COVARIANT FORMULATION OF THE ELECTRO-MAGNETO-MECHANICS COUPLING

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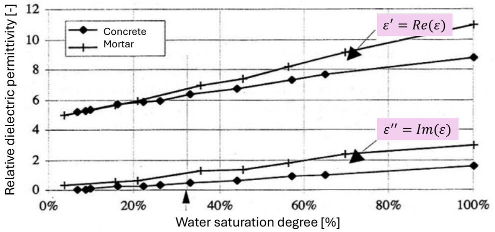
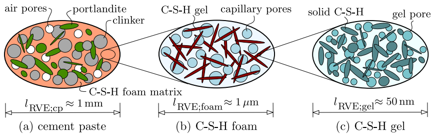
INTRODUCTION

- For cementitious materials:
  - Use of the electro-(magneto)-mechanics coupling to do non destructive monitoring (Andrade et al. 1999; Guihard 2018).
  - Link between hydration and dielectric permittivity ε.
  - Multi-scale models, from the cement paste to the microstructure.
    - ★ CSH[1] = products of hydration.
    - ★ Their microstructure layout is linked to dielectric permittivity. (Ait Hamadouche et al. 2023)



Dielectric permittivity function of water saturation *(Robert 1997)*

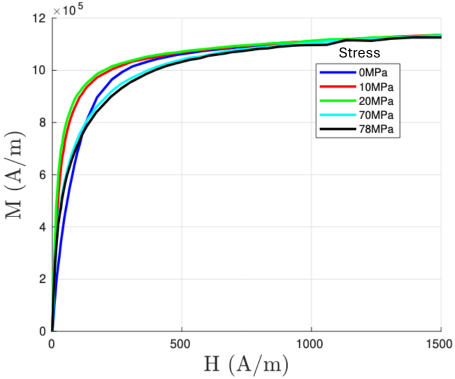


Cement paste at different scales *(Königsberger et al. 2020)*

[1] Calcium Silicate Hydrate

INTRODUCTION

- Magneto-elastic coupling for metals.
  - Various types:
    - ★ Magnetostrictive materials (Dapino 2004),
    - ★ Shape-memory alloys (Lexcellent 2013),
    - ★ Multiferroics (Corcolle et al. 2008).
  - Two effects:
    - ★ Magnetic field $H \to$ strain ϵ: magnetostriction,
    - ★ Stress $\sigma \to$ magnetization M: Villari effect.
  - Multiscale approaches to reach crystal scale.
- For polymers (Danas 2024; Bastola and Hossain 2020):
  - Depending on their range of magnetostrictive strain:
    - ★ mechanically-hard $\to 10^{-6} - 10^{-3}$,
    - ★ mechanically-soft $\to$ up to 10^{-1}.
    - $\to$ Finite strain theory.



Effect of uniaxial stress on magnetization behavior at constant stress, for non-oriented silicon-iron alloy

(Hubert 2019)

VARIATIONAL RELATIVITY (SOURIAU 1958)

- Lagrangian and variational principle
  - Gravitation (g), electromagnetism ($\boldsymbol{A}$), matter (Ψ) (and hyperelasticity) and their coupling are described by fields defined on a 4-dimensional Universe $\mathscr{M}$.
  - Sum of 4 Lagrangians

$$\mathscr{L}[g, \boldsymbol{A}, \Psi] = \int L(\dots)\,\mathrm{vol}_g$$

    depending on these fields.
    - ★ The Lagrangian density L depends on the fields and a finite number of their partial derivatives.
  - Least action principle:

$$\delta\mathscr{L} = 0\,.$$

    - ★ $\delta_g\mathscr{L} = 0 \rightarrow$ Einstein equation and definition of the stress-energy tensor,
    - ★ $\delta_{\boldsymbol{A}}\mathscr{L} = 0 \rightarrow$ Maxwell equations,
    - ★ $\delta_\Psi\mathscr{L} = 0 \rightarrow$ Conservation of the stress-energy tensor and generalization of equilibrium equations.
- Goal: **Propose a coupling Lagrangian for non-conducting elastic continuous media.**

Why General Relativity ?

- Metric tensor g in the arguments of $\mathscr{L} = \mathscr{L}[g, \ldots]$
  - Definition of the stress-energy tensor in the sense of Hilbert (1915).
  - In presence of matter, generalization of the 3D stress tensor.
    - ⋆ Symmetric by definition.
    - ⋆ Other definitions in literature (Eringen and Maugin 1990).
    - → question around symmetry.
- General covariance: invariance under local reparametrizations of the Universe (Einstein 1921).
  - **Our Lagrangian will have to be general covariant.**
  - Used to determine suitable arguments for L (theorem).

→ **Work in 4D but later separation of space and time and obtention of 3D equilibrium equations.**

Summary

SUMMARY

1. Electro-gravitational coupling in general relativity
   - Gravitation
   - 4D formulation of electromagnetism without current
   - Electro-gravitational coupling in Variational Relativity

Gravitation in vacuum described by the metric tensor

- Gravitation phenomena are described using the metric tensor. The Hilbert-Einstein functional is used as Lagrangian:

$$\mathscr{H}_{\mathscr{U}}[g] = \int_{\mathscr{U}} \frac{1}{2\kappa} R^g \, \mathrm{vol}_g \; .$$

  - R^g denotes the scalar curvature and κ the Einstein constant.
  - $\mathscr{H}$ satisfies general covariance:

$$\boxed{\mathscr{H}_{\mathscr{U}}[\varphi^* g] = \mathscr{H}_{\bar{\mathscr{U}}}[g]}$$

  for any diffeomorphism $\varphi : \mathscr{U} \to \bar{\mathscr{U}}$ (both open sets of the Universe $\mathscr{M}$).

