Théorie variationnelle des poutres de Cosserat et tenseurs affines

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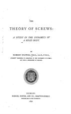
State of the Art

Screw theory
(Robert Stawell Ball, 1876)



THE

THEORY OF SCREWS:

A STUDY IN THE DYNAMICS OF A RIGID BODY.

BY

ROBERT STAWELL BALL, LL.D., F.R.S.,

DUBLIN:

HODGES, FOSTER, AND CO., GRAFTON-STREET.

Generalized continua
(Eugène and François Cosserat, 1909)

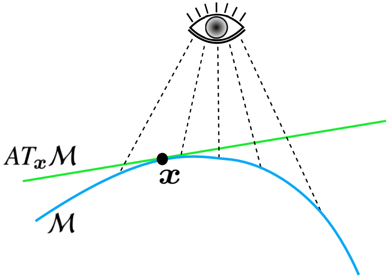
Affine tensors
(Tulczyjew, Urbañski, Grabowski, 1988)

Euler-Poincaré equations
(1744 and 1901)

Affine tensors

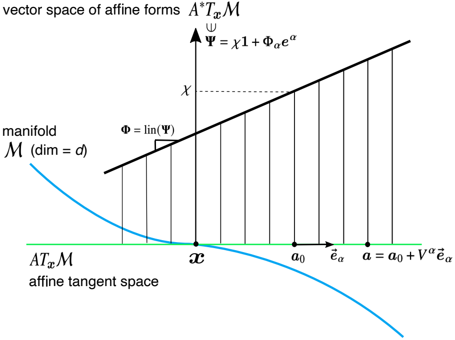
Élie Cartan, Sur les variétés à connexion affine et la théorie de la relativité généralisée (première partie). *Annales de l'École Normale Supérieure*, 40, 325-412 (1923)

"On pourrait enfin regarder l'espace affine attaché à $\boldsymbol{x}$ comme la variété elle-même qui serait perçue d'une manière affine par un observateur placé en $\boldsymbol{x}$"



Affine tensors

Affine tensors are maps which are affine or linear with respect to their arguments



Simplest affine tensors :	Their components :
affine form $\boldsymbol{\Psi}$	(χ, Φ_α) in the affine coframe $f^* = (1, (\boldsymbol{e}^\alpha))$
point $\boldsymbol{a}$	V^α in the affine frame $f = (\boldsymbol{a}_0, (\vec{\boldsymbol{e}}_\alpha))$

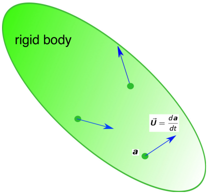
Co-momenta

Motivation : In the screw theory, a **twist** is an object which assigns to a point of a rigid body its velocity.

Co-momentum

map $\quad \bar{\boldsymbol{\theta}} : AT_{\boldsymbol{x}}\mathcal{M} \rightarrow T_{\boldsymbol{x}}\mathcal{M} : \boldsymbol{a} \mapsto \vec{\boldsymbol{U}} = \bar{\boldsymbol{\theta}}(\boldsymbol{a})$

next $\boxed{\boldsymbol{\theta}(\boldsymbol{\Phi}, \boldsymbol{a}) = \boldsymbol{\Phi}(\bar{\boldsymbol{\theta}}(\boldsymbol{a}))}$ linear wrt $\boldsymbol{\Phi}$ and affine wrt $\boldsymbol{a}$



Decomposed in f and f^* : $\boldsymbol{\theta} = \vec{\mathrm{e}}_\beta \otimes (\Upsilon^\beta 1 + K^\beta_\alpha \boldsymbol{e}^\alpha)$,
we assign to $\boldsymbol{\theta}$ its **components** stored
in the $d \times (d+1)$ matrix $\theta = (\Upsilon \mid K)$

