Kinematic locomotion of snakes: from discrete to continuous models

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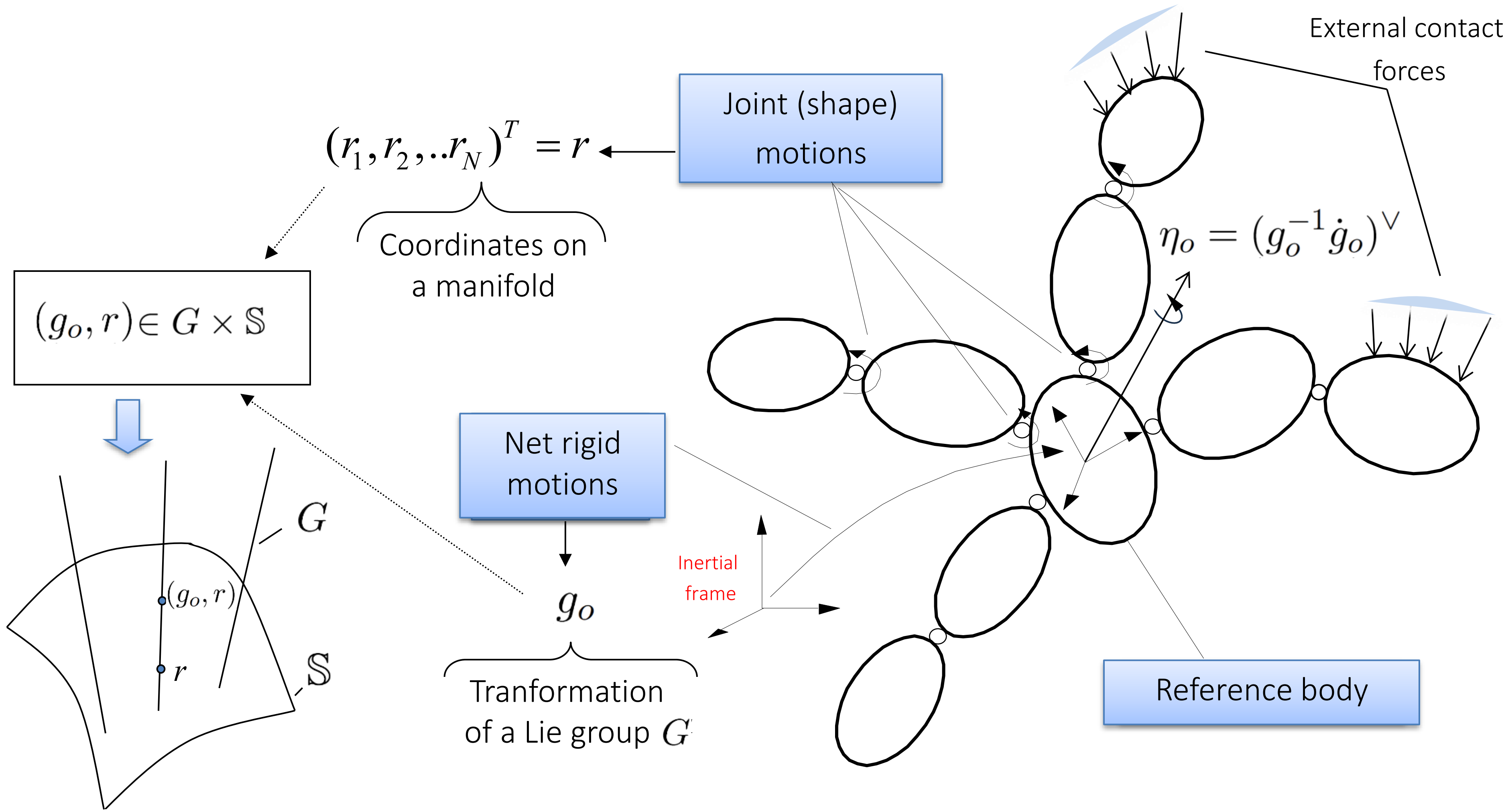
Over the course of evolution, snakes have developed a wide range of locomotion modes, adapted to a wide range of environments...

Side-winding
(dynamic locomotion)

Lateral undulation
(kinematic locomotion)
« follower the leader »

