Hamiltonian mechanics of Timoshenko model on the moving frame

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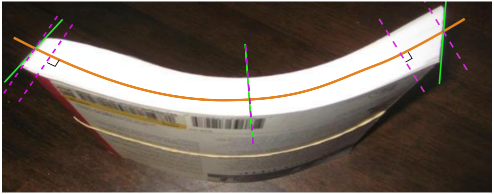
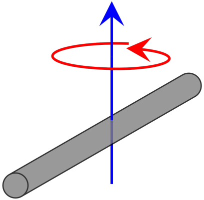
Timoshenko model in solid mechanics fig: Wikipedia

S. Timoshenko (1878-1972)

Shear



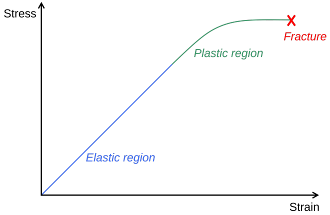
Timoshenko model



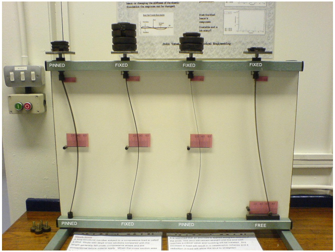
Moment of inertia

Elasticity and buckling

This more general model has motivated numerous studies on various aspects of structural mechanics, including investigations into elasticity and buckling.



Elasticity



Buckling

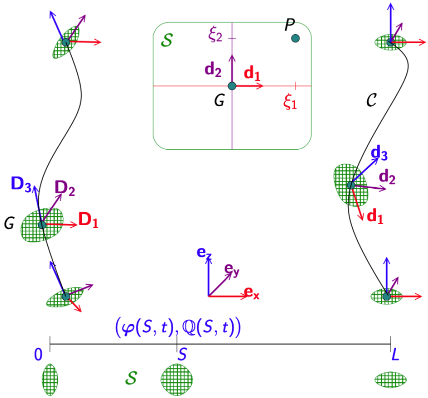
fig: Wikipedia

Solutions of the non-linear dynamical approach were mainly given using numerical methods.

Geometrically exact model

$\approx$ 1970 : geometrically exact formulation of the model (Reissner, Whithman, Simo)

- *placement*: $\varphi(S,t) \in \mathbb{R}^3$
- *velocity*: $\boldsymbol{v} = \frac{\partial \varphi}{\partial t}$
- *rotation tensor*: $\boldsymbol{Q}(S,t) \in SO(3)$
- orthonormal frame basis: $(\boldsymbol{d_1}, \boldsymbol{d_2}, \boldsymbol{d_3}) = \boldsymbol{Q}(S,t)$
- *curvature*: $\boldsymbol{\kappa} = \mathrm{axial}(\frac{\partial \boldsymbol{Q}}{\partial S}\boldsymbol{Q}^T)$
- *spin*: $\boldsymbol{\omega} = \mathrm{axial}(\frac{\partial \boldsymbol{Q}}{\partial t}\boldsymbol{Q}^T)$
- $\frac{\partial \boldsymbol{d_i}}{\partial S} = \boldsymbol{\kappa} \wedge \boldsymbol{d_i}$, $\frac{\partial \boldsymbol{d_i}}{\partial t} = \boldsymbol{\omega} \wedge \boldsymbol{d_i}$,
- *strain-vector*: $\boldsymbol{\varepsilon} = \frac{\partial \varphi}{\partial S} - \boldsymbol{d_3}$



Geometrically exact beam

fig: Le Marrec

Parameters of the model: potential energy

Definition (Strain-energy density)

For linear stress-strain relations, the intern energy density is quadratic:

$$U(\boldsymbol{\varepsilon}, \boldsymbol{\kappa}) = \frac{1}{2}\boldsymbol{\varepsilon}\mathbb{G}\boldsymbol{\varepsilon} + \frac{1}{2}\boldsymbol{\kappa}\mathbb{H}\boldsymbol{\kappa}$$

where the rigidity tensors are diagonal matrices in the $\{\boldsymbol{d_i}\}$-frame:

$$\mathbb{G} = \begin{pmatrix} GA & 0 & 0 \\ 0 & GA & 0 \\ 0 & 0 & EA \end{pmatrix} \text{ and } \mathbb{H} = \begin{pmatrix} EI_1 & 0 & 0 \\ 0 & EI_2 & 0 \\ 0 & 0 & GI_3 \end{pmatrix}$$

with G and E shear and bulk modulus, A area and I_i quadratic moment along $\boldsymbol{d_i}$ of the cross-section.