Fact : A change of frame $f' \rightarrow f$ given by $\mathrm{g} = (C, P) \in \mathbb{GA}(d) = \mathbb{R}^d \rtimes \mathbb{GL}(d)$
leads to $\theta = Ad(\mathrm{g})\, \theta'$

then the systems θ of components of co-momentum tensors live in the Lie algebra of the affine group and their transformation law is the adjoint representation

Euclidean co-momenta (twists in the screw theory)

Decomposition of a co-momentum depending only on the origin $\boldsymbol{a}_0$ (but not on the basis)

- $(\mathrm{lin}(\theta))\,(\Phi, \overrightarrow{\boldsymbol{a}_0\boldsymbol{a}}) \;=\; \theta(\Phi, \boldsymbol{a}) - \theta(\Phi, \boldsymbol{a}_0)$ is bilinear,
  thus $\exists$ an endomorphism $\boldsymbol{K}$ of $T_{\boldsymbol{x}}\mathcal{M}$ such that $(\mathrm{lin}(\theta))(\Phi, \vec{\boldsymbol{V}}) = \Phi(\boldsymbol{K}(\vec{\boldsymbol{V}}))$
  $\boldsymbol{K}$ is represented in the affine frame by the matrix K
- $\theta(\Phi, \boldsymbol{a}_0) = \Phi\, \vec{\boldsymbol{\Upsilon}}_{\boldsymbol{a}_0}$
  $\vec{\boldsymbol{\Upsilon}}_{\boldsymbol{a}_0}$ is represented by the column Υ

Motivation : to a Riemannian manifold $\mathcal{M}$ for the scalar product $\langle \bullet, \bullet \rangle$, we can associate the corresponding Euclid group $\mathbb{SE}(d) = \mathbb{R}^d \rtimes \mathbb{SO}(d)$.

The elements $Z = (dC, dP)$ of its Lie algebra $\mathfrak{se}(d)$ are such that dP is skew-adjoint.

Definition : the co-momentum θ is **Euclidean** if $\boldsymbol{K}$ is skew-adjoint :
$\left\langle \boldsymbol{K}(\vec{\boldsymbol{U}}), \vec{\boldsymbol{V}} \right\rangle = -\left\langle \vec{\boldsymbol{U}}, \boldsymbol{K}(\vec{\boldsymbol{V}}) \right\rangle$

Momenta

reminder : co-momentum $\vec{U} = \bar{\theta}(a)$

A **momentum** is a linear map $\bar{\mu} : A^* T_x\mathcal{M} \to T_x^*\mathcal{M} : \Psi \mapsto \Phi = \bar{\mu}(\Psi)$

or equivalently $\mu(\vec{V}, \Psi) = (\bar{\mu}(\Psi))\,\vec{V}$ is bilinear

Decomposed in f and f^* : $\mu = e^\beta \otimes (\Pi_\beta\, a_0 + L^\alpha_\beta\, \vec{e}_\alpha)$,

we assign to μ its **components** stored in the $(d+1) \times d$ matrix $\mu = \begin{pmatrix} \Pi \\ L \end{pmatrix}$

In screw theory, a **wrench** represents the action of a force acting on a rigid body. To define it, we need

a decomposition of the momentum depending only on the origin a_0

It is based on the direct sum $A^* T_x\mathcal{M} = A_c^* T_x\mathcal{M} \oplus A_{a_0}^* T_X\mathcal{M}$

- of the subspace $A_{a_0}^* T_x\mathcal{M}$ of affine forms vanishing at the origin
- and that $A_c^* T_x\mathcal{M}$ of constant affine forms

Euclidean momenta (wrenches in the screw theory)