    - ★ A change of variable under the integral leads to general covariance.

Definition of the Einstein tensor

- Variations of $\mathscr{H}$ with respect to the metric tensor:

$$\mathrm{d}\mathscr{H}.\delta g = \int (\mathbf{G}^g)^\sharp : \delta g \,\mathrm{vol}_g \,,$$

  - With the Einstein tensor defined as:

$$\boxed{(\mathbf{G}^g)^\sharp = 2\kappa \frac{\delta \mathscr{H}}{\delta g} \,.}$$

  - $\sharp$ raises the indices: $(\mathbf{G}^g)^\sharp = g^{-1}\,\mathbf{G}^g g^{-1}$, or $(G^g)^{\mu\nu} = g^{\mu\alpha} g^{\beta\nu} G^g_{\alpha\beta}$.
  - The Einstein equation in vacuum is:

$$\delta_g \mathscr{H} = 0 \to \boxed{(\mathbf{G}^g)^\sharp = 0 \,.}$$

Conservation of the Einstein tensor

- Conservation of $\mathbf{G}^g$ assured by general covariance $\mathscr{H}[\varphi^* g] = \mathscr{H}[g]$ (Noether 1918):
- Let us derive the general covariance:
  - For $\varphi(s)$ a path of diffeomorphisms, and with $\varphi(0) = \mathbf{Id}$ and $\dot{\varphi}(0) = X$ with X a vector field.
  - Then

$$
\begin{aligned}
\frac{d}{ds}\mathscr{H}[\varphi^*(s)g] &= 0 \\
&= \mathrm{d}\mathscr{H} \,.\, \mathrm{L}_X\, g \\
&= \int (\mathbf{G}^g)^\sharp :\ \mathrm{L}_X\, g \,\mathrm{vol}_g \\
&= \int (\mathbf{G}^g)^\sharp :\ \nabla^g X \,\mathrm{vol}_g \\
&= -\int \mathrm{div}^g \left((\mathbf{G}^g)^\sharp \right) \cdot X^\flat \,\mathrm{vol}_g \,,
\end{aligned}
$$

by **integrating by parts**, with $\flat$ lowering the index of X. Finally

$$
\mathrm{div}^g \left((\mathbf{G}^g)^\sharp \right) = 0 \,.
$$

Summary

Faraday tensor

- The electromagnetic field $(\boldsymbol{e}, \boldsymbol{b})$ is modelled by the Faraday tensor $\mathbf{F}$, a differential 2-form in 4 dimensions.
- Maxwell-Faraday and Maxwell-Thomson (homogenous Maxwell equations) are recasted using the exterior derivative of $\mathbf{F}$:

$$\boxed{\mathrm{d}\mathbf{F} = 0\,.}$$

Flat Minkowski Spacetime

In a flat Minkowski Spacetime: Components of the Faraday tensor

- The coupling between the electromagnetic pertubations and the gravitationnal field are assumed to be neglectible $\rightarrow$ passive coupling.
- Therefore we place ourselves in the flat Minkowski Spacetime, with the canonical coordinates system (x^μ), and where the metric tensor can be evaluated as $g = \eta = \mathrm{diag}(-1, 1, 1, 1)$.
- In the canonical coordinates system $(x^\mu) = \left(ct, x^1, x^2, x^3\right)$, the components of $\mathbf{F}$ are:

$$\mathbf{F} = (F_{\mu\nu}) = \begin{pmatrix} 0 & \frac{1}{c}e^1 & \frac{1}{c}e^2 & \frac{1}{c}e^3 \\ -\frac{1}{c}e^1 & 0 & -b^3 & b^2 \\ -\frac{1}{c}e^2 & b^3 & 0 & -b^1 \\ -\frac{1}{c}e^3 & -b^2 & b^1 & 0 \end{pmatrix}$$

with c the speed of light.

Faraday tensor

- The electromagnetic field $(\boldsymbol{e}, \boldsymbol{b})$ is modelled by the Faraday tensor $\mathbf{F}$, a differential 2-form in 4 dimensions.
- Maxwell-Faraday and Maxwell-Thomson (homogenous Maxwell equations) are recasted using the exterior derivative of $\mathbf{F}$:

$$\boxed{\mathrm{d}\mathbf{F} = 0\,.}$$

In a flat Minkowski Spacetime: Maxwell Faraday and Maxwell Thomson equations

$$\mathrm{d}\mathbf{F} = 0 \qquad \Longleftrightarrow \qquad \begin{cases} \operatorname{curl} \boldsymbol{e} = -\dfrac{\partial \boldsymbol{b}}{\partial t} \\ \operatorname{div} \boldsymbol{b} = 0 \end{cases}$$

Four-potential

- Under topological and regularity assumptions, $\mathrm{d}\mathbf{F} = 0$ leads to

$$\mathbf{F} = \mathrm{d}\boldsymbol{A}$$

where $\boldsymbol{A}$ is a 1-form called the four-potential.