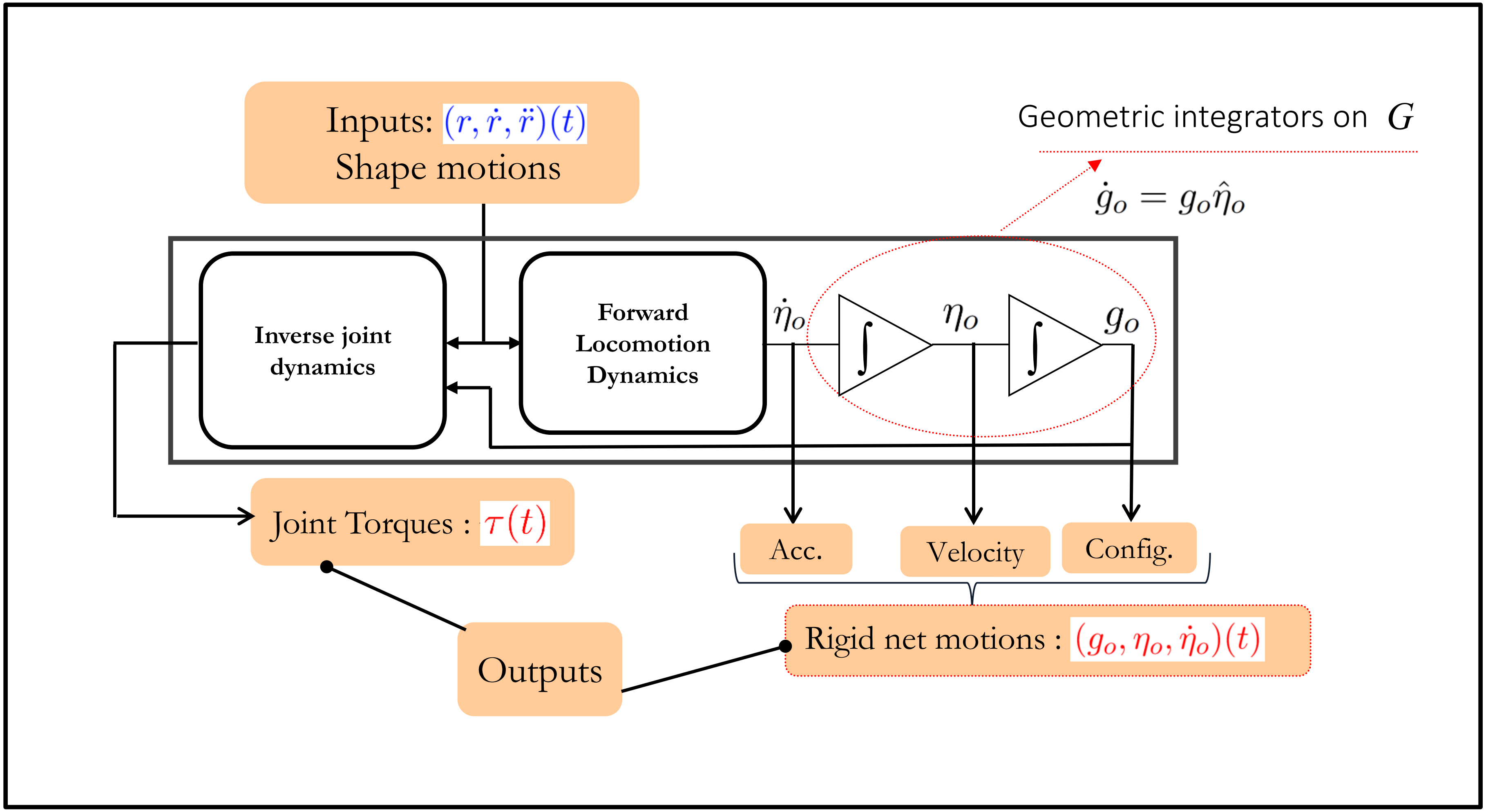
To investigate snake locomotion, we use the model of MMS, i.e. a set of rigid bodies connected by joints and submitted to shape variations and overall rigid net motions…



Configuration Space =
Principal Fiber Bundle
with a left *G*-action

From this definition…

We consider the following (inverse) dynamic problem…



In the general case, a dynamic model is required...

To get it, take the Lagrangian:

$$l(g_o, r, \eta_o, \dot{r}) = \frac{1}{2}(\eta_o^T, \dot{r}^T) \begin{pmatrix} \mathcal{M}_o & M \\ M^T & m \end{pmatrix} \begin{pmatrix} \eta_o \\ \dot{r} \end{pmatrix} - U(g_o, r)$$

Dynamics on G we need Poincaré eq. [Poincaré, 1901]:

$$\frac{d}{dt}\left(\frac{\partial l}{\partial \eta_o}\right) - ad^T_{\eta_o}\left(\frac{\partial l}{\partial \eta_o}\right) = F_{\text{ext}}$$

Locomotion dynamics :

Depends on $(\eta_o, r, \dot{r}, \ddot{r})$

$$\boxed{\begin{pmatrix} \dot{\eta}_o \\ \dot{g}_o \end{pmatrix} = \begin{pmatrix} \mathcal{M}_o^{-1}(F_{\text{inert}} + F_{\text{ext}}) \\ g_o\hat{\eta}_o \end{pmatrix}}$$

External forces...

reconstruction eq. of g_o from η_o

Ex: Lighthill theory of swimming

F. Boyer, M. Porez, Leroyer A., Poincaré–Cosserat Equations for the Lighthill Three-dimensional Large Amplitude Elongated Body Theory: Application to Robotics, JNLS, 2009.

The dynamic model can turn into kinematics, when we have a linear relation :

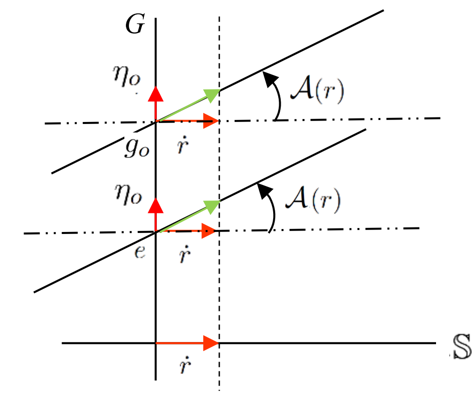
Net vel. Shape vel.

$$\eta_o + \mathcal{A}(g_o, r)\dot{r} = 0$$

Defines a connexion on $G \times \mathbb{S}$ [Ehresmann,1950]