- $A^*_{a_0} T_x\mathcal{M}$ is composed of maps of the form $\boldsymbol{a} \mapsto \boldsymbol{\Phi}(\overrightarrow{\boldsymbol{a}_0\boldsymbol{a}}) = ((p_{\boldsymbol{a}_0})^*\boldsymbol{\Phi})(\boldsymbol{a})$
  with the pullback of $\boldsymbol{\Phi}$ by $p_{\boldsymbol{a}_0} : AT_X\mathcal{M} \to T_X\mathcal{M} : \boldsymbol{a} \mapsto \overrightarrow{\boldsymbol{a}_0\boldsymbol{a}}$
  so we define $\boldsymbol{\mu}_{\boldsymbol{a}_0}(\vec{\boldsymbol{V}}, \boldsymbol{\Phi}) = \boldsymbol{\mu}(\vec{\boldsymbol{V}}, (p_{\boldsymbol{a}_0})^*\boldsymbol{\Phi})$
  thus $\exists$ an endomorphism $\boldsymbol{\Lambda}_{\boldsymbol{a}_0}$ of $T^*_{\boldsymbol{x}}\mathcal{M}$ such that $\boldsymbol{\mu}_{\boldsymbol{a}_0}(\vec{\boldsymbol{V}}, \boldsymbol{\Phi}) = (\boldsymbol{\Lambda}_{\boldsymbol{a}_0}(\boldsymbol{\Phi}))\vec{\boldsymbol{V}}$
  The endomorphism $\boldsymbol{L}_{\boldsymbol{a}_0} = (\boldsymbol{\Lambda}_{\boldsymbol{a}_0})^t$ of $T_X\mathcal{M}$ is represented by the matrix L
- $A^*_c T_{\boldsymbol{x}}\mathcal{M}$ is composed of the maps $\boldsymbol{a} \mapsto \boldsymbol{\Psi}(\boldsymbol{a}) = \chi\, 1$
  so we define $\boldsymbol{\mu}_c(\vec{\boldsymbol{V}}, \chi) = \boldsymbol{\mu}(\vec{\boldsymbol{V}}, \chi\, 1) = \chi\, \boldsymbol{\mu}(\vec{\boldsymbol{V}}, 1)$
  with $\boldsymbol{\mu}(\vec{\boldsymbol{V}}, 1) = \boldsymbol{\Pi}\, \vec{\boldsymbol{V}}$ where the linear form $\boldsymbol{\Pi}$ is represented by the row Π

Definition : for a Riemannian manifold, $\boldsymbol{\mu}$ is **Euclidean** if $\boldsymbol{L}_{\boldsymbol{a}_0}$ is skew-adjoint.

Dual pairing : $\boldsymbol{\mu}\,\boldsymbol{\theta} = \boldsymbol{\Pi}\,\vec{\boldsymbol{\Upsilon}}_{\boldsymbol{a}_0} - \frac{1}{2}\,Tr(\boldsymbol{L}_{\boldsymbol{a}_0}\boldsymbol{K}) = \Pi\,\Upsilon - \frac{1}{2}\,Tr(L\,K)$

Fact : A change of frame $f' \to f$ leads to $\mu = Ad^*(\mathrm{g})\,\mu'$, then

The systems μ of components of momentum tensors live in the dual of the Lie algebra of $\mathbb{SE}(d)$ and their transformation law is the co-adjoint representation

Examples of applications

rigid body

beam, arch
string or rod

Co-momenta and momenta : example 1

Dynamics of rigid bodies : motion given by $t \mapsto \mathrm{g}(t) = (x(t), R(t)) \in \mathbb{SE}(3)$
The Maurer-Cartan 1-forms ϑ_R and ϑ_L are good candidates
to be components of co-momenta because $\vartheta_L(\dot{\mathrm{g}}) = Ad(\mathrm{g})\vartheta_R(\dot{\mathrm{g}})$

spatial description :

$$\theta = \vartheta_R(\dot{\mathrm{g}}) = \dot{\mathrm{g}}\,\mathrm{g}^{-1} = (\dot{x} - \varpi \times x \mid j(\varpi))$$

$$\mu = \begin{pmatrix} p^T \\ j(l) \end{pmatrix}$$

$$\boldsymbol{\mu}\,\boldsymbol{\theta} = p\cdot(\dot{x} - \varpi \times x) + l\cdot\varpi = p\cdot\dot{x} + l_{pr}\cdot\varpi$$