- The four-potential is defined **up to a gauge transformation** $\boldsymbol{A} \to \boldsymbol{A} + \mathrm{d}\chi$, with χ a scalar function.
  - Since $\mathrm{d}^2 = 0$, $\mathbf{F} = \mathrm{d}(\mathbf{A} + \mathrm{d}\chi) = \mathrm{d}\mathbf{A}$.

  $\rightarrow$ $\mathbf{F}$ is **gauge invariant**: it does not change when the gauge changes.

  $\rightarrow$ Our full Lagrangian will have to be **gauge invariant** too.

In a flat Minkowski Spacetime: Components of the four-potential

$\boldsymbol{A}$ includes the electric scalar potential ϕ and the magnetic vector potential $\boldsymbol{a}$:

$$\mathbf{A} = (A_\mu) = (\phi, a_1, a_2, a_3) \ .$$

SUMMARY

1. Electro-gravitational coupling in general relativity
   - Gravitation
   - 4D formulation of electromagnetism without current
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Electro-gravitational Lagrangian

- Electromagnetism is described by the four-potential $\boldsymbol{A}$.
- The electromagnetic Lagrangian is added to the Hilbert-Einstein functional to form $\mathscr{L}$:

$$\mathscr{L}[g,\boldsymbol{A}] = \mathscr{H}[g] + \mathscr{L}^{\mathrm{EM}}[g,\boldsymbol{A}] = \int \frac{1}{2\kappa} R^g \operatorname{vol}_g + \int -\frac{1}{4\mu_0} ||\mathbf{F}||_g^2 \operatorname{vol}_g \,.$$

  - μ_0 denotes the magnetic permeability of vacuum.
  - $||\mathbf{F}||_g^2 = \mathbf{F}^\sharp : \mathbf{F} = F^{\mu\nu} F_{\mu\nu}$ (with $\mathbf{F} = \mathrm{d}\boldsymbol{A}$).
  - It is general covariant:
  $$\mathscr{L}[\varphi^* g, \varphi^* \boldsymbol{A}] = \mathscr{L}[g,\boldsymbol{A}] \,,$$
  for any local diffeomorphism of the Universe φ.
  - As well as gauge invariant:
  $$\mathscr{L}[g,\boldsymbol{A} + \mathrm{d}\chi] = \mathscr{L}[g,\boldsymbol{A}] \,,$$
  for any function χ.

Variation with respect to the four-potential $\boldsymbol{A}$

- The two last Maxwell equations are derived from the variations of $\boldsymbol{A}$.
  - Indeed $\delta_{\boldsymbol{A}}\mathscr{L} = 0$ yields

$$
\begin{aligned}
\delta_{\boldsymbol{A}}\mathscr{L} &= \mathrm{d}\mathscr{L}\,.\delta\boldsymbol{A} \\
&= \int -\frac{1}{2\mu_0}\mathbf{F}^\sharp : (\delta\mathrm{d}\boldsymbol{A})\,\mathrm{vol}_g = \int -\frac{1}{2\mu_0}\mathbf{F}^\sharp : (\mathrm{d}\delta\boldsymbol{A})\,\mathrm{vol}_g \\
&= \int -\frac{1}{\mu_0}\,\mathrm{div}^g\,\mathbf{F}^\sharp \cdot \delta\boldsymbol{A}\,\mathrm{vol}_g\;.
\end{aligned}
$$

  - Such that finally

$$
\mathrm{div}^g\left(\frac{1}{\mu_0}\mathbf{F}^\sharp\right) = 0\,.
$$

In a flat Minkowski Spacetime: Maxwell-Gauss and Maxwell-Ampere

$$
\mathrm{div}^g\left(\frac{1}{\mu_0}\mathbf{F}^\sharp\right) = 0 \iff \begin{cases} \mathrm{curl}\,\boldsymbol{b} = \dfrac{1}{c^2}\dfrac{\partial\boldsymbol{e}}{\partial t} \\ \mathrm{div}\,\boldsymbol{e} = 0 \end{cases}
$$

Variation with respect to the metric tensor g

- Variation of the Lagrangian with respect to g leads to

$$\delta_g \mathscr{L} = 0 \to \underbrace{\delta_g \mathscr{H}}_{\frac{1}{2\kappa}(\mathbf{G}^g)^\sharp} + \underbrace{\delta_g \mathscr{L}^{\text{EM}}}_{-\frac{1}{2}\mathbf{T}^{\text{EM}}} = 0\,,$$

  - with the electromagnetic stress-energy tensor $\mathbf{T}^{\text{EM}}$ defined as:

$$\mathbf{T}^{\text{EM}} = -2\frac{\delta \mathscr{L}^{\text{EM}}}{\delta g}\,.$$

  - Along the extremum lines the electromagnetic Einstein equation is

$$(\mathbf{G}^g)^\sharp = \kappa \mathbf{T}^{\text{EM}}\,.$$

  - The conservation of $\mathbf{T}^{\text{EM}}$ is obtained using the Einstein equation along the extremum lines:

$$\operatorname{div}^g \mathbf{T}^{\text{EM}} = \operatorname{div}^g\left((\mathbf{G}^g)^\sharp\right) = 0\,.$$