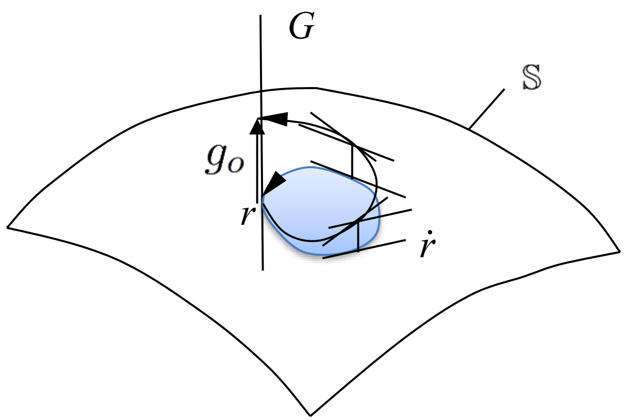
$$\mathcal{A}(\cancel{g_o}, r)$$

Equivariance $\Leftrightarrow$ Left invariance



$$\dot{g}_o = -g_o(\mathcal{A}(r)\dot{r})^\wedge$$

Enjoys nice geometric properties usefull for control…



E.g.: For a cyclic change of shape (gait) the net displacement is the integral of the curvature across the enclosed area:

G abelian

$$d\mathcal{A} + \cancel{\mathcal{A} \wedge \mathcal{A}}$$

… Non-Abelian Stokes theorem.

In bio-inspired robotics there are two well known cases where locomotion is modelled by a connexion: Conservation moment law, NH platforms [Marsden, 78]

First case: conservation law (ex. falling cat…)

$$\sigma = \sigma_{ref} + \sigma_{sh} = R(J\Omega + \alpha\dot{r}) = 0$$

$$\Downarrow$$

$$\Omega = -(J^{-1}\alpha)(r)\dot{r}$$

$$\Downarrow$$

$$\mathcal{A}(r) = J^{-1}(r)\alpha(r)$$

« Mechanical connexion »

[Marsden 78, Montgomery 93]

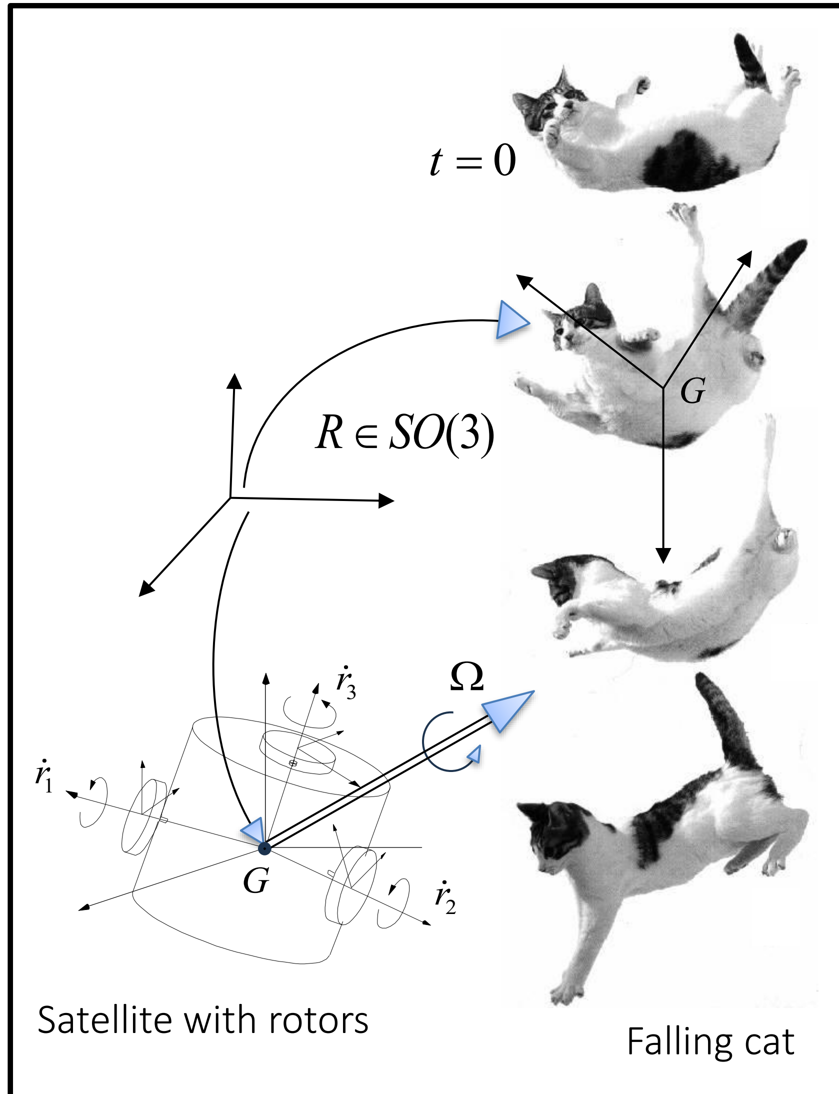
Remarks:

- Applying the same idea to translations:

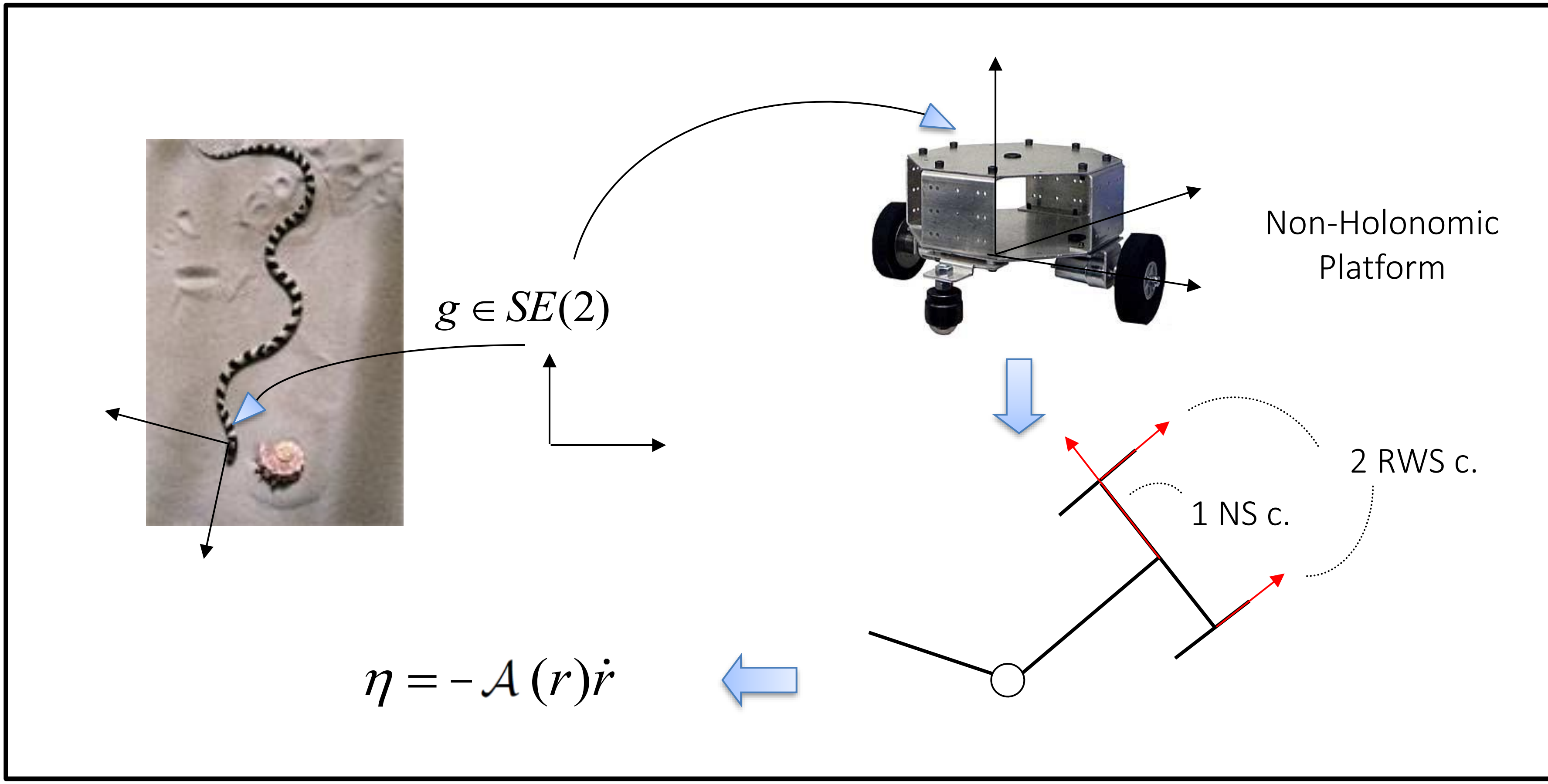
$$\mathcal{A}(r) = 0$$

- When immersed in an ideal fluid:

$$\mathcal{A}(r) \neq 0$$



Second case: Kinematic non-holonomic constraints (wheeled platforms, snakes…)



$\mathcal{A}(r)$: « Principal kinematic connexion » [Kelly & Murray, 95]

I Basic formulation

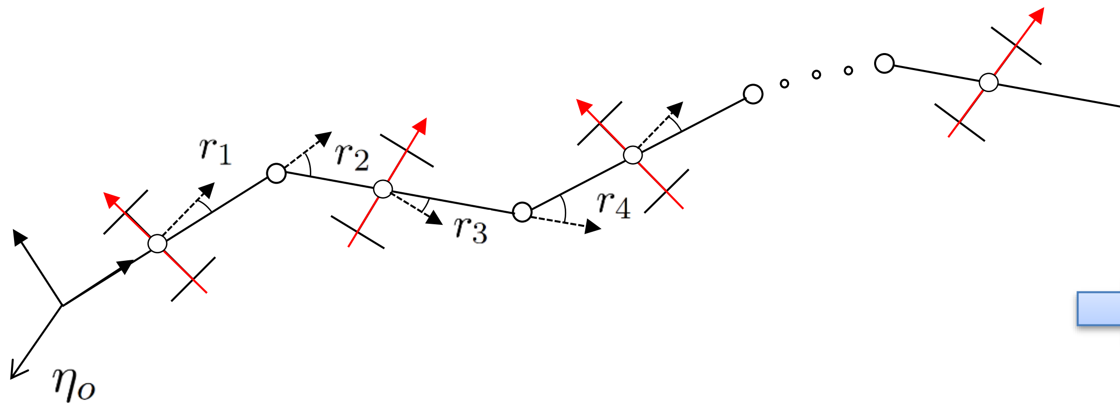
II Discrete model of snake lateral undulation

III Continuous model of snake lateral undulation

Wheeled snake robots

Hirose’s ACM

The ACM is a serial assembly of passive wheeled axels connected by active joints...



