D_{\mathrm{v}}) - \frac{1}{4}\kappa(2-\operatorname{tr}\mathbf{D}) \qquad \widetilde{\mathbf{H}} = H\,\mathbf{D}' * \mathbf{D}'$$

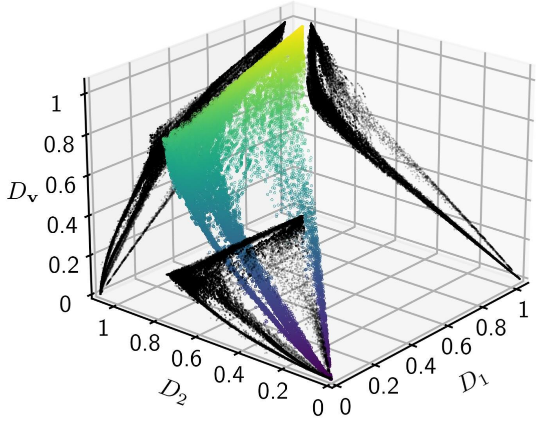
Since $\operatorname{tr}\mathbf{di} = 4\kappa$ and $\operatorname{tr}\mathbf{vo} = 2(2\mu+\kappa)$ and where we have set

$$D_{\mathrm{v}} = \frac{\operatorname{tr}\mathbf{vo} - \operatorname{tr}\widetilde{\mathbf{vo}}}{\operatorname{tr}\mathbf{vo}}. \qquad H = \frac{8\kappa^2\tilde{K}_3}{\tilde{I}_2^2}$$

MODELING BY A SINGLE SECOND ORDER DAMAGE TENSOR

Damage D_{v} – Not an independent variable

Plotted as a function of the eigenvalues D_1 and D_2 ($D_2 > D_1$)

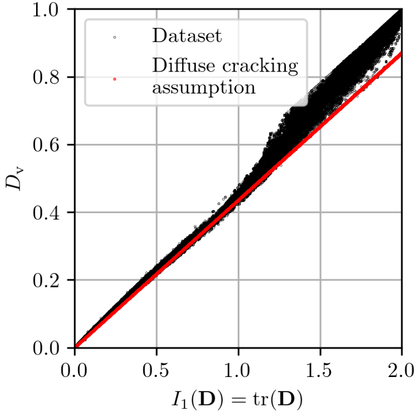


The points $(D_1, D_2, D_{\mathrm{v}})$ of the dataset are grouped around a surface.

$$D_{\mathrm{v}}(\mathbf{D}) \approx c_1\, I_1(\mathbf{D}) + c_2\, I_2(\mathbf{D}) + c_3\, I_3(\mathbf{D}), \qquad I_k(\mathbf{D}) = \mathrm{tr}(\mathbf{D}^k) = D_1^k + D_2^k.$$

where $2 \sum c_k = 1$ by the rupture constraint $D_{\mathrm{v}}(\mathbf{D} = \mathbf{1}) = 1$.

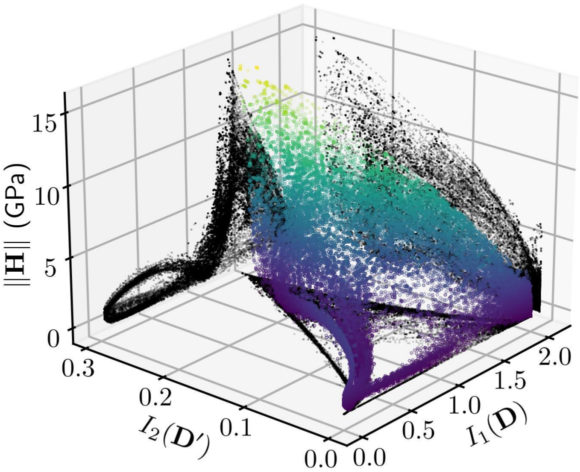
Check of Kachanov diffuse micro-cracking assumption



$$
\begin{aligned}
D_v(\mathbf{D}) &= c_1\, I_1(\mathbf{D}) + c_2\, I_2(\mathbf{D}) + c_3\, I_3(\mathbf{D}) \\
&= \frac{\kappa}{2\mu + \kappa}\left(I_1(\mathbf{D}) - I_2(\mathbf{D})\right) + \frac{1}{2} I_2(\mathbf{D}) + m\left(I_3(\mathbf{D}) - I_2(\mathbf{D})\right).
\end{aligned}
$$

Modeling of the harmonic part

To identify the constitutive equation $H(\mathbf{D})$ (function of the damage variable), let us check if the norm $\|\mathbf{H}\|$ of the harmonic part can be represented by a function of $\mathbf{D}$ (through its invariants $I_1(\mathbf{D})$ and $I_2(\mathbf{D}') = \mathbf{D}' : \mathbf{D}'$).