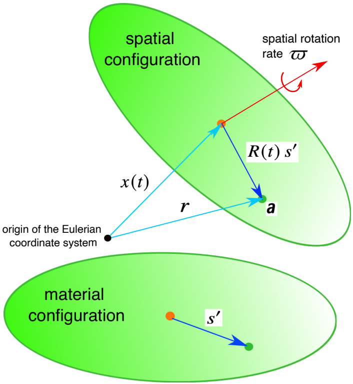
with the spin $l_{pr} = l - x \times p$

material description :

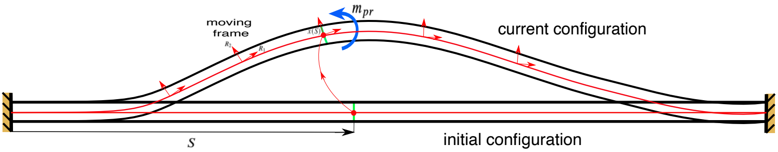
$$\theta' = Ad(\mathrm{g}^{-1})\,\theta = \mathrm{g}^{-1}\dot{\mathrm{g}} = \vartheta_L(\dot{\mathrm{g}})$$

$$\theta' = (R^T\dot{x} \mid R^T\varpi) = (\dot{x}' \mid \varpi')$$

$$\mu' = \begin{pmatrix} (R^T p)^T \\ j(R^T l_{pr}) \end{pmatrix} = \begin{pmatrix} p'^T \\ j(l'_{pr}) \end{pmatrix}$$



Co-momenta and momenta : example 2



Static of rods : deformation given by $S \mapsto (x(S), R(S)) \in \mathbb{SE}(3)$
$\frac{dR}{dS} = \mathring{R} = R\, j(\kappa')$ with $\kappa'^T = (\phi' \quad \kappa'_2 \quad \kappa'_3)$
twist bending curvatures

- **spatial description :**

  $\theta = (\mathring{x} - \kappa \times x \mid j(\kappa))$ $\mu = \begin{pmatrix} f^T \\ j(m) \end{pmatrix}$

- **material description :**

  $\theta' = (\mathring{x}' \mid j(\kappa'))$ $\mu' = \begin{pmatrix} f'^T \\ j(m'_{pr}) \end{pmatrix}$

Euler-Poincaré equation

- **Constitutive relation :** $\boldsymbol{\mu} = \frac{\partial \mathfrak{L}}{\partial \boldsymbol{\theta}} \qquad \left(\boldsymbol{\Pi} = \frac{\partial \mathfrak{L}}{\partial \boldsymbol{\Upsilon}_{a_0}}, \qquad \boldsymbol{L}_{a_0} = \frac{\partial \mathfrak{L}}{\partial \boldsymbol{K}} \right)$
- **Rigid body dynamics** (material description)
  $\delta \int_{t_0}^{t_1} \mathfrak{L}(\vartheta_L(\dot{\mathrm{g}}))\,\mathrm{d}t = \int_{t_0}^{t_1} \mu'\,\delta(\vartheta_L(\dot{\mathrm{g}}))\,\mathrm{d}t = 0$
  Maurer-Cartan eqn. $\mathrm{d}\vartheta_L(\delta \mathrm{g}, \dot{\mathrm{g}}) = -[\vartheta_L(\delta \mathrm{g}), \vartheta_L(\dot{\mathrm{g}})]$ next integrate by part
  exterior derivative d, Lie bracket $[,]$, infinitesimal coadjoint representation ad^*

  Euler-Poincaré equations $\boxed{\dot{\mu}' + ad^*(\vartheta_L(\dot{\mathrm{g}}))\mu' = 0}$