Recovering the Maxwell stress tensor

In a flat Minkowski Spacetime

- The 4D stress-energy tensor falls back to

$$\mathbf{T}^{\mathrm{EM}} = \eta^{-1}\,\mathbf{F}\eta^{-1}\,\mathbf{F}\eta^{-1} + \|\mathbf{F}\|_{\eta}^{2}\eta^{-1} = \left(\left(T^{\mathrm{EM}}\right)^{\mu\nu}\right) = -\begin{pmatrix} \varepsilon^{\mathrm{EM}} & \frac{1}{c}\mathbf{S} \\ \frac{1}{c}\mathbf{S} & -\boldsymbol{\sigma}^{\mathrm{EM}} \end{pmatrix},$$

with

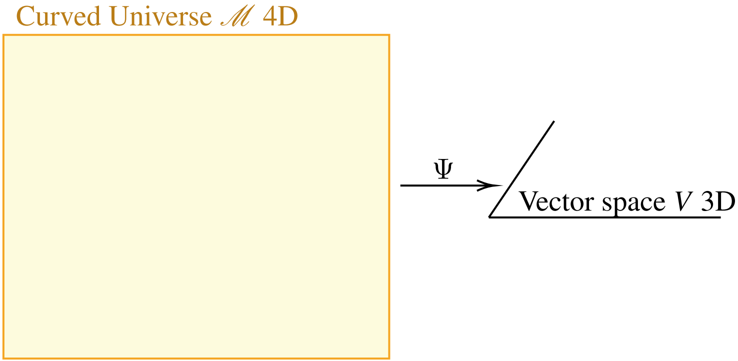
- Electromagnetic energy density $\varepsilon^{\mathrm{EM}} = \frac{1}{2}(\varepsilon_0||\boldsymbol{e}||_{\mathbf{q}}^2 + \frac{1}{\mu_0}||\boldsymbol{b}||_{\mathbf{q}}^2)$,
- Poynting vector $\mathbf{S} = \frac{1}{\mu_0}\boldsymbol{e} \times \boldsymbol{b}$,
- Maxwell electromagnetic stress tensor $\boldsymbol{\sigma}^{\mathrm{EM}} = \varepsilon_0\,\boldsymbol{e} \otimes \boldsymbol{e} + \frac{1}{\mu_0}\boldsymbol{b} \otimes \boldsymbol{b} - \varepsilon^{\mathrm{EM}}\mathbf{q}^{-1}$, where ε_0 is the vacuum permittivity, $\varepsilon_0\mu_0 c^2 = 1$, and $\mathbf{q} = \mathrm{diag}(1, 1, 1)$ the Euclidean metric tensor.

Summary

Summary

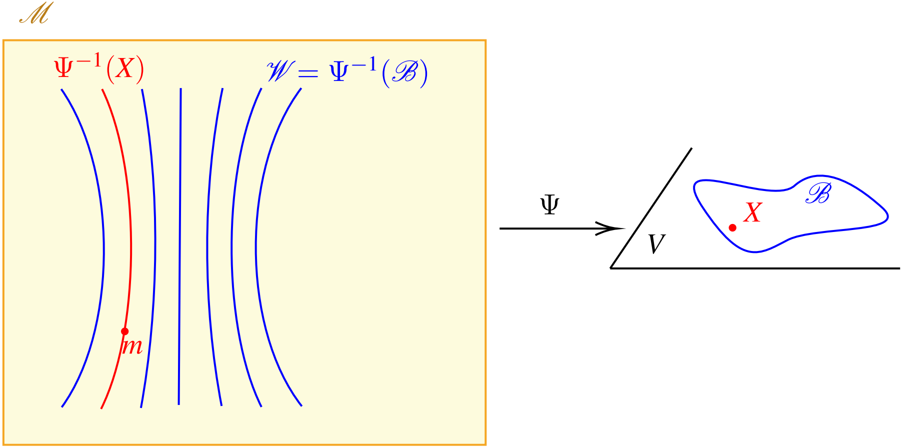
Matter field

- In classical 3D mechanics, the **deformation** p sends the reference configuration onto the deformed one.
  - Its linear tangent map $F = Tp$ is the **deformation gradient**.
- In our approach, matter is described through the matter field Ψ (Souriau 1958).
  - Same idea as p but opposite direction: $\Psi : \mathscr{M} \to V$ (deformed to reference),
  - with $\mathscr{M}$ the 4-dimensional Universe and V the 3-dimensional space of labels.