LASSO sparse regression method (Tibshirani, 1996) aims at fitting a parametrized function with respect to data while penalizing the number of non-zero parameters.

It is applied to the (multivariate) polynomial in the damage invariants $I_1(\mathbf{D})$ and $I_2(\mathbf{D}')$.

$$\min_{h_{n_1 n_2}} \sum_{i=1}^{76\,356} \left(\sqrt{2}\|\mathbf{H}_i\| - \sum_{n_1,n_2} h_{n_1 n_2} I_1(\mathbf{D}_i)^{n_1} I_2(\mathbf{D}'_i)^{n_2} \right)^2 + \alpha \sum_{n_1,n_2} |h_{n_1 n_2}|,$$

- $h_{n_1 n_2}$ are the coefficients of the sought polynomial, *i.e.*, the parameters,
- α is an arbitrary hyper-parameter of the method (the higher α, the fewer the non-zero parameters).

Best compromise $\alpha = 0.0043$ GPa (Loiseau, 2023) between accuracy (evaluated via the coefficient of determination r^2) and the number of non-zero parameters :

$$H(\mathbf{D}) = h\, I_1(\mathbf{D})^4\,, \qquad h = h_{41} = 17 \text{ GPa : harmonic prefactor.}$$

Summary of the constitutive equations

Quantity	Model
Shear modulus	$\tilde{\mu}(\mathbf{D}) = \mu - \frac{1}{4}\kappa\,\mathrm{tr}\,\mathbf{D} + \frac{1}{4}\,(\kappa - 2\mu)\,\mathbf{D} : \mathbf{D} + m\left(\mathbf{D} : \mathbf{D} - \mathrm{tr}(\mathbf{D}^3)\right)$
Bulk modulus	$\tilde{\kappa}(\mathbf{D}) = \kappa\left(1 - \frac{1}{2}\,\mathrm{tr}\,\mathbf{D}\right)$
Dilatation tensor	$\tilde{\mathbf{di}} = 2\kappa\,(\mathbf{1} - \mathbf{D}) \qquad \longrightarrow \qquad \mathbf{D} = \mathbf{1} - \dfrac{\widetilde{\mathbf{di}}}{2\kappa}$
Harmonic part	$\widetilde{\mathbf{H}}(\mathbf{D}) = h\,(\mathrm{tr}\,\mathbf{D})^4\,\mathbf{D}' * \mathbf{D}'$

Elasticity coupled with 2nd order damage damage tensor $\mathbf{D}$

$$\begin{aligned}\widetilde{\mathbf{E}} =& 2\Big(\mu - \frac{1}{4}\kappa\,\mathrm{tr}\,\mathbf{D} + \frac{1}{4}(\kappa - 2\mu)\mathbf{D} : \mathbf{D} + m\left(\mathbf{D} : \mathbf{D} - \mathrm{tr}(\mathbf{D}^3)\right)\Big)\mathbf{J} \\ &+ \kappa\Big(1 - \frac{1}{2}\,\mathrm{tr}\,\mathbf{D}\Big)\,\mathbf{1}\otimes\mathbf{1} + \kappa\left(\mathbf{1}\otimes(\mathbf{1}-\mathbf{D})' + (\mathbf{1}-\mathbf{D})'\otimes\mathbf{1}\right) \\ &+ h\,(\mathrm{tr}\,\mathbf{D})^4\,\mathbf{D}' * \mathbf{D}',\end{aligned}$$

where $\mathbf{J} = \mathbf{I} - \frac{1}{2}\mathbf{1}\otimes\mathbf{1}$.