Parameters of the model: kinetic energy

Definition (Kinetic energy)

$$T(\boldsymbol{v}, \boldsymbol{\omega}) = \frac{1}{2}\boldsymbol{v}\mathbb{A}\boldsymbol{v} + \frac{1}{2}\boldsymbol{\omega}\mathbb{J}\boldsymbol{\omega}$$

where $\mathbb{A}$ and $\mathbb{J}$ are diagonal inertial tensors in the mobile frame $\{\boldsymbol{d_i}\}$:

$$\mathbb{A} = \begin{pmatrix} \rho A & 0 & 0 \\ 0 & \rho A & 0 \\ 0 & 0 & \rho A \end{pmatrix} \text{ and } \mathbb{J} = \begin{pmatrix} \rho I_1 & 0 & 0 \\ 0 & \rho I_2 & 0 \\ 0 & 0 & \rho I_3 \end{pmatrix},$$

ρ being the mass density.

Parameters live on the moving frame $\{d_i\}$

Parameters of Timoshenko models	
$\mathbb{G}$	rigidity tensor
$\mathbb{H}$	rigidity tensor
$\mathbb{A}$	inertial tensor
$\mathbb{J}$	inertial tensor

- Material invariance $\Longrightarrow$ the parameters of the model are naturally expressed in material coordinates.

Therefore, it is natural to study the Timoshenko model **in Lagrangian coordinates**: by avoiding any decomposition of tensors on the Cartesian frame associated to the ambient space. This is the main motivation of this presentation.

Algebraic notation vs. moving frame

Let $\boldsymbol{u} = u_i \boldsymbol{d_i}$ be a vector expressed in the moving frame $\{\boldsymbol{d_i}\}$. The components $(u_i)_i$ are said to be Lagrangian coordinates. In these coordinates:

$$u := \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \qquad \text{and} \quad d_1 := \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad d_2 := \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad d_3 := \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

For any vector $\boldsymbol{u}(S, t)$:

$$\frac{\partial \boldsymbol{u}}{\partial S} = \frac{\partial u_i}{\partial S} \boldsymbol{d_i} + \boldsymbol{\kappa} \wedge \boldsymbol{u}, \qquad \frac{\partial \boldsymbol{u}}{\partial t} = \frac{\partial u_i}{\partial t} \boldsymbol{d_i} + \boldsymbol{\omega} \wedge \boldsymbol{u} \tag{1}$$

Differential calculus on the moving frame

Let us consider, for any vectors $\boldsymbol{u}(S,t)$ and $\boldsymbol{v}(S,t)$, the quadratic form:

$$f(\boldsymbol{u},\boldsymbol{v}) := \boldsymbol{u}\mathbb{X}\boldsymbol{v},$$

where $\mathbb{X} = \mathbb{X}^T$ is symmetric and has time and space independent components when expressed in $\{\boldsymbol{d_i}\}$. Thanks to equation (1):

Proposition (Derivations of quadratic forms on the moving frame)

Time and space derivations of the quadratic form f are respectively related to corotational time and space derivations of associated vectors. In equations:

$$\begin{aligned}
\frac{\partial f}{\partial t} &= \left(\frac{\partial \boldsymbol{u}}{\partial t} - \boldsymbol{\omega}\wedge\boldsymbol{u}\right)\mathbb{X}\boldsymbol{v} + \boldsymbol{u}\mathbb{X}\left(\frac{\partial \boldsymbol{v}}{\partial t} - \boldsymbol{\omega}\wedge\boldsymbol{v}\right)\\
\frac{\partial f}{\partial S} &= \left(\frac{\partial \boldsymbol{u}}{\partial S} - \boldsymbol{\kappa}\wedge\boldsymbol{u}\right)\mathbb{X}\boldsymbol{v} + \boldsymbol{u}\mathbb{X}\left(\frac{\partial \boldsymbol{v}}{\partial S} - \boldsymbol{\kappa}\wedge\boldsymbol{v}\right)
\end{aligned} \tag{2}$$