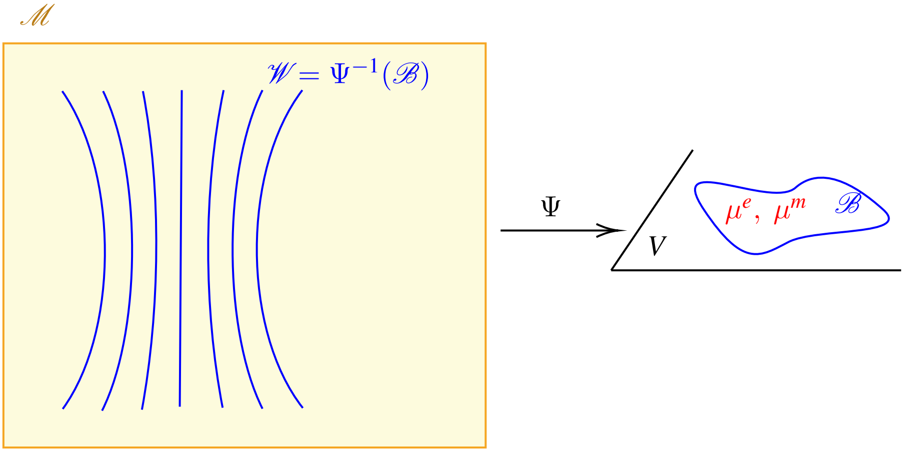
World Lines

- X is a label of a particle.
- $\Psi^{-1}(X)$ spans the World line associated to the particule labelled by X: from its past to its future.
- All world lines form the body's World tube $\mathscr{W} = \Psi^{-1}(\mathscr{B})$.



Electric charge and mass measure

- Firstly the electric charge measure μ^e and the mass measure μ^m are defined on the body. Integrated over the body, they give the electric charge $Q = \int_{\mathscr{B}} \mu^e$ and mass $m = \int_{\mathscr{B}} \mu^m$.

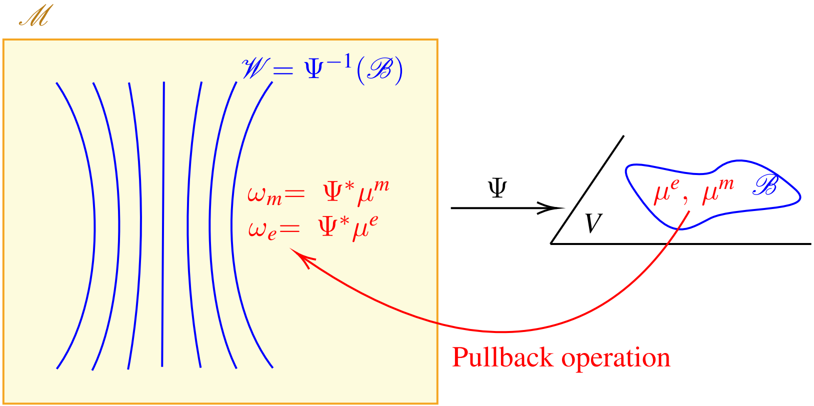


Currents definition

- Their pullback using the matter field are 3-forms in a 4D space.
  They can be completed by a four-vector in order to come back to the volume form vol_g:

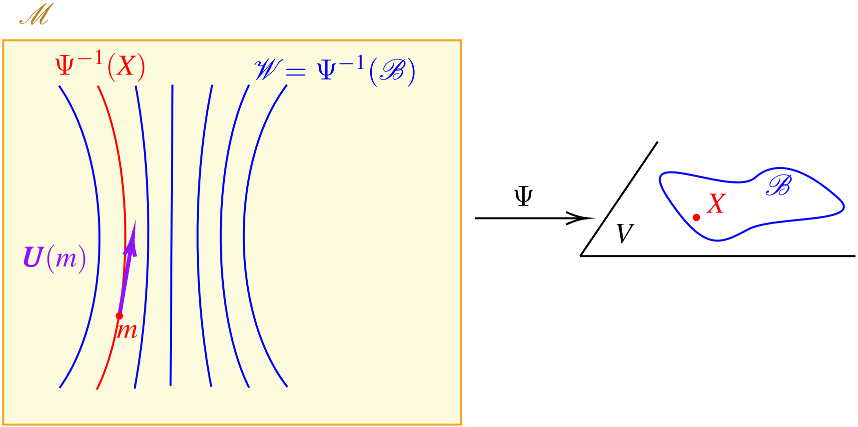
$$i_{\boldsymbol{J}_e}\,\mathrm{vol}_g = \Psi^*\mu^e\,, \qquad i_{\boldsymbol{J}_m}\,\mathrm{vol}_g = \Psi^*\mu^m\,,$$

with $\boldsymbol{J}_e$ the electric current and $\boldsymbol{J}_m$ the matter current.



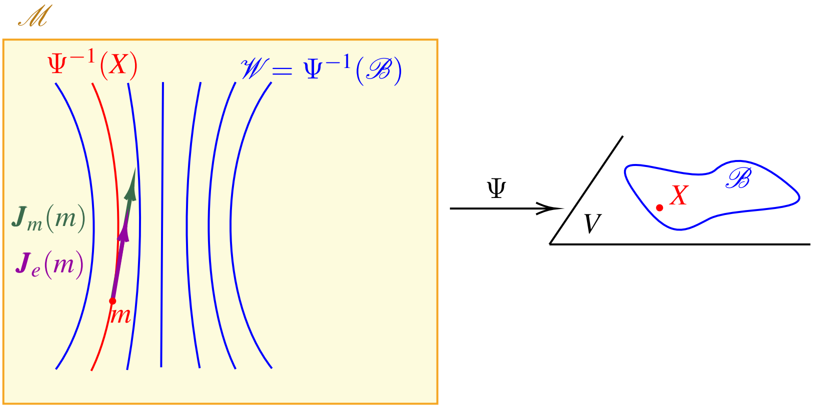
Unit time-like four-vector

- For perfect matter, the linear tangent map of the matter field $T\Psi$ is of rank 3, so it has a 1-dimensional kernel.
- $\ker(T\Psi)$ is spanned by a unit four-vector $\boldsymbol{U}$.
- It is time-like: $g(\boldsymbol{U},\boldsymbol{U}) = -1$.