Summary of the constitutive equations

Name of material parameter	Symbol	Value in GPa
Initial shear modulus	μ	19.4
Initial bulk modulus	κ	30
Nonlinear shear-damage coupling parameter	m	3.35
Harmonic prefactor	h	17

Elasticity coupled with second order damage damage tensor $\mathbf{D}$

$$\widetilde{\mathbf{E}} = \widetilde{\mathbf{Iso}}(\mathbf{D}) + \widetilde{\mathbf{Dil}}(\mathbf{D}) + \widetilde{\mathbf{H}}(\mathbf{D}).$$

Only one damage variable ($\mathbf{D}$) is introduced in this final state coupling for quasi-brittle materials between 2D elasticity and anisotropic damage.

ASSESSMENT OF THE PROPOSED ANISOTROPIC DAMAGE STATE COUPLING

Assessment over the 76 356 tensors dataset

Each computed (micro-cracked) elasticity tensor $\mathbf{E}_i$ from the dataset has

- an isotropic part $\mathbf{Iso}_i$,
- a dilatation part $\mathbf{Dil}_i$
- and an harmonic part $\mathbf{H}_i$.

The damage $\mathbf{D} = \mathbf{D}_i$ is taken (measures) as equal to the damage variable

$$\mathbf{D}_i = \mathbf{1} - \frac{\mathbf{di}_i}{2\kappa} = \mathbf{1} - \frac{\mathbf{E}_i : \mathbf{1}}{2\kappa}$$

measured for $\widetilde{\mathbf{E}} = \mathbf{E}_i$ in the dataset.

TWO DIFFERENT DAMAGE STATE COUPLINGS

1. Elasticity–damage coupling with isotropic damage variable d,

$$\boldsymbol{\sigma} = \mathbf{E}(1-d) : \boldsymbol{\varepsilon}, \qquad d = 1 - \frac{\|\widetilde{\mathbf{E}}\|}{\|\mathbf{E}\|},$$

i.e.

(1) $\widetilde{\mathbf{E}} = \mathbf{E}(1-d)$.

2. Elasticity–(anisotropic) damage couplings with damage variable $\mathbf{D}$,

$$\boldsymbol{\sigma} = \widetilde{\mathbf{E}}(\mathbf{D}) : \boldsymbol{\varepsilon},$$

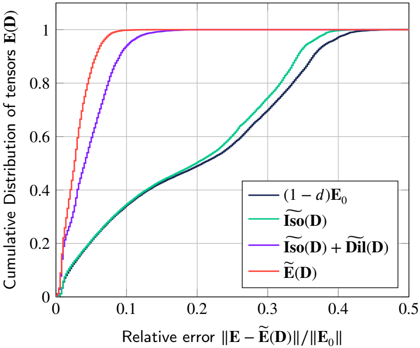
$(2a)$ $\widetilde{\mathbf{E}}(\mathbf{D}) = \widetilde{\mathbf{Iso}}(\mathbf{D})$, with $m = 0$,

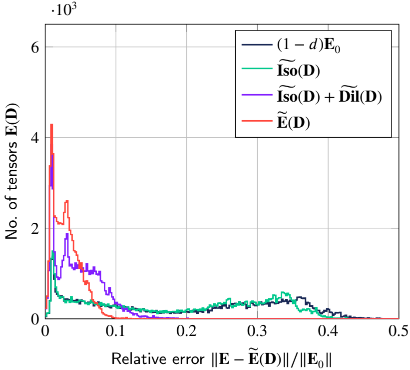
$(2b)$ $\widetilde{\mathbf{E}}(\mathbf{D}) = \widetilde{\mathbf{Iso}}(\mathbf{D}) + \widetilde{\mathbf{Dil}}(\mathbf{D})$, with $m = 0$,

$(2c)$ $\widetilde{\mathbf{E}}(\mathbf{D}) = \widetilde{\mathbf{Iso}}(\mathbf{D}) + \widetilde{\mathbf{Dil}}(\mathbf{D}) + \widetilde{\mathbf{H}}(\mathbf{D})$, with $m = 3.35$ GPa and $h = 17$ GPa.

Relative error $\|\mathbf{E}_i - \widetilde{\mathbf{E}}(\mathbf{D}_i)\|/\|\mathbf{E}_0\|$ for full elasticity tensor

Cumulative Distribution Function – Histogram –





(Mechanical) damage is anisotropic !