Linear perturbation on the moving frame

Corollary (Linear perturbation of quadratic forms on the moving frame)

The infinitesimal perturbation of the quadratic form f follows in the same way:

$$\delta f = (\delta \boldsymbol{u} - \delta \boldsymbol{\theta} \wedge \boldsymbol{u})\, \mathbb{X} \boldsymbol{v} + \boldsymbol{u} \mathbb{X}\, (\delta \boldsymbol{v} - \delta \boldsymbol{\theta} \wedge \boldsymbol{v}) \tag{3}$$

where $\boxed{\delta \boldsymbol{\theta} = \mathit{axial}(\boldsymbol{Q}^{-1} \delta \boldsymbol{Q}).}$

- $\delta f \neq \delta \boldsymbol{u} \mathbb{X} \boldsymbol{v} + \boldsymbol{u} \mathbb{X} \delta \boldsymbol{v}$
- $\delta \boldsymbol{u} - \delta \boldsymbol{\theta} \wedge \boldsymbol{u}$ is sometimes called the *corotational perturbation* of $\boldsymbol{u}$

Lagrangian density on the moving frame

Lagrangian density $\ell(\boldsymbol{v}, \boldsymbol{\omega}, \boldsymbol{\varepsilon}, \boldsymbol{\kappa})$ and Lagrangian $\mathcal{L}(\boldsymbol{v}, \boldsymbol{\omega}, \boldsymbol{\varepsilon}, \boldsymbol{\kappa})$:

$$\ell(\boldsymbol{v}, \boldsymbol{\omega}, \boldsymbol{\varepsilon}, \boldsymbol{\kappa}) := \frac{1}{2}\boldsymbol{v}\mathbb{A}\boldsymbol{v} + \frac{1}{2}\boldsymbol{\omega}\mathbb{J}\boldsymbol{\omega} - \frac{1}{2}\boldsymbol{\varepsilon}\mathbb{G}\boldsymbol{\varepsilon} - \frac{1}{2}\boldsymbol{\kappa}\mathbb{H}\boldsymbol{\kappa},$$

$$\mathcal{L}(\boldsymbol{v}, \boldsymbol{\omega}, \boldsymbol{\varepsilon}, \boldsymbol{\kappa}) := \int_0^L \ell(\boldsymbol{v}, \boldsymbol{\omega}, \boldsymbol{\varepsilon}, \boldsymbol{\kappa})\, \mathrm{d}S$$

Accordingly, the *action* $\mathcal{S}$ is:

$$\mathcal{S} := \int_{t_1}^{t_2} \mathcal{L}\, \mathrm{d}t.$$

Variational principle of the dynamical problem on the moving frame

Theorem (Variational principle, Le Marrec et al., 2017)

Under suitable boundary conditions, Hamilton's Principle $\delta\mathcal{S} = 0$ is equivalent to

$$\int_{t_1}^{t_2}\int_0^L \delta\boldsymbol{\varphi}\cdot\left(\frac{\partial\,\mathbb{G}\boldsymbol{\varepsilon}}{\partial S} - \frac{\partial\,\mathbb{A}\boldsymbol{v}}{\partial t}\right) + \delta\boldsymbol{\theta}\cdot\left(\frac{\partial\,\mathbb{H}\boldsymbol{\kappa}}{\partial S} + \frac{\partial\boldsymbol{\varphi}}{\partial S}\wedge(\mathbb{G}\boldsymbol{\varepsilon}) - \frac{\partial\,\mathbb{J}\boldsymbol{\omega}}{\partial t}\right)dS\,dt = 0. \tag{4}$$

The variable $\delta\boldsymbol{\theta}$ – involved in the weak formulation (4) – is related to Euler-Poincaré reduction on the group $\{\boldsymbol{Q}\colon [0, L] \to SO(3)\}$.

