Covariance et thermodynamique pour les milieux continus

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Plan

Principle of covariance

Einstein has postulated that *the laws governing our universe should be in their general formulation covariant under arbitrary substitutions of space-time variables.*
In this document, we wish to explore the implications of such a principle when applied to material continua with regards to thermodynamics.

Context

- Object of study : compact four-volume $\Omega \subset \mathcal{M}$
  $\mathcal{M}$: four-dimensional pseudo-Riemannian manifold endowed with a Lorentzian metric $\boldsymbol{g}$ of signature $(+,-,-,-)$
- A point $P \in \mathcal{M}$: event identified by a set of four coordinates $(x^\mu)_{\mu\in\{0,1,2,3\}}$ denoted x
- $T_P\mathcal{M}$ the tangent space and $T_P^*\mathcal{M}$ the cotangent space at P.
- Define tensor fields.
- Lie derivative along a vector field X noted $\mathscr{L}_X$
- Covariant derivative ∇ (metric connection : $\nabla \boldsymbol{g} = 0$)

General covariance

Consider an energy functional of the form

$$\mathscr{E}_\Omega(\phi_\mathcal{I}) = \int_\Omega A(\phi_\mathcal{I}) \sqrt{g}\; dx^4 \tag{1}$$

$\sqrt{g}\; dx^4$ is the volume form associated with the metric $\boldsymbol{g}$, with $g = -\det(\boldsymbol{g})$ where $\det(\boldsymbol{g})$ is the determinant of $\boldsymbol{g}$.

- The functional $\mathscr{E}_\Omega$ is general covariant if it is invariant under the action of G, group of local diffeomorphisms, meaning that for all $\varphi_\epsilon \in G$,
$$\mathscr{E}_\Omega\left(\varphi_\epsilon^*(\phi_\mathcal{I})\right) = \mathscr{E}_\Omega(\phi_\mathcal{I}) \tag{2}$$
- G : group of local diffeomorphisms φ_ϵ tangent to the identity
- $\varphi_\epsilon : x \mapsto \tilde{x} = \varphi_\epsilon(x)$ defined by $\varphi_0(x) = x$ and $\varphi_\epsilon(x) = x + \epsilon X(x) + o(\epsilon)$ with X a vector field.
- $\varphi_\epsilon^*(\phi)$ the pull-back of ϕ by the diffeomorphism φ_ϵ

General covariance : infinitesimal version

Consider an energy functional of the form

$$\mathscr{E}_\Omega(\phi_\mathcal{I}) = \int_\Omega A(\phi_\mathcal{I})\sqrt{g}\, dx^4$$

$\mathscr{E}_\Omega$ is general covariant if there exists a vector field Y such that

$$\left(\frac{\partial A}{\partial \phi_\mathcal{I}}\mathscr{L}_X\phi_\mathcal{I} + \frac{A}{2}\boldsymbol{g}^{-1}\mathscr{L}_X\boldsymbol{g}\right)\sqrt{g} + \partial_\mu Y = 0 \qquad \forall X \in T\Omega \tag{3}$$

Example of application

Define $\mathscr{E}_\Omega^{R\phi}$ as

$$\mathscr{E}_\Omega^{R\phi}(\boldsymbol{g},\phi) = \int_\Omega (\frac{k}{2}\boldsymbol{R} : \boldsymbol{g}^{-1} + A^M(\boldsymbol{g},\phi))\sqrt{g}\; dx^4 \qquad (4)$$

k a constant scalar, $\boldsymbol{R}$ the Ricci curvature tensor.
$\mathscr{E}_\Omega^{R\phi}$ is general covariant if, with $\boldsymbol{G} = \boldsymbol{R} - \frac{R}{2}\boldsymbol{g}$ Einstein's tensor :