  Invariant Lagrangian $\mathfrak{L}(\theta') = \frac{1}{2}\, m \parallel \dot{x}' \parallel^2 + \frac{1}{2}\,\varpi' \cdot (\mathcal{J}'\,\varpi')$
  Momentum components $p' = \mathrm{grad}_{\dot{x}'}\mathfrak{T} = m\,\dot{x}', \qquad l'_{pr} = \mathrm{grad}_{\varpi'}\mathfrak{T} = \mathcal{J}'\,\varpi'$
  Equations of motion $\dot{p}' + \varpi' \times p' = 0, \qquad \dot{l}'_{pr} + \varpi' \times l'_{pr} = 0$ (Euler)

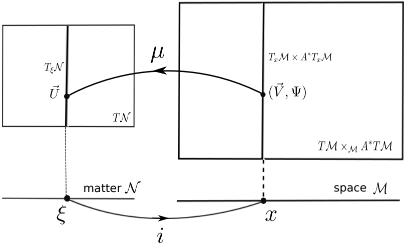
  Coming back to the spatial representation $\dot{p} = 0, \qquad \dot{l}_{pr} = 0$ (Poinsot)
- **Rod equilibrium**
  $\mathring{f}' + \kappa' \times f' = 0, \qquad \mathring{m}'_{pr} + \kappa' \times m'_{pr} + \mathring{x}' \times f' = 0$

  or in the spatial representation $\mathring{f} = 0, \qquad \mathring{m}_{pr} + \mathring{x} \times f = 0$

Generalization to continuous media of arbitrary dimension

The **matter manifold** $\mathcal{N}$ is the set of material particles $\boldsymbol{\xi}$

Its behaviour is described by an embedding i from $\mathcal{N}$ into the surrounding space $\mathcal{M}$



Bundle maps over the matter manifold $\mathcal{N}$:

Vector valued momentum $\boldsymbol{\mu} : i^*(T\mathcal{M} \times_{\mathcal{M}} A^*T\mathcal{M}) \rightarrow T\mathcal{N}$

Vector valued co-momentum $\boldsymbol{\theta} : i^*(T^*\mathcal{M} \times_{\mathcal{M}} AT\mathcal{M}) \rightarrow T^*\mathcal{N}$

Decomposition in an affine frame and a basis $({}_\gamma\vec{\boldsymbol{\eta}})$ of $T_\xi\mathcal{N}$

$$\boldsymbol{\mu} = {}_\gamma\vec{\boldsymbol{\eta}} \otimes \boldsymbol{e}^\beta \otimes \left({}^\gamma\Pi_\beta\, \boldsymbol{a}_0 + {}^\gamma L^\alpha_\beta\, \vec{\boldsymbol{e}}_\alpha \right), \quad \boldsymbol{\theta} = {}^\gamma\boldsymbol{\eta} \otimes \vec{\mathrm{e}}_\beta \otimes \left({}_\gamma\Upsilon^\beta 1 + {}_\gamma K^\beta_\alpha \boldsymbol{e}^\alpha \right)$$

Rod dynamics $\xi = ({}^0\xi, {}^1\xi) = (t, S)$

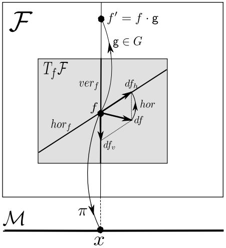
Equation of motion $\partial_\gamma\, {}^\gamma\mu' + ad^*(\vartheta_L(\partial_\gamma \mathrm{g}))\, {}^\gamma\mu' = 0$

Principal connection (Ehresmann)

G-principal bundle of affine frames $\mathcal{F}$

$(\mathrm{g}, f) \mapsto f' = f \cdot \mathrm{g}$ free right action
$(Z, f) \mapsto df = f \cdot Z$ infinitesimal action

$ver_f = Ker\,(T\pi)$ vertical space
hor_f horizontal subspace *such that*
$T_f\mathcal{F} = ver_f \oplus hor_f$



Connection 1**-form** $\Gamma : T\mathcal{F} \to \mathfrak{g}$ of kernel $hor_f = Ker\,\Gamma$ *such that*

- ♡ $\Gamma(Z \cdot f) = Z$
- ♠ Γ is Ad-**equivariant** : $R_{\mathrm{h}}\Gamma = Ad(\mathrm{h}^{-1})\,\Gamma$