Closure relations

Out of the construction of the variables $(\boldsymbol{v}, \boldsymbol{\omega}, \boldsymbol{\varepsilon}, \boldsymbol{\kappa})$, one proves:

Proposition

- *Time derivative of strain-vector and space derivative of velocity are related by:*

$$\frac{\partial \boldsymbol{\varepsilon}}{\partial t} = \frac{\partial \boldsymbol{v}}{\partial S} - \boldsymbol{\omega} \wedge \boldsymbol{d_3}. \tag{5}$$

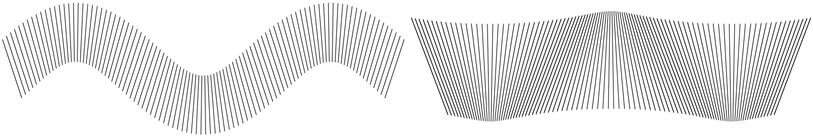
- *Time derivative of curvature and space derivative of spin are related by:*

$$\frac{\partial \boldsymbol{\kappa}}{\partial t} = \frac{\partial \boldsymbol{\omega}}{\partial S} + \boldsymbol{\omega} \wedge \boldsymbol{\kappa}. \tag{6}$$

Equations of Timoshenko model on the moving frame

Out of the variational principle (thm 4) and closure relations (6) and (5), we obtain 4 first-order differential equations with 4 3-dimensional unknowns:

$$\left\{\begin{array}{rcl} \dfrac{\partial\,\mathbb{G}\boldsymbol{\varepsilon}}{\partial S} & = & \dfrac{\partial\,\mathbb{A}\boldsymbol{v}}{\partial t} \\ \dfrac{\partial\,\mathbb{H}\boldsymbol{\kappa}}{\partial S} + (\boldsymbol{\varepsilon} + \boldsymbol{d_3}) \wedge (\mathbb{G}\boldsymbol{\varepsilon}) & = & \dfrac{\partial\,\mathbb{J}\boldsymbol{\omega}}{\partial t} \\ \dfrac{\partial\,\boldsymbol{v}}{\partial S} - \boldsymbol{\omega} \wedge \boldsymbol{d_3} & = & \dfrac{\partial\,\boldsymbol{\varepsilon}}{\partial t} \\ \dfrac{\partial\,\boldsymbol{\omega}}{\partial S} + \boldsymbol{\omega} \wedge \boldsymbol{\kappa} & = & \dfrac{\partial\,\boldsymbol{\kappa}}{\partial t} \end{array}\right. \tag{7}$$



Two beam configurations

fig: Le Marrec

Choice of variables: replace Lagrangian coordinates by local coordinates

Set space and rotation momenta $p = \mathbb{A}v$ and $\sigma = \mathbb{J}\omega$. In the *local chart*, the system (7) with variables $(p, \sigma, \varepsilon, \kappa)$ gets written as

$$\left\{\begin{array}{rcl}
\dfrac{\partial\,\mathbb{G}\varepsilon}{\partial S} + \kappa \wedge (\mathbb{G}\varepsilon) - (\mathbb{J}^{-1}\sigma) \wedge p & = & \dfrac{\partial\, p}{\partial t} \\
\dfrac{\partial\,\mathbb{H}\kappa}{\partial S} + \kappa \wedge (\mathbb{H}\kappa) + (\varepsilon + d_3) \wedge (\mathbb{G}\varepsilon) - (\mathbb{J}^{-1}\sigma) \wedge \sigma & = & \dfrac{\partial\,\sigma}{\partial t} \\
\dfrac{\partial\,\mathbb{A}^{-1}p}{\partial S} + \kappa \wedge (\mathbb{A}^{-1}p) + (\varepsilon + d_3) \wedge (\mathbb{J}^{-1}\sigma) & = & \dfrac{\partial\,\varepsilon}{\partial t} \\
\dfrac{\partial\,\mathbb{J}^{-1}\sigma}{\partial S} + \kappa \wedge (\mathbb{J}^{-1}\sigma) & = & \dfrac{\partial\,\kappa}{\partial t}
\end{array}\right. \tag{8}$$

A first Hamiltonian formulation in the ambient space

Theorem (Marsden et. al., 1987)