"ACMechanism"

Reproduces the principle of Snake creeping:

Body undulations

⇩

Lateral reaction forces

⇩

Axial resultant

Gathering all the constraint equations on $\overset{SE(2)}{\overset{||}{G}} \times \mathbb{S}$:

⇨ $A(r)\eta_o + B(r)\dot{r} = 0 \ (\star)$ ⇨ $A(r) =$ Matrix of locked constraints

⇨ 2 cases depending upon $\mathrm{Rank}(A)$ w.r.t. $\dim(G) = 3$ ...

- Case 1 (under constrained): $\mathrm{Rank}(A) < 3$

⇨ The system has not enough constraints to be governed by kinematics only!

Generalized Inversion of $(\star)$ ⇨ $$\eta = \underbrace{H(r)\,\eta_r}_{\in \ker(A)} + \underbrace{J(r)}_{= -A^{(-1)}B}\dot{r}$$

Where: η_r is a “reduced velocity” kinematically undetermined!

⇨ To determine η_r we need a dynamic model

⇨ Projection of the dynamics of η in $\ker(A)$

⇨

⇨ This case also occurs in some singular configurations…

- Case 2 (fully constrained): $\mathrm{Rank}(A) = 3$

➡ The system has enough constraints to be governed by kinematics only!

Ex. ACM: 3 first axels ← $\begin{pmatrix} \bar{A}(r) \\ \tilde{A}(r) \end{pmatrix} \eta_o + \begin{pmatrix} \bar{B}(r) \\ \tilde{B}(r) \end{pmatrix} \dot{r} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ → 1 → 2

1 ➡ $\eta_o = \bar{A}^{-1}(r)\bar{B}(r)\dot{r} = -\mathcal{A}(r)\dot{r}$ ➡ Kinematic connexion [Ostrowsky,1999]

The forward locomotion dynamics turn into a kinematic model (of contacts)

2 ➡ Compatibility eq.: give the motion of other joints in order to preserve mobility

➡ Each axle follows the track of its predecessor (follower-leader)

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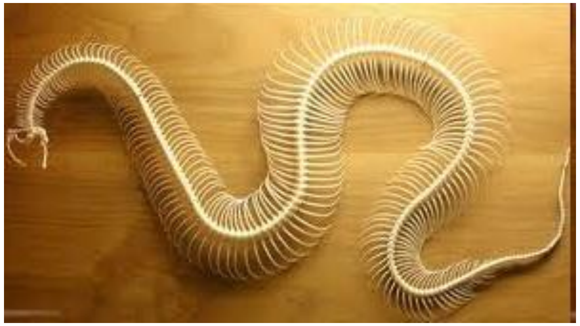
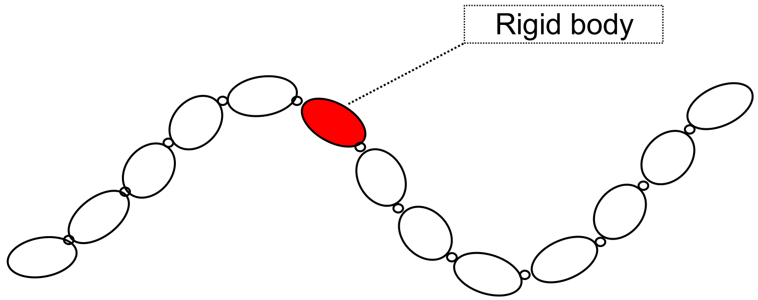
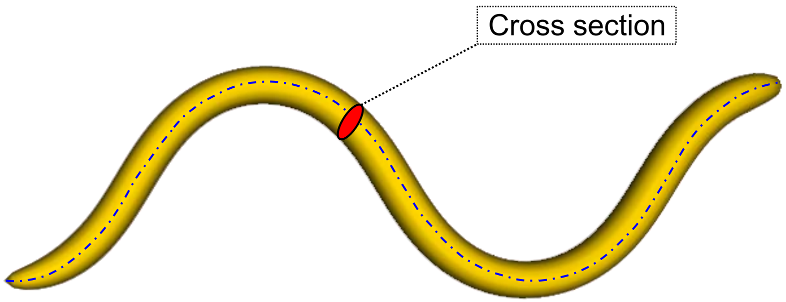