Currents as colinear to $\boldsymbol{U}$

- Definition of electric current $\boldsymbol{J}_e$ and matter current $\boldsymbol{J}_m$ as four-vectors colinear to $\boldsymbol{U}$ (Carter 1980):
  - $\boldsymbol{J}_e = \rho_e \boldsymbol{U}$ with ρ_e the rest electric charge density,
  - $\boldsymbol{J}_m = \rho_r \boldsymbol{U}$ with ρ_r the rest mass density.



Currents - conservation

- Both currents are conserved:

$$(\mathrm{div}^g \boldsymbol{J})\,\mathrm{vol}_g = \mathrm{d}\left(i_{\boldsymbol{J}}\,\mathrm{vol}_g\right) = \mathrm{d}\left(\Psi^*\mu\right) = \Psi^*\left(\mathrm{d}\mu\right) = 0 \quad \rightarrow \quad \mathrm{div}^g \boldsymbol{J}_e = 0\,, \quad \mathrm{div}^g \boldsymbol{J}_m = 0\,.$$

In a flat Minkowski Spacetime: conservation equations

When **at rest**:

$$\boldsymbol{J}_e = ({J_e}^{\mu}) = (\rho_e, 0, 0, 0) \ ,$$

with ρ_e the rest electric charge density.

$$\boldsymbol{J}_m = ({J_m}^{\mu}) = (\rho_r, 0, 0, 0) \ ,$$

with ρ_r the rest mass density. Their conservation is interpreted respectively as **electric charge conservation** and **mass conservation**.

Summary

2. Perfect electrised matter
   - Description of matter
   - Perfect electrised matter in Variational Relativity

Variational formulation of perfect electrised matter

- The total Lagrangian $\mathscr{L}$ is the sum of one Lagrangian per phenomenon:

$$\mathscr{L}[g,\Psi,\boldsymbol{A}] = \mathscr{H}[g] + \mathscr{L}^{\mathrm{EM}}[g,\boldsymbol{A}] + \mathscr{L}^{\mathrm{Elas}}[g,\Psi] + \mathscr{L}^{\boldsymbol{J}}[g,\Psi,\boldsymbol{A}]$$

  with
  - $\mathscr{H}$ for gravitation,
  - $\mathscr{L}^{\mathrm{EM}}$ for vacuum electromagnetism,
  - $\mathscr{L}^{\mathrm{Elas}}$ for hyperleasticity,
  - $\mathscr{L}^{\boldsymbol{J}}$ for the electro-magneto-mechanics coupling.

  $\rightarrow$ Each term is taken from literature.
- Let us recall the role of each field:
  - g the metric tensor: gravitational and inertial properties,
  - Ψ the matter field: description of the matter (inverse of the deformation),
  - $\boldsymbol{A}$ the four-potential: combination of electric and magnetic potentials.

Hyperelastic Lagrangian $\mathscr{L}^{\text{Elas}}$

$$\mathscr{L}^{\text{Elas}}[g, \Psi] = \int L_0^{\text{Elas}}\left(g_{\mu\nu}, \partial_\rho\, g_{\mu\nu}, \Psi, \partial_\rho \Psi\right) \text{vol}_g$$

- $\mathscr{L}^{\text{Elas}}$ general covariant $\iff$ its density comes down to a function of Ψ and $\mathbf{K}$ (Souriau 1958):

$$L_0^{\text{Elas}}(g, \Psi, \partial\Psi) = L^{\text{Elas}}(\Psi, \mathbf{K})\,.$$

  - $\mathbf{K}$ the conformation, defined as

$$\mathbf{K} = T\Psi g^{-1}\,(T\Psi)^{\star}\ .$$

$\rightarrow$ Relativistic inverse of the right Cauchy-Green tensor $\mathbf{C} = F^t F$ (with F the deformation gradient) (Souriau 1958; Maugin 1978).

  - For example $\mathscr{L}^{\text{Elas}}[g, \Psi] = \int \left(\rho_r(\Psi, \mathbf{K})c^2 + E\,(\Psi, \mathbf{K})\right) \text{vol}_g$, with E an internal energy density.

Coupling Lagrangian $\mathscr{L}^{\boldsymbol{J}}$

$$\mathscr{L}^{\boldsymbol{J}}[g, \Psi, \boldsymbol{A}] = \int -\boldsymbol{A} \cdot \boldsymbol{J}_e \, \mathrm{vol}_g \ .$$

- It is general covariant: $\mathscr{L}^{\boldsymbol{J}}[\varphi^* g, \varphi^* \Psi, \varphi^* \boldsymbol{A}] = \mathscr{L}^{\boldsymbol{J}}[g, \Psi, \boldsymbol{A}]$.
- The Lagrangian is gauge invariant only up to a boundary term:

$$\mathscr{L}[g, \Psi, \boldsymbol{A} + \mathrm{d}\chi] - \mathscr{L}[g, \Psi, \boldsymbol{A}] = \int_{\mathscr{U}} \mathrm{div}^g \left(\chi \boldsymbol{J}_e\right) \mathrm{vol}_g = \int_{\partial \mathscr{U}} \chi \boldsymbol{J}_e \cdot \boldsymbol{n}^\flat \, \boldsymbol{i}_{\boldsymbol{n}} \mathrm{vol}_g \ ,$$

and therefore it is called **weakly** gauge invariant.