The system (8) *is Hamiltonian for*

$$H(p,\sigma,\varepsilon,\kappa) = \int_0^L \frac{1}{2}p\mathbb{A}^{-1}p + \frac{1}{2}\sigma\mathbb{J}^{-1}\sigma + \frac{1}{2}\varepsilon\mathbb{G}\varepsilon + \frac{1}{2}\kappa\mathbb{H}\kappa \, dS \quad (9)$$

and the Poisson bracket

$$\begin{aligned}
\{f,g\} = \int_0^L & < \frac{\partial f}{\partial p}, \frac{\partial}{\partial S}\left(\frac{\partial g}{\partial \varepsilon}\right) > - < \frac{\partial g}{\partial p}, \frac{\partial}{\partial S}\left(\frac{\partial f}{\partial \varepsilon}\right) > \\
& + < \frac{\partial f}{\partial \sigma}, \frac{\partial}{\partial S}\left(\frac{\partial g}{\partial \kappa}\right) > - < \frac{\partial g}{\partial \sigma}, \frac{\partial}{\partial S}\left(\frac{\partial f}{\partial \kappa}\right) > \\
& + < \sigma, \frac{\partial g}{\partial \sigma} \wedge \frac{\partial f}{\partial \sigma} > \\
& + < \kappa, \frac{\partial g}{\partial \varepsilon} \wedge \frac{\partial f}{\partial p} - \frac{\partial f}{\partial \varepsilon} \wedge \frac{\partial g}{\partial p} > + < p, \frac{\partial g}{\partial \sigma} \wedge \frac{\partial f}{\partial p} - \frac{\partial f}{\partial \sigma} \wedge \frac{\partial g}{\partial p} > \\
& + < \varepsilon + d_3, \frac{\partial g}{\partial \varepsilon} \wedge \frac{\partial f}{\partial \sigma} - \frac{\partial f}{\partial \varepsilon} \wedge \frac{\partial g}{\partial \sigma} > + < \kappa, \frac{\partial g}{\partial \kappa} \wedge \frac{\partial f}{\partial \sigma} - \frac{\partial f}{\partial \kappa} \wedge \frac{\partial g}{\partial \sigma} > dS.
\end{aligned}$$

Differential geometry

Definition (Configuration and velocity spaces)

The configuration space is

$$C = \{(\boldsymbol{\varphi}, \boldsymbol{Q})\colon [0, L] \to \mathbb{R}^3 \times SO(3)\}.$$

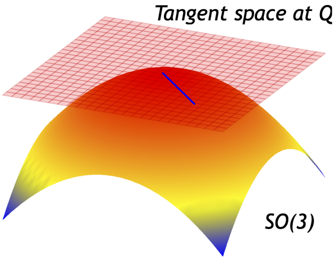
The velocity space is the tangent bundle TC of C:

$$TC = \{(\delta\boldsymbol{\varphi}, \delta\boldsymbol{Q})\colon [0, L] \to \mathbb{R}^3 \times T_{\boldsymbol{Q}}SO(3), (\boldsymbol{\varphi}, \boldsymbol{Q}) \in C\}.$$

Remark (Tangent space of a matrix Lie group)

Any tangent vector $\delta\boldsymbol{Q} \in T_{\boldsymbol{Q}}SO(3)$ at $\boldsymbol{Q}$ is of the form $\delta\boldsymbol{Q} = \frac{\partial \boldsymbol{Q}(\nu)}{\partial \nu}_{|\nu=0}$ where

$$\begin{cases} \boldsymbol{Q}(\nu) \in SO(3) & \forall \nu \in \mathbb{R} \\ \boldsymbol{Q}(0) = \boldsymbol{Q} \end{cases}$$



Position-momenta space: cotangent bundle

We define momenta $\boldsymbol{p}, \Sigma \in T^*C = \{(\varphi, \boldsymbol{Q}, \boldsymbol{p}, \Sigma)\}$ as dual variables.