Curvature 2**-form** $\mathcal{R} = \mathrm{D}\Gamma$ (Covariant exterior derivative)

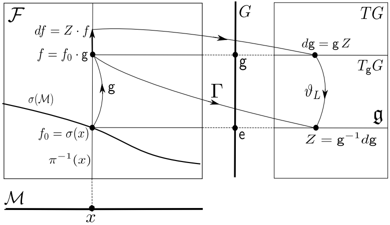
Flat connection $\mathcal{R} = 0$

The Maurer-Cartan 1-form as a principal connection

Construction of a principal connection from ϑ_L

By the choice of a section $\boldsymbol{x} \mapsto f_0 = \sigma(\boldsymbol{x})$, we identify

- $f = f_0 \cdot \mathrm{g}$ with g
- $df = f \cdot Z \in ver_f$ with $d\mathrm{g} = \mathrm{g}Z \in T_{\mathrm{g}}G$



Through this identification, $\Gamma = \vartheta_L$ is a principal connection because

♡ $\Gamma(f \cdot Z) = \mathrm{g}^{-1} d\mathrm{g} = \mathrm{g}^{-1}(\mathrm{g}\, Z) = Z$

♠ $R_{\mathrm{h}}\Gamma = (\mathrm{g}\,\mathrm{h})^{-1} d(\mathrm{g}\,\mathrm{h}) = \mathrm{h}^{-1}(\mathrm{g}^{-1} d\mathrm{g})\,\mathrm{h} = Ad(\mathrm{h}^{-1})\,\Gamma$

As the 1-form ϑ_L is left-invariant, Γ **is independent of the choice of** σ

Maurer-Cartan equation $\Rightarrow \mathcal{R} = 0$ and the connection is **flat**

Covariant derivative of a tensor

As for differential forms on $\mathcal{F}$,
we would like to do the same for sections of tensor bundles

Let us consider a tensor $\boldsymbol{u}$ of component system u

We start with the manifold $\mathcal{U} \ni u \mapsto u' = \mathrm{g} \cdot u$ (left action)

On $\mathcal{F} \times \mathcal{U}$, $(f, u) \cdot \mathrm{g} = (f \cdot \mathrm{g}, \mathrm{g}^{-1} \cdot u)$ (free right action)

We identifiy a tensor with an orbit in $\mathcal{F} \times \mathcal{U}$

$\boldsymbol{u} = orb((f, u)) = \{(f', u') \text{ s.t. } \exists \mathrm{g} \in G,\ (f', u') = (f, u) \cdot \mathrm{g}\}$

Orbit manifold $(\mathcal{F} \times \mathcal{U})/G = \mathcal{F} \times^G \mathcal{U}$, called the associated bundle

Covariant derivative

$\nabla_{\overrightarrow{d\boldsymbol{x}}}\ \boldsymbol{u} = orb((f, \nabla_{dx} u)) = orb((f,\ du - u \cdot (\Gamma(df))))$

Covariant derivative of a co-momentum

The present approach is closed to the mathematical framework proposed in

M. Castrillón López, T. Ratiu, S. Shkoller, Reduction in principal fiber bundles : covariant Euler-Poincaré equations, Proceedings of the American Mathematical Society, Vol. 128, No. 7, 2155-2164 (2000)

Derivative of a co-momentum and torsion 2-form

Bundle of **Euclidean frames** (s.t. the basis is orthonormal)
Manifold $\mathcal{F} \times^{\mathbb{SE}(3)} \mathfrak{se}(3)$ for the action $(f, \theta) \cdot \mathrm{g} = (f \cdot \mathrm{g}, Ad(\mathrm{g}^{-1})\,\theta)$

The orbit $\boldsymbol{\theta} = orb((f, \theta))$ is identified with the co-momentum tensor $\boldsymbol{\theta}$