Multiaxial non-proportional loading case

Periodic boundary conditions, shear → (plane) tension loading

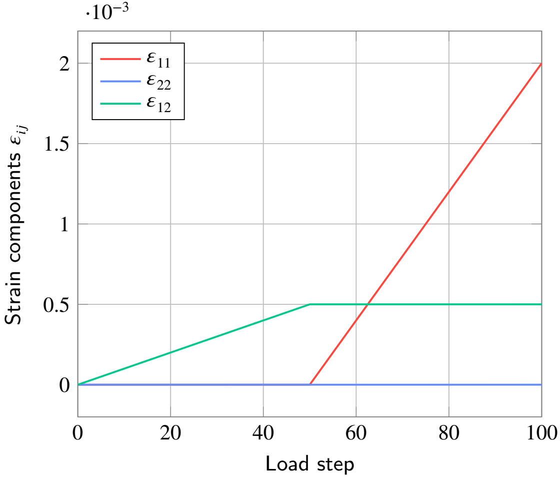
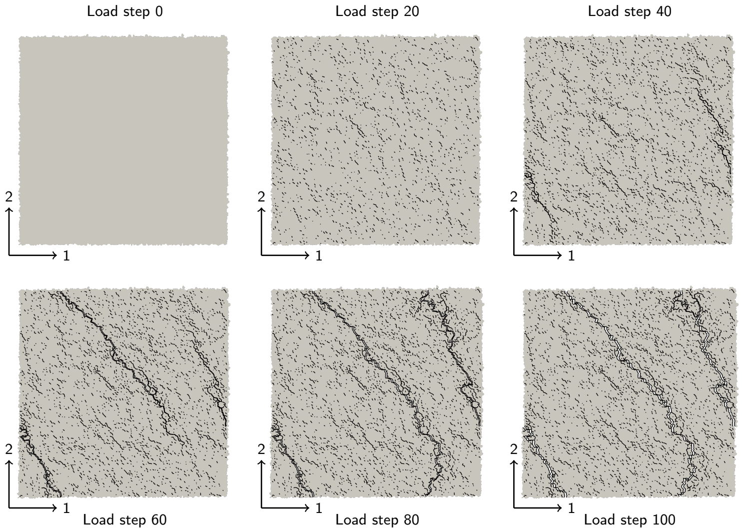


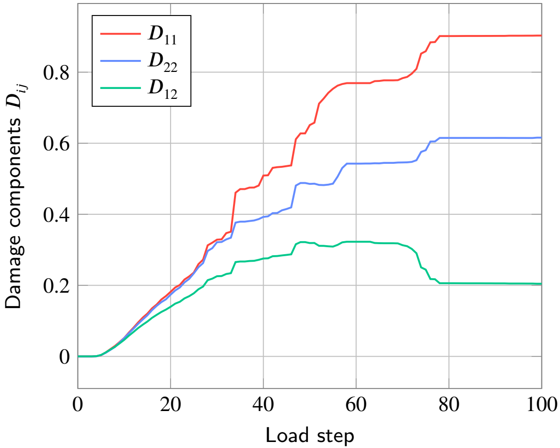
Figure – Applied macroscopic strain (on the Area Element)

Micro-cracking patterns at different load steps



(area elements are deformed, displacements are scaled with a factor of 10)

Macroscopic damage evolution

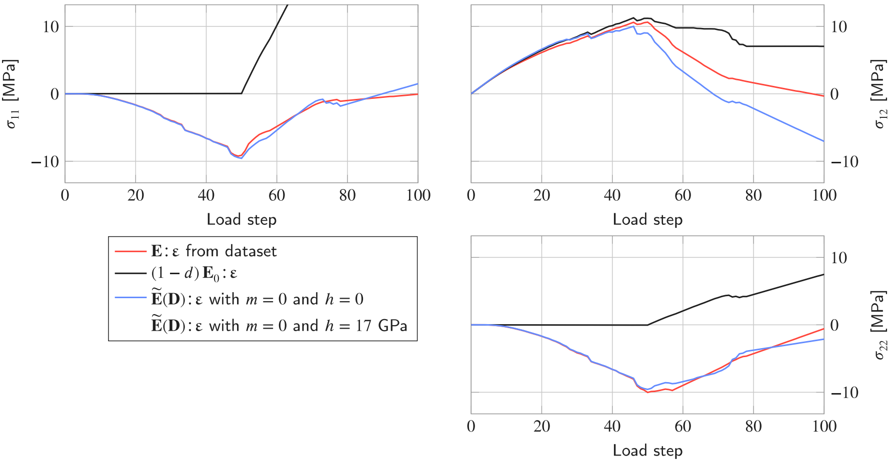


Remark

Shear component D_{12} decreases but damage rate $\dot{\mathbf{D}}$ remains definite positive