$$\left(\frac{\partial A^M}{\partial \boldsymbol{g}} - \frac{k}{2}\boldsymbol{G} + \frac{A^M}{2}\boldsymbol{g}^{-1}\right)\mathscr{L}_X\boldsymbol{g} + \frac{\partial A^M}{\partial \phi}\mathscr{L}_X\phi = 0 \quad \forall X \in T\Omega. \qquad (5)$$

that is if

- $\frac{\partial A^M}{\partial \phi}\mathscr{L}_X\phi = 0 \quad \forall X \in T\Omega$
- $k\boldsymbol{G} = \boldsymbol{T}$ hence $Div\,\boldsymbol{T} = 0$
  - Energy-momentum tensor $\boldsymbol{T} = 2\frac{\partial A^M}{\partial \boldsymbol{g}} + A^M\boldsymbol{g}^{-1}$

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Entities

Given a vector field X and its induced flow $\varphi(l, x)$:

- X is spacelike if $X.X < 0$
- X is time-like if $X.X > 0$
  - define $\beta > 0$ the norm of this time-like vector field (temperature)
  - Define u its velocity, u being the unitary vector such that $X = \beta u$
- Define $\boldsymbol{D}_X$ the rate of deformation induced by the time-like vector field X :

$$\boldsymbol{D}_X = \frac{1}{2}\mathscr{L}_X(\boldsymbol{g}) \tag{6}$$

Components of $\boldsymbol{D}_X$ with $X = \beta u$ time-like :

$$2D_{X_{\mu\nu}} = \beta\mathscr{L}_u(g_{\mu\nu}) + u_\nu\partial_\mu\beta + u_\mu\partial_\nu\beta. \tag{7}$$

Orthogonal decomposition

Two projectors are defined :

- The projection on n evaluates the contribution of a tensor in the direction of the unitary vector n.
- The projection on $\underline{\underline{\boldsymbol{g}}} = \boldsymbol{g} - n \otimes n$ evaluates the contribution of a tensor in the space orthogonal to n

Orthogonal decomposition of $\boldsymbol{T}$, second rank tensor :

$$\boldsymbol{T} = \mathcal{U}_T n \otimes n + \underline{\boldsymbol{T}} \otimes n + \underline{\boldsymbol{T}} \otimes n + \underline{\underline{\boldsymbol{T}}}. \quad (8)$$

with

- the scalar field $\mathcal{U}_T = T_{\alpha\beta} n^\alpha n^\beta$
- the vector field $\underline{\boldsymbol{T}}$: $\underline{T}^\mu = T_{\alpha\beta} \underline{\underline{g}}^{\mu\alpha} n^\beta$
- the second rank tensor field $\underline{\underline{\boldsymbol{T}}}$: $\underline{\underline{T}}^{\mu\nu} = T_{\alpha\beta} \underline{\underline{g}}^{\alpha\mu} \underline{\underline{g}}^{\beta\nu}$.

Orthogonal decomposition

Useful examples of application :

- the projection twice on n of the Lie derivative along n of the metric tensor vanishes :

$$n^{\mu} n^{\nu} \mathscr{L}_n(g_{\mu\nu}) = 0 \tag{9}$$

because $n \cdot n = 1$.

- Decomposition of a contraction :
consider two tensors $\boldsymbol{A}$ and $\boldsymbol{B}$ of type two, then :

$$A^{\mu\nu} B_{\mu\nu} = \mathcal{U}_A \mathcal{U}_B + \underline{A}^{\mu} \underline{B}_{\mu} + \underline{A}^{\nu} \underline{B}_{\nu} + \underline{\underline{T}}^{\mu\nu} \underline{\underline{B}}_{\mu\nu}. \tag{10}$$

Proper coordinate system

Given a time-like vector field X (that is never null on Ω), define (Straightening theorem)

- the proper coordinate system noted $\hat{x}$ such that :

$$\hat{X}^{\mu}(1,0,0,0) \ \forall \hat{x} \in \Omega. \tag{11}$$

- the proper basis noted $\hat{\boldsymbol{e}}_{\boldsymbol{\mu}}$
- the transformation $\varphi_X : \mathbb{R}^4 \to \mathbb{R}^4,\ \hat{x} \mapsto x = \varphi_X(\hat{x})$
- the proper time τ such that $\hat{x}_0 = c\tau$ where c is the constant speed of light in vacuum.