When the number of internal DoFs increases …





Discrete Mobile MS … ⇨ Continuous MMS = Cosserat beam

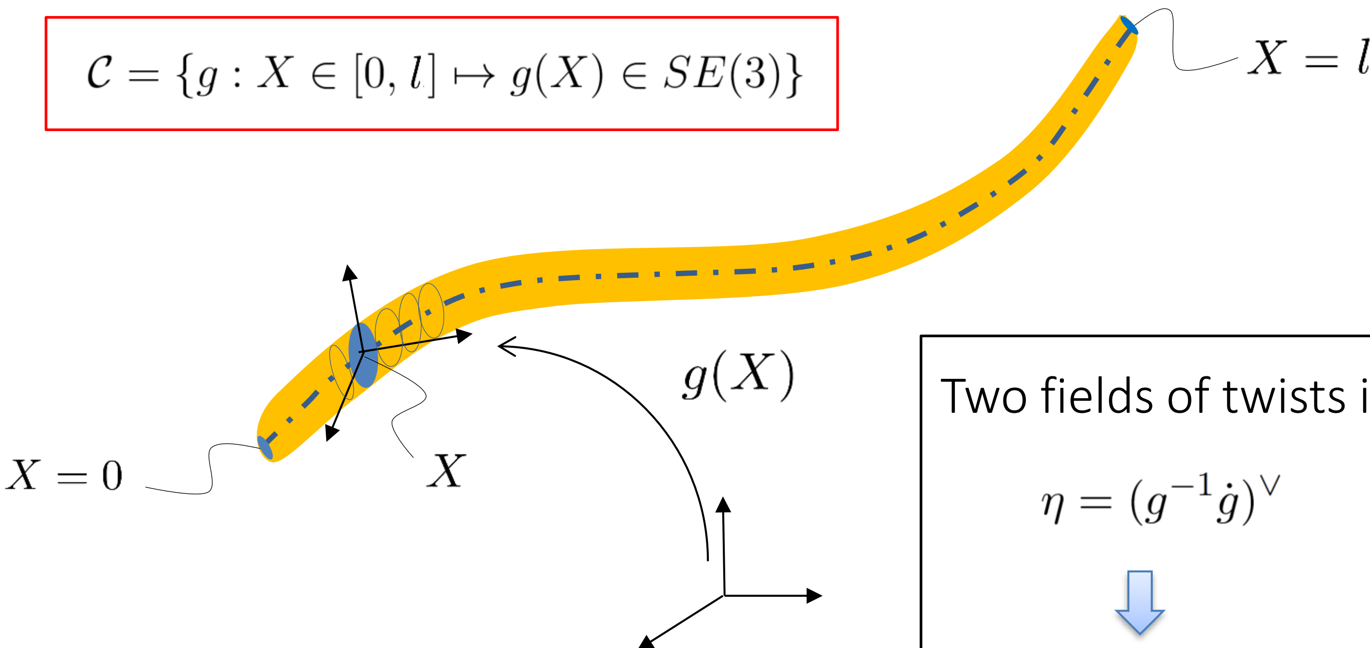
- Rigid bodies ⇨ Cross sections
- Vertebral column (backbone) ⇨ Beam centroidal line
- Discret body indices j ⇨ Beam sections label X
- Joint coordinates r_j ⇨ Strain field $\xi(X)$
- Interbody wrenches F_j ⇨ Stress field $\Lambda(X)$

Model of Cosserat rods:

A Cosserat rod is a continuous stacking of rigid cross-sections…

Configuration space of a Cosserat beam:

$$\boxed{\mathcal{C} = \{g : X \in [0, l] \mapsto g(X) \in SE(3)\}}$$



Two fields of twists in $se(3) \cong \mathbb{R}^6$:

$$\eta = (g^{-1}\dot{g})^{\vee} \qquad \xi = (g^{-1}g')^{\vee}$$

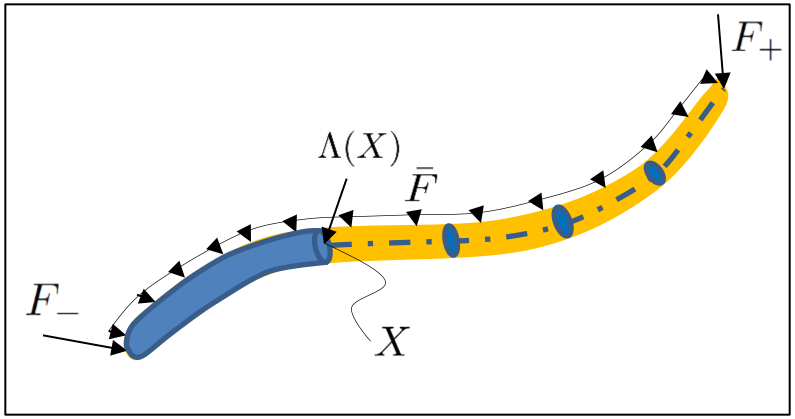
Field of time-twist

Field of space-twist (strains $\epsilon = \xi - \xi_o$).

For dynamic locomotion of elongated animals:

Dynamic balance + space-time-reconstruction eq.

- $$\begin{cases} \mathcal{M}\dot{\eta} - ad_{\eta}^{T}\mathcal{M}\eta = \Lambda' - ad_{\xi}^{T}\Lambda + \bar{F} \\ \dot{g} = g\hat{\eta} \ , \quad g' = g\hat{\xi} \end{cases}$$



+ I.C. and B.C:

- $$\Lambda(0) = -F_- \ , \ \Lambda(l) = F_+.$$

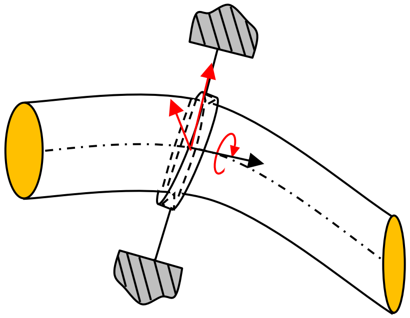
+ Active constitutive law (Hookean):