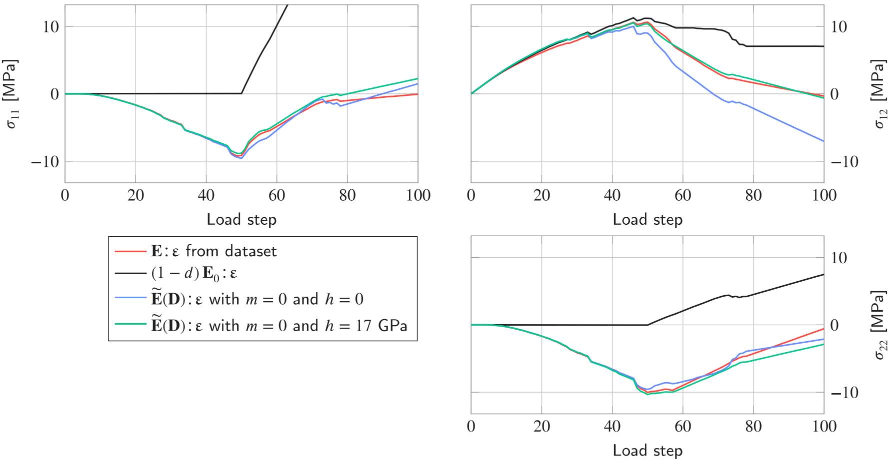
Macroscopic stress components

Non-proportional shear→tension loading, Comparison to DE data



Macroscopic stress components

Non-proportional shear→tension loading, Comparison to DE data



PARTIAL CONCLUSION

- We have performed 2D beam-particle simulations of Area Elements submitted to various complex loadings and have generated a dataset of 76 356 effective (damaged) elasticity tensors.
- High levels of anisotropic damage have been reached.
  Those correspond to discrete computations with strong micro-cracks interactions and multiple coalescence.
- Thanks to a (covariant) reconstruction formula of 2D elasticity tensors :
  - A definition of second order damage tensor has been given.
  - The coupling $\widetilde{\mathbf{E}}(\mathbf{D}) = \widetilde{\mathbf{Iso}}(\mathbf{D}) + \widetilde{\mathbf{Dil}}(\mathbf{D}) + \widetilde{\mathbf{H}}(\mathbf{D})$ is modelled with accuracy.
  - To set $h = 17$ GPa brings the isotropic part and the harmonic part to the same level of accuracy.

Conclusion

CONCLUSION

- Damage as well as its mechanical consequences are anisotropic for quasi-brittle materials.
- Thanks to a reconstruction formula of the orthotropic bidimensional elasticity tensor by means of its covariants, we have proposed an anisotropic damage state coupling by a single second-order damage tensor.
- The proposed anisotropic damage state coupling models 85.8% of the effective elasticity tensors in the dataset with less than 5% of error, including those with strong micro-cracks interactions and multiple coalescence.
- It remains to generalize to 3D the work presented in Part B.
- to poromechanics of (anisotropically) damaged materials and structures.

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Distance to a symmetry class

See (Gazis, 1963, François et al, 1995, 1998, Vianello, 1997)

This usually consists,

(*i*) in finding elasticity tensor $\mathbf{E}^*$ in the symmetry stratum $\overline{\Sigma}$ ($\overline{\Sigma}_{iso}$ or $\overline{\Sigma}_{ortho}$) which is the closest to the measured elasticity tensor $\mathbf{E}$,

(*ii*) in calculating the distance between $\mathbf{E}$ and $\mathbf{E}^*$.