Proper tensor fields

A tensor field α_X is said to be *proper to a time-like vector field* X if it is such that

$$\mathscr{L}_X(\alpha_X) = 0. \tag{12}$$

for all X time-like. In the proper coordinate system :

$$\mathscr{L}_X(\hat{\alpha}_X) = \frac{\partial \hat{\alpha}_X}{\partial c\tau} \tag{13}$$

because $\hat{X}^\mu(1, 0, 0, 0)$.
Thus if $\hat{\alpha}_X$ is independent of τ, then

$$\mathscr{L}_X(\alpha_X) = 0 \tag{14}$$

for all X time-like.

Examples of proper tensor fields

- X
- $\boldsymbol{b}$ the deformation tensor induced by X such that $\hat{b}_{\mu\nu} = \eta_{\mu\nu}$ where $\eta_{\mu\nu}$ are the components of Minkowski's metric and :

$$\boldsymbol{b} = \eta_{\kappa\lambda} \hat{\boldsymbol{e}}^{\boldsymbol{\kappa}} \otimes \hat{\boldsymbol{e}}^{\boldsymbol{\lambda}} \quad \text{of components} \quad b_{\mu\nu} = \frac{\partial \hat{x}^{\kappa}}{\partial x^{\mu}} \frac{\partial \hat{x}^{\lambda}}{\partial x^{\nu}} \eta_{\kappa\lambda} \tag{15}$$

  Space-time counter part of $B^{-1} = F^{-T} F^{-1}$.
- the rest mass per unit volume noted $\omega_{\rho_c} = \rho_c \sqrt{g}$, a scalar density (volume form), with ρ_c corresponding to the mass density at rest, i.e. in the proper coordinate system. Then, mass conservation implies

$$\mathscr{L}_X(\omega_{\rho_c}) = Div(\omega_{\rho_c} X) = 0. \tag{16}$$

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Definition

The functionnal

$$\mathscr{E}_\Omega(\phi_\mathcal{I}) = \int_\Omega A\left(x, \phi_\mathcal{I}\right)\sqrt{g}\; dx^4 \tag{17}$$

is said to be time-covariant if

$$\frac{\partial A}{\partial \phi_\mathcal{I}}\mathscr{L}_X \phi_\mathcal{I} + \frac{A}{2}\boldsymbol{g}^{-1}\mathscr{L}_X\boldsymbol{g} = 0 \qquad \forall X = \beta u \in T\Omega \text{ time-like} \tag{18}$$

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Example for a time-covariant functional

Define the energy functional

$$\mathscr{E}_\Omega(\boldsymbol{g}, \beta, \alpha_X) = \int_\Omega \left(\frac{k}{2} R + A^M(\boldsymbol{g}, \beta, \alpha_X) \right) \sqrt{g}\; dx^4 \tag{19}$$

with α_X, tensor field proper to the time-like vector field X.
Then $\mathscr{E}_\Omega$ is time-covariant if $\forall(\beta u) \in T\Omega$:

$$\left(\frac{\partial A^M}{\partial g} - \frac{k}{2} \boldsymbol{G} + \frac{A^M}{2} g^{-1} \right) \mathscr{L}_{\beta u}(\boldsymbol{g}) + \frac{\partial A^M}{\partial \beta} \mathscr{L}_{\beta u}(\beta) = 0 \tag{20}$$