Variation with respect to the four-potential $\boldsymbol{A}$

- Variation of the Lagrangian with respect to $\boldsymbol{A}$ are written

$$\delta_{\boldsymbol{A}}\mathscr{L} = \int_{\mathscr{U}} \left(\frac{1}{\mu_0} \operatorname{div}^g \left(\mathbf{F}^\sharp \right) - \boldsymbol{J}_e \right) \cdot \delta \boldsymbol{A} \ \operatorname{vol}_g ,$$

and therefore one has

$$\operatorname{div}^g \left(\frac{1}{\mu_0} \mathbf{F}^\sharp \right) = \boldsymbol{J}_e .$$

In a flat Minkowski Spacetime: Maxwell-Gauss and Maxwell-Ampere equations

$$\operatorname{div}^g \left(\frac{1}{\mu_0} \mathbf{F}^\sharp \right) = \boldsymbol{J}_e \iff \begin{cases} \operatorname{curl} \boldsymbol{b} = \dfrac{1}{c^2} \dfrac{\partial \boldsymbol{e}}{\partial t} \\ \operatorname{div} \boldsymbol{e} = \dfrac{1}{\varepsilon_0} \rho_e \end{cases}$$

VARIATION WITH RESPECT TO THE METRIC TENSOR g

- Variation of the Lagrangian with respect to g, $\delta_g \mathscr{L} = 0$, leads to

$$(\mathbf{G}^g)^\sharp = \kappa \mathbf{T} \quad \text{with} \quad \mathbf{T} = \mathbf{T}^{\text{EM}} + \mathbf{T}^{\text{Elas}} + \mathbf{T}^{\boldsymbol{J}} = -2\frac{\delta \mathscr{L}^{\text{EM}}}{\delta g} - 2\frac{\delta \mathscr{L}^{\text{Elas}}}{\delta g} - 2\frac{\delta \mathscr{L}^{\boldsymbol{J}}}{\delta g}\,.$$

  - $\mathbf{T}^{\boldsymbol{J}} = 0$ since $\mathscr{L}^{\boldsymbol{J}}[g, \Psi, \boldsymbol{A}] = \int -\boldsymbol{A} \cdot \boldsymbol{J}_e \,\mathrm{vol}_g = \int -\boldsymbol{A} \wedge \star \boldsymbol{J}_e$, with $\star$ the Hodge star operator.
  - The remaining part is expressed as

$$\mathbf{T} = \underbrace{\frac{1}{\mu_0}\mathbf{F}^\sharp \mathbf{F} g^{-1} + \frac{1}{4\mu_0}\|\mathbf{F}\|_g^2 g^{-1}}_{\text{Electromagnetic part}} + \underbrace{\rho_r \left(c^2 + e\right) \boldsymbol{U} \otimes \boldsymbol{U}}_{\text{Hyperelastic energetic part}} + \underbrace{2\rho_r g^{-1} (T\Psi)^\star \frac{\partial e}{\partial \mathbf{K}} (T\Psi) g^{-1}}_{\text{Relativistic 4D stress tensor}}\,,$$

where $e = E/\rho_r$ is a specific internal energy density (Kolev and Desmorat 2023).

Conservation of the stress-energy tensor and Lorentz force

- Let us get back to

$$\mathbf{T} = \mathbf{T}^{\mathrm{EM}} + \mathbf{T}^{\mathrm{Elas}} .$$

- Conservation of the Einstein tensor yields

$$\operatorname{div}^g \mathbf{T} = 0 = \operatorname{div}^g \mathbf{T}^{\mathrm{EM}} + \operatorname{div}^g \mathbf{T}^{\mathrm{Elas}}$$

  for $\mathbf{T}$ solution of the Einstein equation $(\mathbf{G}^g)^\sharp = \kappa \mathbf{T}$.
- The Lorentz force is defined as $\boldsymbol{f}^L = g^{-1}\mathbf{F} \cdot \boldsymbol{J}_e$.
- For $\mathbf{T}$ solution of the Einstein equation, generalization of the equilibrium equations:

$$\operatorname{div}^g \mathbf{T}^{\mathrm{EM}} = \boldsymbol{f}^L = -\operatorname{div}^g \mathbf{T}^{\mathrm{Elas}} .$$

In a flat Minkowski Spacetime: Equilibrium

$$\operatorname{div} \boldsymbol{\sigma} + \boldsymbol{f} = 0$$

Summary

Matter rest frame

- The rest matter frame $\mathscr{R}_{rest}$ is a frame where matter is at rest.
  - Frame of the tangent space to the Universe $T_m\mathscr{M}$.
  - Defined for each $m \in \mathscr{M}$:

$$\mathscr{R}_{rest} = (\boldsymbol{U}, \boldsymbol{E}_1, \boldsymbol{E}_2, \boldsymbol{E}_3) \ ,$$

  - Time-like part: $\boldsymbol{U}$,
  - Space-like part: $\boldsymbol{E}_1, \boldsymbol{E}_2$ and $\boldsymbol{E}_3$ form a basis of the orthogonal to $\boldsymbol{U}$ (so $\boldsymbol{U}(\boldsymbol{E}_I) = 0$ for $I = 1, 2, 3$).