Definition (Riemannian metric for dual variables)

For any $\begin{cases} (\varphi, \boldsymbol{Q}) \in C \\ \Big((\delta\boldsymbol{\varphi}, \delta\boldsymbol{Q}), (\widetilde{\delta\boldsymbol{Q}}, \widetilde{\delta\boldsymbol{\varphi}})\Big) \in T_{(\boldsymbol{\varphi},\boldsymbol{Q})}C \times T_{(\boldsymbol{\varphi},\boldsymbol{Q})}C \end{cases}$,

$$g\Big((\delta\boldsymbol{\varphi}, \widetilde{\delta\boldsymbol{\varphi}}), (\delta\boldsymbol{Q}, \widetilde{\delta\boldsymbol{Q}})\Big) = \int_0^L < \delta\boldsymbol{\varphi}(S), \widetilde{\delta\boldsymbol{\varphi}}(S) > + \ll \delta\boldsymbol{Q}(S), \widetilde{\delta\boldsymbol{Q}}(S) \gg \mathrm{d}S$$

with

$< \cdot, \cdot >$	usual scalar product on $\mathbb{R}^3$
$\ll \boldsymbol{A}, \boldsymbol{B} \gg = \frac{1}{2}\mathrm{Tr}(\boldsymbol{A} \cdot \boldsymbol{B}^T)$	the Frobenius scalar product of matrices.

Legendre transform

A fundamental tool to recover equations of motion in terms of the momenta $\boldsymbol{p}$ and Σ is the Legendre transform.

Proposition (From velocities to momenta)

The Legendre transform induced by $\mathcal{L}$ is

$$(q, \delta q) \in TC \mapsto (q, \pi) \in T^*C$$

with $q = (\varphi, \boldsymbol{Q}) \in C$ and

$$\pi = D_q \mathcal{L} = \begin{pmatrix} \boldsymbol{p} \\ \Sigma \end{pmatrix} = \begin{pmatrix} \mathbb{A}\delta\boldsymbol{\varphi} \\ \boldsymbol{Q} j^{-1}\left(\mathbb{J} j(\boldsymbol{Q}^{-1}\delta\boldsymbol{Q})\right) \end{pmatrix} \in T_q^* C$$

where $j(\cdot) = axial(\cdot)$.

Induced Hamiltonian

Proposition (Hamiltonian on T^*C)

The Legendre transform induces the Hamiltonian $H\colon T^*C \to \mathbb{R}$ defined as

$$\begin{aligned} H(q,\pi) = \frac{1}{2}\int_0^L &< \boldsymbol{p}, \mathbb{A}^{-1}\boldsymbol{p} > + < j(\boldsymbol{Q}^{-1}\Sigma), \mathbb{J}^{-1} j(\boldsymbol{Q}^{-1}\Sigma) > \\ &+ < \frac{\partial \boldsymbol{\varphi}}{\partial S} - \boldsymbol{d_3}, \mathbb{G}(\frac{\partial \boldsymbol{\varphi}}{\partial S} - \boldsymbol{d_3}) > \\ &+ < j(\boldsymbol{Q}^{-1}\frac{\partial \boldsymbol{Q}}{\partial S}), \mathbb{H} j(\boldsymbol{Q}^{-1}\frac{\partial \boldsymbol{Q}}{\partial S}) > \; dS. \end{aligned} \tag{10}$$

Remember : strain-vector $\boldsymbol{\varepsilon} = \frac{\partial \boldsymbol{\varphi}}{\partial S} - \boldsymbol{d_3}$, curvature $\boldsymbol{\kappa} = j(\boldsymbol{Q}^{-1}\frac{\partial \boldsymbol{Q}}{\partial S})$.

Equations of motion in position-momenta variables

Set $\boldsymbol{\sigma} = j(\boldsymbol{Q}^{-1}\Sigma)$ and recall $\boldsymbol{\varepsilon} = \frac{\partial \boldsymbol{\varphi}}{\partial S} - \boldsymbol{d_3}$ and $\kappa = j(\frac{\partial \boldsymbol{Q}}{\partial S}\boldsymbol{Q}^T)$.