For $\Gamma = \vartheta_L$, its covariant derivative is $\nabla_{\dot{x}}\,\theta = \dot{\theta} + ad(\vartheta_L(\dot{\mathrm{g}}))\,\theta$

Using Maurer-Cartan equation, its **torsion** 2-form vanishes
$T(\delta x, \dot{x}) = \nabla_{\delta x}\,\vartheta_L(\dot{\mathrm{g}}) - \nabla_{\dot{x}}\,\vartheta_L(\delta \mathrm{g}) - [\vartheta_L(\delta \mathrm{g}), \vartheta_L(\dot{\mathrm{g}})] = 0$

Covariant derivative of a momentum

Interpretation of the variation equation

Manifold $\mathcal{F} \times^{\mathbb{SE}(3)} (\mathfrak{se}(3))^*$ for the action $(f, \mu) \cdot \mathrm{g} = (f \cdot \mathrm{g}, Ad^*(\mathrm{g}^{-1})\, \mu)$

The orbit $\boldsymbol{\mu} = orb((f, \mu))$ is identified with the momentum tensor $\boldsymbol{\mu}$

For $\Gamma = \vartheta_L$, its covariant derivative is $\nabla_{\dot{x}}\, \mu' = \dot{\mu}' + ad^*(\vartheta_L(\dot{\mathrm{g}}))\, \mu'$

Interpretation of the **Euler-Poincaré equation** in terms of covariant derivative

for the natural evolution, the momentum tensor is parallel-transported

$$\boxed{\nabla_{\dot{x}}\, \mu' = 0}$$

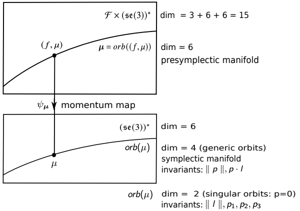
Euler-Poincaré equation in higher dimension

introducing the covariant divergence operator
$\nabla_\gamma {}^\gamma\mu' = \partial_\gamma {}^\gamma\mu' + ad^*(\vartheta_L(\partial_\gamma \mathrm{g}))\, {}^\gamma\mu' = 0$

A symplectic viewpoint

Theorem (Kirillov-Kostant-Souriau)

For the action $\mu = Ad^*(g)\,\mu'$
on the orbit $orb(\mu) \subset \mathfrak{g}^*$,
there exists a symplectic form ω_{KKS},
invariant by the action of G
and the identity map is a momentum map.



Theorem

- the restriction $\psi_{\boldsymbol{\mu}}$ of $(f, \mu) \mapsto \mu$ to the orbit $\boldsymbol{\mu}$ is a submersion
- on $orb((f, \mu))$, the pull-back by $\psi_{\boldsymbol{\mu}}$ of the symplectic form ω_{KKS} is a presymplectic form ω and it is G-invariant
- $\psi_{\boldsymbol{\mu}}$ is a momentum map and $\psi_{\boldsymbol{\mu}} \circ L_a = Ad^*(a)\,\psi_{\boldsymbol{\mu}}$
- The covariant equations of motion are
  $d(f, \mu) \in Ker(\omega) \quad \Leftrightarrow \quad \boxed{\nabla_{dx}\mu = d\,\mu + ad^*(\Gamma)\mu = 0}$ (Euler-Poincaré Eqn.)

G. de Saxcé, C. Vallée, *Galilean Mechanics and Thermodynamics of Continua*, Wiley-ISTE (2016)

Conclusions

- We revisited the **screw theory** with the affine tensor calculus, the counterparts of the Euclidean co-momentum and momentum tensors being in screw theory respectively the twist and the wrench

- We interpreted the **Euler-Poincaré equation** in terms of parallel-transport of the momentum tensor thanks to the concept of Ehresmann connection on the principal bundle of Euclidean affine frames

- We showed that the **left Maurer-Cartan** 1-**form** defines a connection of which the curvature 2-form is null

- In the future, we hope **to extend the present work** to problems of mechanics with a non-vanishing curvature

Merci pour votre attention !