- $$\underbrace{\Lambda = \Lambda_a(t)}_{\text{Actuation stress-field}} + \mathcal{H}\epsilon \quad \text{with ,} \quad \mathcal{H} = \mathrm{diag}(GI_1, EI_2, EI_3, EA, GA, GA)$$

Actuation stress-field

Model of contacts for lateral undulation…

Contacts modelled by ideal (rigid, friction-less…) annular supports:



- Define non-holonomic kinematic constraints
- Each scalar constraint introduces one reaction unknown
- Reaction unknowns $(F_-, F_+, \bar{F})$ define F_{ext}

When the total number of independent constraints $\geq \dim(G)$

- Locomotion entirely ruled by kinematics…

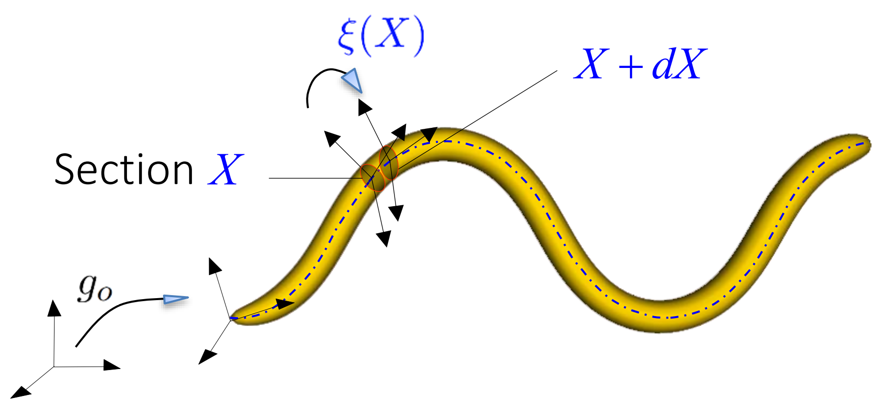
For the study of kinematic locomotion…

Second def. of the configuration space: as a Principal fiber bundle

$$\mathcal{C} = SE(3) \times \mathbb{S} \quad , \quad \mathbb{S} := \{\xi(.) : X \in [0, l] \mapsto \xi(X) \in se(3)\}$$

Space reconstruction of g :

$$g' = g\hat{\xi}$$
$$g(0) = g_o$$



Shape – control : $\xi = (g^{-1}g')^{\vee} = \xi_d(t) = \begin{pmatrix} K_d(t) \\ \Gamma_d(t) \end{pmatrix}$ ⇨ Λ = Lagrange Xtipliers

For modelling snake lateral undulation in 2/3D:

$$\underbrace{K_d}_{} = (\underbrace{K_{dX}}_{\text{Twist}}, \underbrace{K_{dY}, K_{dZ}}_{\text{Curvature}})^T \quad , \quad \Gamma_d = (\underbrace{1}_{\text{No Stretching}}, \underbrace{0, 0}_{\text{No shearing}})^T$$

Resolution of motions…

- As in the ACM, we consider kinematic constraints distributed along the snake:

- Force $V_\perp = 0$ in the continuous kinematics : $\eta' = -ad_{\xi_d}\eta + \dot{\xi}_d$

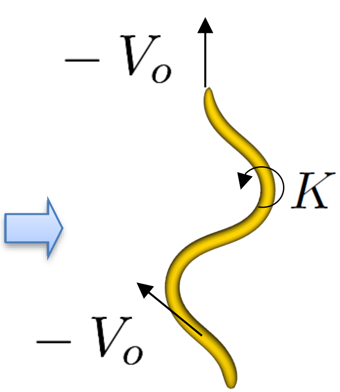
Net motions are governed by a connection:

$$\boxed{\eta_o = -\mathcal{A}(K_o, K_o')\dot{K}_o \ , \ K(0) \triangleq K_o}$$

- Compatibility equations ⇒ P.D.Es…

E.g. for snakes:

$$\frac{\partial K}{\partial t} - V_o \frac{\partial K}{\partial X} = 0$$



- Advection of curvature : Follower-the-leader motion :

$$K_d(X,t) = f\left(X + \int_0^t V_o(\tau)d\tau\right)$$

Given a compatible shape time- law: $\xi_d(X,t) = (K_d(X,t), 1, 0, 0)^T$

One can calculate the head kinematics with the connection:

$$\eta_o = -\mathcal{A}(K_{o,d}, K'_{o,d})\dot{K}_{o,d} \Rightarrow \int_0^t \cdot \, , \; \frac{d\,\cdot}{dt} \Rightarrow (g_o, \eta_o, \dot{\eta}_o)(t)$$