Definition (Relative distance to the symmetry stratum $\overline{\Sigma}$)

$$\Delta_{\overline{\Sigma}}(\mathbf{E}) = \min_{\mathbf{E}^* \in \overline{\Sigma}} \frac{\|\mathbf{E} - \mathbf{E}^*\|}{\|\mathbf{E}\|}.$$

In the case of isotropy, the harmonic decomposition provides, by orthogonal projection (Vianello, 1997), the closest isotropic tensor $\mathbf{E}^*$ to $\mathbf{E}$ as its isotropic part $\mathbf{Iso}(\mathbf{E})$,

$$\Delta_{iso}(\mathbf{E}) = \min_{\mathbf{E}^* \text{ isotropic}} \frac{\|\mathbf{E} - \mathbf{E}^*\|}{\|\mathbf{E}\|} = \frac{\|\mathbf{E} - \mathbf{Iso}(\mathbf{E})\|}{\|\mathbf{E}\|}, \quad \text{since} \quad \mathbf{E}^* = \mathbf{Iso}(\mathbf{E}).$$

Distance to a symmetry class

See (Gazis, 1963, François et al, 1995, 1998, Vianello, 1997)

This usually consists,

(*i*) in finding elasticity tensor $\mathbf{E}^*$ in the symmetry stratum $\overline{\Sigma}$ ($\overline{\Sigma}_{iso}$ or $\overline{\Sigma}_{ortho}$) which is the closest to the measured elasticity tensor $\mathbf{E}$,

(*ii*) in calculating the distance between $\mathbf{E}$ and $\mathbf{E}^*$.

Definition (Relative distance to the symmetry stratum $\overline{\Sigma}$)

$$\Delta_{\overline{\Sigma}}(\mathbf{E}) = \min_{\mathbf{E}^* \in \overline{\Sigma}} \frac{\|\mathbf{E} - \mathbf{E}^*\|}{\|\mathbf{E}\|}.$$

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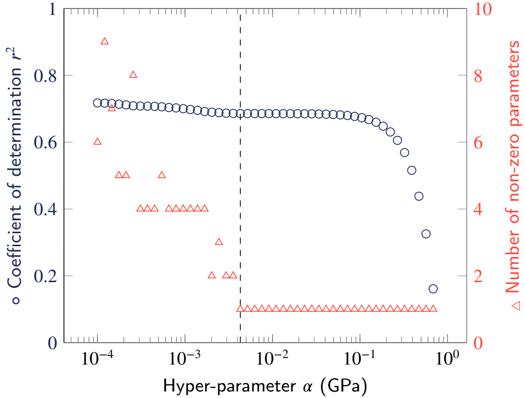
$$\Delta_{iso}(\mathbf{E}) = \min_{\mathbf{E}^* \text{ isotropic}} \frac{\|\mathbf{E} - \mathbf{E}^*\|}{\|\mathbf{E}\|} = \frac{\|\mathbf{E} - \mathbf{Iso}(\mathbf{E})\|}{\|\mathbf{E}\|}, \quad \text{since} \quad \mathbf{E}^* = \mathbf{Iso}(\mathbf{E}).$$

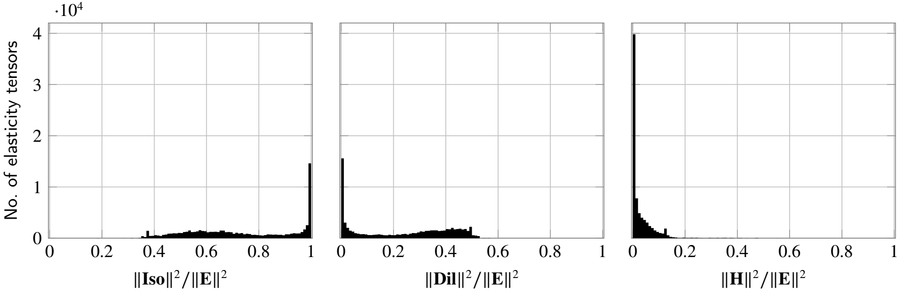
Compromise between accuracy and non-zero parameters

Coefficient of determination of the LASSO regression (an indicator of the accuracy) :

$$r^2 = 1 - \frac{\sum_{i=1}^{76\,356}\left(\sqrt{2}\|\mathbf{H}_i\| - \sum_{n_1,n_2} h_{n_1 n_2} I_1(\mathbf{D}_i)^{n_1} I_2(\mathbf{D}'_i)^{n_2}\right)^2}{\sum_{i=1}^{76\,356}\left(\sqrt{2}\|\mathbf{H}_i\| - \text{mean}\left(\sum_{n_1,n_2} h_{n_1 n_2} I_1(\mathbf{D}_i)^{n_1} I_2(\mathbf{D}'_i)^{n_2}\right)\right)^2}.$$