About the matter rest frame - Relativistic gradient hyperelasticity work

- Developed with L. Darondeau, R. Desmorat, C. Ecker and B. Kolev.
- Presented at the IRCAM meeting, November 2024.
- On HAL : (Chapon et al. 2024, hal-04792877).

Theorem: arguments for the coupling Lagrangian density

- The coupling Lagrangian

$$\mathscr{L}^{\text{EMM}}[g, \Psi, \boldsymbol{A}] = \int L_0^{\text{EMM}}\left(g_{\mu\nu}, \partial_\rho g_{\mu\nu}, \Psi^I, \partial_\rho \Psi^I, A_\mu, \partial_\rho A_\mu\right) \text{vol}_g$$

is general covariant if and only if its density can be written as

$$L_0^{\text{EMM}} = L_1^{\text{EMM}}\left(\Psi, [g^{-1}]_{\mathscr{R}_{rest}}, [\boldsymbol{A}^\sharp]_{\mathscr{R}_{rest}}, [(\nabla^g \boldsymbol{A})^\sharp]_{\mathscr{R}_{rest}}\right) .$$

- $[\mathbf{T}]_{\mathscr{R}_{rest}}$ are the components of $\mathbf{T}$ in the rest frame $\mathscr{R}_{rest} = (\boldsymbol{U}, \boldsymbol{E}_1, \boldsymbol{E}_2, \boldsymbol{E}_3)$.
  - For example, $[g^{-1}]_{\mathscr{R}_{\textbf{rest}}} = \begin{pmatrix} -1 & 0 \\ 0 & \mathbf{K} \end{pmatrix}$, with $\mathbf{K}$ the relativistic right Cauchy-Green tensor $\mathbf{C} = F^t F$.
- One can check how the previous Lagrangian fits this result:

$$\mathscr{L}^{\mathbf{J}}[g, \Psi, \boldsymbol{A}] = \int -\boldsymbol{A} \cdot \boldsymbol{J}_e \,\text{vol}_g = \int -\rho_e \,\boldsymbol{A} \cdot \boldsymbol{U} \,\text{vol}_g \ .$$

About gauge invariance

- $\mathscr{L}^{\boldsymbol{J}}[g, \Psi, \boldsymbol{A}] = \int -\boldsymbol{A} \cdot \boldsymbol{J}_e \,\mathrm{vol}_g$ is not strongly gauge invariant.
- The Lagrangian is strongly gauge invariant if and only if its density is written as

$$L_0^{\mathrm{EMM}} = L_2^{\mathrm{EMM}} \left(\Psi, [g^{-1}]_{\mathscr{R}_{rest}}, [\mathbf{F}^\sharp]_{\mathscr{R}_{rest}} \right) ,$$

with $\mathbf{F} = \mathrm{d}\boldsymbol{A}$.

→ Adds a new term to $\mathscr{L}$: $\mathscr{L} = \mathscr{H} + \mathscr{L}^{\mathrm{EM}} + \mathscr{L}^{\mathrm{Elas}} + \mathscr{L}^{\mathrm{J}} + \mathscr{L}^{\mathrm{EMM}}$.

Components of the Faraday tensor in the matter rest frame

- Let us name the two main components of the Faraday tensor:
  - $\alpha^I = \mathbf{F}^\sharp\left(\boldsymbol{U}^\flat, \boldsymbol{E}^I\right)$,
  - $\beta^{IJ} = \mathbf{F}^\sharp\left(\boldsymbol{E}^I, \boldsymbol{E}^J\right)$.
- Then, the coupling Lagrangian is general covariant and gauge invariant if and only if its density comes down to

$$L_2^{\text{EMM}} = L_2^{\text{EMM}}\left(\Psi, \mathbf{K}, \boldsymbol{\alpha}, \boldsymbol{\beta}\right) .$$

- And this way, if we consider L_2^{EMM} as an internal energy density E^{EMM}, the whole Lagrangian becomes

$$\mathscr{L}[g, \boldsymbol{A}, \Psi] = \int \frac{1}{2\kappa} R^g \operatorname{vol}_g - \int \frac{1}{4\mu_0} ||\mathbf{F}||_g^2 \operatorname{vol}_g + \int \left(\rho_r c^2 + E\left(\Psi, \mathbf{K}\right)\right) \operatorname{vol}_g + \int \left(E^{\text{EMM}}\left(\Psi, \mathbf{K}, \boldsymbol{\alpha}, \boldsymbol{\beta}\right) - \boldsymbol{A} \cdot \boldsymbol{J}_e\right) \operatorname{vol}_g$$

Summary

Conclusion

- What has been done so far ?
  - Formulation of the electro-magneto-mechanics coupling in Variational Relativity.
  - Proper definition of the 4D variables of the Lagrangian density.
  - Theorem to obtain the expression of general covariant coupling Lagrangians that depend on first partial derivative of fields.

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