Proposition

In the coordinates $\{\boldsymbol{\varphi}, \boldsymbol{p}, \boldsymbol{Q}, \boldsymbol{\sigma}\}$, *the equations of motion are*

$$\begin{aligned}
\frac{\partial \boldsymbol{\varphi}}{\partial t} &= \mathbb{A}^{-1}\boldsymbol{p} \\
\frac{\partial \boldsymbol{p}}{\partial t} &= \frac{\partial \, \mathbb{G}\boldsymbol{\varepsilon}}{\partial S} \\
\frac{\partial \boldsymbol{Q}}{\partial t} &= \boldsymbol{Q} j^{-1}(\mathbb{J}^{-1}\boldsymbol{\sigma}) \\
\frac{\partial \boldsymbol{\sigma}}{\partial t} &= \frac{\partial \, \mathbb{H}\boldsymbol{\kappa}}{\partial S} + \frac{\partial \boldsymbol{\varphi}}{\partial S} \wedge (\mathbb{G}\boldsymbol{\varepsilon})
\end{aligned} \tag{11}$$

Poisson bracket on the moving frame

Theorem (C., Le Marrec, 2024)

In the coordinates $\{\boldsymbol{\varphi}, \boldsymbol{Q}, \boldsymbol{p}, \boldsymbol{\sigma}\}$, *the Poisson bracket on* T^*C *becomes*

$$\{\bar{f}, \bar{g}\} = \int_0^L < \frac{\partial \bar{f}}{\partial \boldsymbol{\varphi}}, \frac{\partial \bar{g}}{\partial \boldsymbol{p}} > - < \frac{\partial \bar{g}}{\partial \boldsymbol{\varphi}}, \frac{\partial \bar{f}}{\partial \boldsymbol{p}} > + \ll \boldsymbol{Q}^{-1} \frac{\partial \bar{f}}{\partial \boldsymbol{Q}}, j^{-1}(\frac{\partial \bar{g}}{\partial \boldsymbol{\sigma}}) \gg - \ll \boldsymbol{Q}^{-1} \frac{\partial \bar{g}}{\partial \boldsymbol{Q}}, j^{-1}(\frac{\partial \bar{f}}{\partial \boldsymbol{\sigma}}) \gg dS. \tag{12}$$

Remark

$$\ll \boldsymbol{Q}^{-1} \frac{\partial \bar{f}}{\partial \boldsymbol{Q}}, j^{-1}(\frac{\partial \bar{g}}{\partial \boldsymbol{\sigma}}) \gg$$

is the natural pairing of a vector in $T\,SO(3)$ *with a covector in* $T^*SO(3)$.

Hamiltonian formulation of Timoshenko model on the moving frame

Corollary (Hamiltonian formulation of Timoshenko model, C., Le Marrec, 2024)

Under boundary conditions

$$< \delta\boldsymbol{\varphi}, \mathbb{G}\boldsymbol{\varepsilon} >=< \delta\boldsymbol{\theta}, \mathbb{H}\boldsymbol{\kappa} >= 0,$$

the equations of motion (11) *are Hamiltonian for the bracket* (12) *and the Hamiltonian*

$$\begin{aligned} H(\varphi, \boldsymbol{p}, \boldsymbol{Q}, \boldsymbol{\sigma}) = \frac{1}{2}\int_0^L & < \boldsymbol{p}, \mathbb{A}^{-1}\boldsymbol{p} > + < \boldsymbol{\sigma}, \mathbb{J}^{-1}\boldsymbol{\sigma} > \\ & + < (\frac{\partial \varphi}{\partial S} - \boldsymbol{d_3}), \mathbb{G}(\frac{\partial \varphi}{\partial S} - \boldsymbol{d_3}) > \\ & + < j(\boldsymbol{Q}^{-1}\frac{\partial \boldsymbol{Q}}{\partial S}), \mathbb{H} j(\boldsymbol{Q}^{-1}\frac{\partial \boldsymbol{Q}}{\partial S}) > \, dS. \end{aligned}$$

One sip of the proof

We are left to prove the four equations:

$$\{\boldsymbol{\varphi}, H\} = \mathbb{A}^{-1}\boldsymbol{p} \tag{13}$$

$$\{\boldsymbol{p}, H\} = \frac{\partial\, \mathbb{G}\boldsymbol{\varepsilon}}{\partial S} \tag{14}$$

$$\{\boldsymbol{Q}, H\} = \boldsymbol{Q} j^{-1}(\mathbb{J}^{-1}\boldsymbol{\sigma}) \tag{15}$$

$$\{\boldsymbol{\sigma}, H\} = \frac{\partial\, \mathbb{H}\boldsymbol{\kappa}}{\partial S} + \frac{\partial \boldsymbol{\varphi}}{\partial S} \wedge (\mathbb{G}\boldsymbol{\varepsilon}). \tag{16}$$