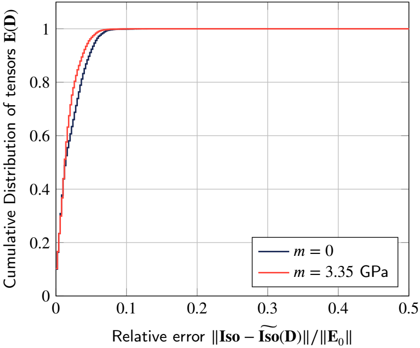
Proportions of each part of the 76 356 elasticity tensors

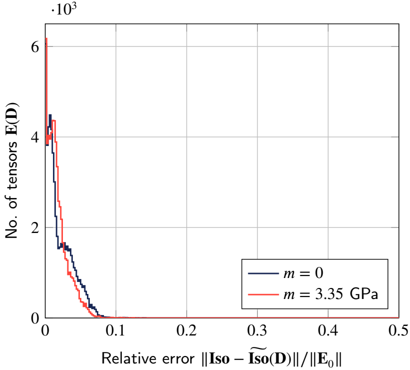


The harmonic part is a small proportion of elasticity tensors in most cases. It can often be neglected (setting $h = 0$, *i.e.*, $\widetilde{\mathbf{H}}(\mathbf{D}) = 0$).

Relative error [5] $\| \mathbf{Iso}_i - \widetilde{\mathbf{Iso}}(\mathbf{D}_i)\| / \|\mathbf{E}_0\|$ for isotropic part

Cumulative Distribution Function – Histogram



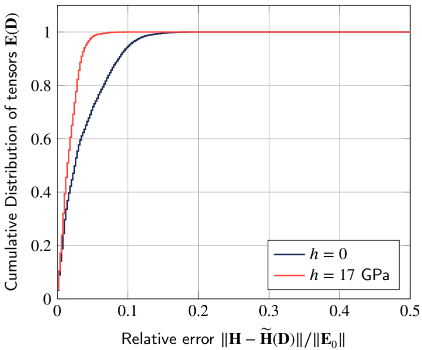


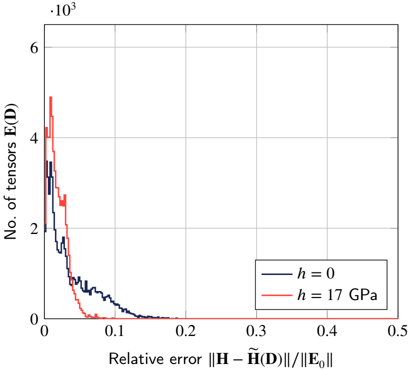
Most isotropic parts are modeled with an error below 10% for both modeling.

5. Duplicated tensors during a loading are filtered out in this plot.

Relative error $\|\mathbf{H}_i - \widetilde{\mathbf{H}}(\mathbf{D}_i)\|/\|\mathbf{E}_0\|$ for harmonic part

Cumulative Distribution Function – Histogram

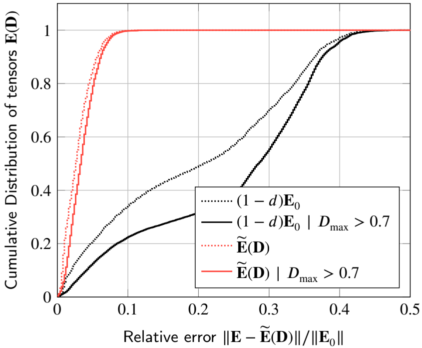


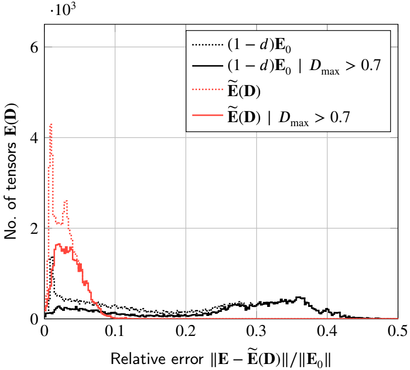


Setting $h = 17$ GPa enables to model most harmonic parts of the dataset with an error below 5% whereas it is not the case for $h = 0$.

Assessment at high level of damage (> 0.7)

Cumulative Distribution Function – Histogram





(Mechanical) damage is anisotropic !
Also in the strong micro-cracks interaction stage.