And reconstruct all the fields along the body thanks to ODEs:

$$\Rightarrow \boxed{\begin{aligned} g' &= g\hat{\xi}_d \;\;, g(0) = g_o(t). \\ \eta' &= -ad_{\xi_d}\eta + \dot{\xi}_d \;\;, \eta(0) = \eta_o(t). \\ \dot{\eta}' &= -ad_{\xi_d}\dot{\eta} - ad_{\dot{\xi}_d}\eta + \ddot{\xi}_d \;\;, \dot{\eta}(0) = \dot{\eta}_o(t). \end{aligned}}$$

$$\Rightarrow \int_0^X \cdot \Rightarrow (g, \eta, \dot{\eta})(t) \quad \ldots$$

Torque computation…

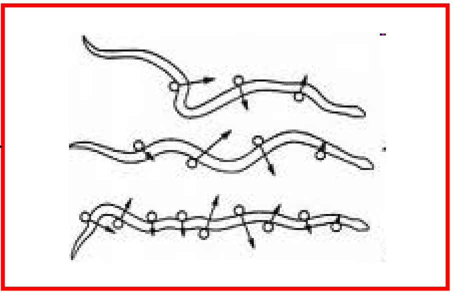
$F_{\rm ext}$ can be computed in:

$$F_{ext} = \mathcal{M}\dot{\eta}_o + F_{\rm inert} = -F_- + Ad^T_{k(l)} F_+ + \int_0^l Ad_k^T \bar{F}\, dX$$

That one solve $/(F_-, F_+, \bar{F})$ with some assumption on distribution of forces

since the solution is not unique (in the over-constrained case)…

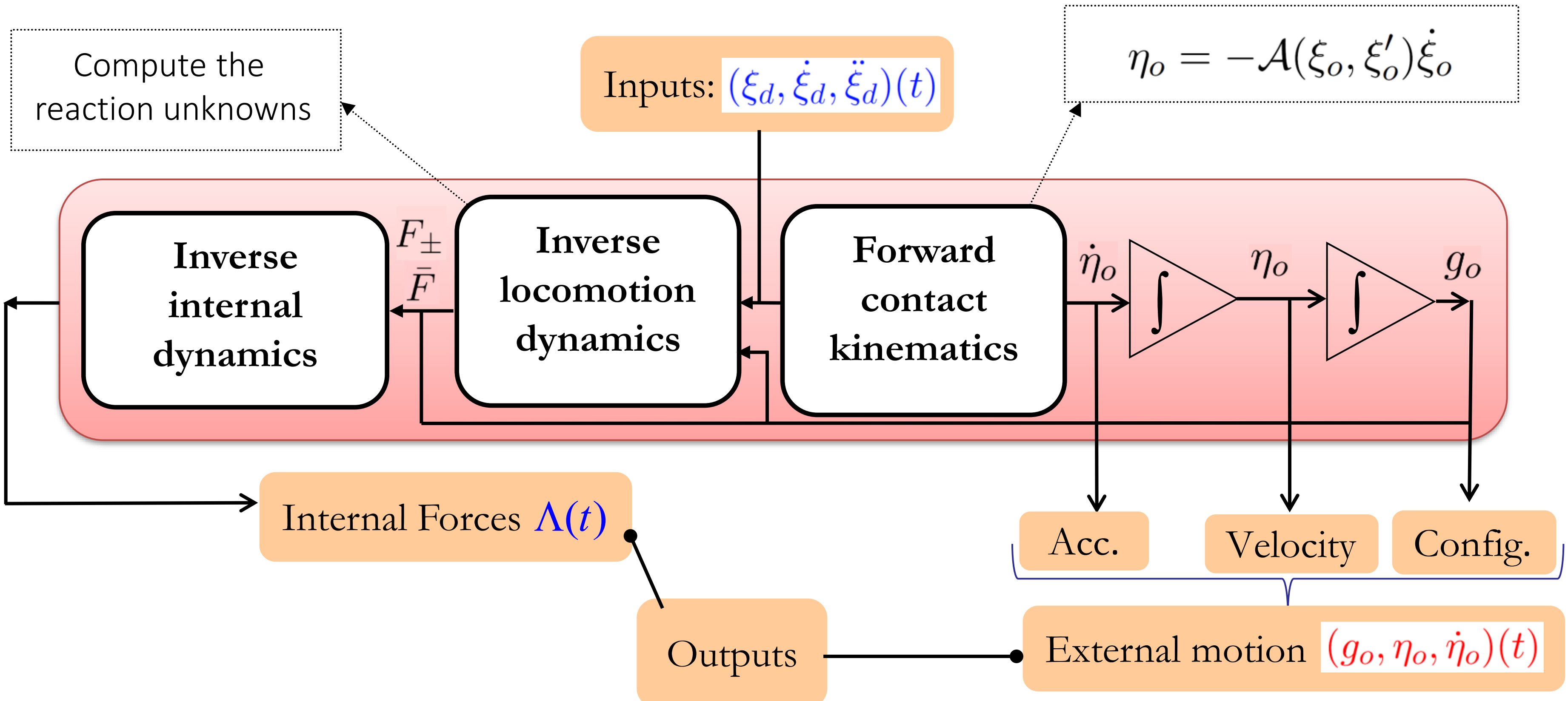
Once $(F_-, F_+, \bar{F})$ fixed…



Computation of $\Lambda(t)$ from the rod p.d.e.s:

$$\boxed{X\text{-ODE: } \Lambda' = -ad_{\xi_d(t)}\Lambda + (\mathcal{M}\dot{\eta} - ad_\eta^T \mathcal{M}\eta + \bar{F})(t) \quad \text{BC: } \Lambda(0) = -F_-(t)}$$

Macro-continuous algorithm for kinematic locomotion…



The forward locomotion model is a kinematic model

F.Boyer, S.Ali, and M. Porez, Macrocontinuous Dynamics for Hyperredundant Robots: Application to Kinematic Locomotion Bioinspired by Elongated Body Animals, IEEE TRO, 2012

Thank you for your attention…