Recover closure relations

Proposition

For any test function $f : (\mathbb{R}^3)^{[0,L]} \to \mathbb{R}$,

$$\{f(\boldsymbol{\varepsilon}), H\} = \frac{\partial f(\boldsymbol{\varepsilon}(t))}{\partial t} = \int_0^L < \frac{\partial f}{\partial \boldsymbol{\varepsilon}}, \frac{\partial \boldsymbol{v}}{\partial S} - \boldsymbol{\omega} \wedge \boldsymbol{d_3} > dS. \tag{17}$$

where $\boldsymbol{v} = \mathbb{A}^{-1}\boldsymbol{p}$ *and* $\boldsymbol{\omega} = \mathbb{J}^{-1}\boldsymbol{\sigma}$.

Proposition

For any test function $f : (\mathbb{R}^3)^{[0,L]} \to \mathbb{R}$,

$$\{f(\boldsymbol{\kappa}), H\} = \frac{\partial f(\boldsymbol{\kappa}(t))}{\partial t} = \int_0^L < \frac{\partial f}{\partial \boldsymbol{\kappa}}, \frac{\partial \boldsymbol{\omega}}{\partial S} + \boldsymbol{\omega} \wedge \boldsymbol{\kappa} > dS. \tag{18}$$

Conclusion

Differential calculus is more complicated on the moving frame...

$$\delta(\boldsymbol{u}\mathbb{X}\boldsymbol{v}) = (\delta\boldsymbol{u} - \delta\boldsymbol{\theta}\wedge\boldsymbol{u})\,\mathbb{X}\boldsymbol{v} + \boldsymbol{u}\mathbb{X}\,(\delta\boldsymbol{v} - \delta\boldsymbol{\theta}\wedge\boldsymbol{v})$$

versus

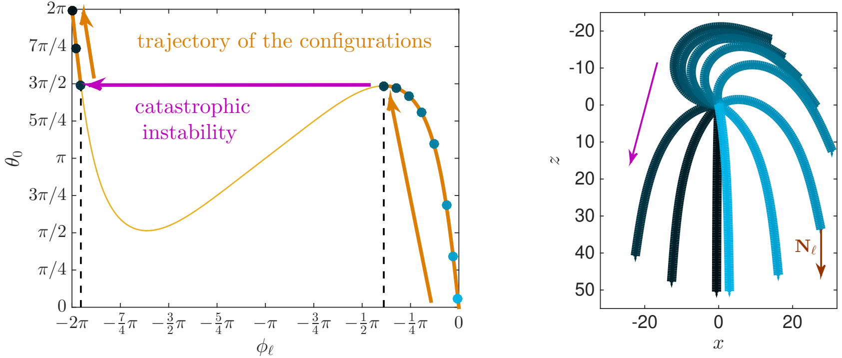
$$\delta\,(u\mathbb{X}v) = (\delta u)\,\mathbb{X}v + u\mathbb{X}\,(\delta v)$$

... but Poisson geometry is simpler there:

$$\{\bar{f},\bar{g}\} = \int_0^L < \frac{\partial\bar{f}}{\partial\boldsymbol{\varphi}}, \frac{\partial\bar{g}}{\partial\boldsymbol{p}} > - < \frac{\partial\bar{g}}{\partial\boldsymbol{\varphi}}, \frac{\partial\bar{f}}{\partial\boldsymbol{p}} > + \ll \frac{\partial\bar{f}}{\partial\boldsymbol{Q}}, \boldsymbol{Q}j^{-1}(\frac{\partial\bar{g}}{\partial\boldsymbol{\sigma}}) \gg - \ll \frac{\partial\bar{g}}{\partial\boldsymbol{Q}}, \boldsymbol{Q}j^{-1}(\frac{\partial\bar{f}}{\partial\boldsymbol{\sigma}}) \gg \mathrm{d}S.$$

Perspectives

- Numerics: Geometric integrators using the Hamiltonian structure
- Use Poisson geometry for the study of mechanical stability



Catastrophic instability in the Timoshenko model

fig: Le Marrec