Remember

$$\mathscr{L}_{\beta u}(g)_{\mu\nu} = \beta \mathscr{L}_u(g)_{\mu\nu} + u_\nu \partial_\mu \beta + u_\mu \partial_\nu \beta \tag{21}$$

$$\mathscr{L}_{\beta u}(\beta) = \beta u^\nu \partial_\nu \beta \tag{22}$$

$$u^\mu u^\nu \mathscr{L}_u(g_{\mu\nu}) = 0 \tag{23}$$

$$A^{\mu\nu} B_{\mu\nu} = \mathcal{U}_A \mathcal{U}_B + \underline{A}^\mu \underline{B}_\mu + \underline{A}^\nu \underline{B}_\nu + \underline{\underline{T}}^{\mu\nu} \underline{\underline{B}}_{\mu\nu}. \tag{24}$$

Example for a time-covariant functional

$\mathscr{E}_\Omega$ is time-covariant if $\forall(\beta u) \in T\Omega$:

$$\left(\frac{\partial A^M}{\partial g} - \frac{k}{2}\boldsymbol{G} + \frac{A^M}{2}g^{-1}\right)\mathscr{L}_{\beta u}(\boldsymbol{g}) + \frac{\partial A^M}{\partial \beta}\mathscr{L}_{\beta u}(\beta) = 0 \qquad (25)$$

Define the function $\tilde{s}$ (entropy counterpart) as $\tilde{s} = \frac{\partial A^M}{\partial \beta}$,
Define the energy-momentum tensor field $\boldsymbol{T}$ as

$$\boldsymbol{T} = 2\frac{\partial A^M}{\partial \boldsymbol{g}} + A^M \boldsymbol{g}^{-1} + \beta \tilde{s} u \otimes u. \qquad (26)$$

Then $\mathscr{E}_\Omega$ is time-covariant if, for all events in Ω :

$$\left(T^{\mu\nu} - kG^{\mu\nu}\right) u_\mu \partial_\nu \beta = 0 \quad \forall(\beta u) \in T\Omega. \qquad (27)$$

that is $k\boldsymbol{G} = \boldsymbol{T}$. Then the conservation of energy-momentum is :

$$Div\,\boldsymbol{T} = 0. \qquad (28)$$

Time-covariant thermodynamics

We have defined the momentum energy tensor and the entropy as :

$$\boldsymbol{T} = 2\frac{\partial A^M}{\partial \boldsymbol{g}} + A^M \boldsymbol{g}^{-1} + \beta\tilde{s}u \otimes u \qquad \tilde{s} = \frac{\partial A^M}{\partial \beta}$$

Define

- $\underline{T}^\nu = T^{\mu\nu}u_\mu = 2\frac{\partial A^M}{\partial g_{\mu\nu}}u_\mu + (A^M + \beta\tilde{s})u^\nu$
- $\mathcal{U}_T = T^{\mu\nu}u_\mu u_\nu = 2\frac{\partial A^M}{\partial g_{\mu\nu}}u_\mu u_\nu + (A^M + \beta\tilde{s})$
- the heat flux $q^\nu = \underline{T} - \mathcal{U}_T u = 2\left(\frac{\partial A^M}{\partial g_{\mu\nu}}u_\mu - \frac{\partial A^M}{\partial g_{\kappa\lambda}}u_\kappa u_\lambda u^\nu\right)$

Entropy current

Define the entropy current, a vector field as

$$\tilde{S} = \tilde{s}u + \frac{q}{\beta} \tag{29}$$

that may be rewritten as

$$\tilde{S}^{\nu} = (\beta\tilde{s} - \mathcal{U}_T)\frac{u^{\nu}}{\beta} + T^{\mu\nu}\frac{u_{\mu}}{\beta}. \tag{30}$$

This may also be expressed as, for the energy functional to be time covariant :

$$\tilde{S}^{\nu} = \frac{\partial A^M}{\partial \beta}u^{\nu} + \frac{2}{\beta}\left(\frac{\partial A^M}{\partial g_{\mu\nu}}u_{\mu} - \frac{\partial A^M}{\partial g_{\kappa\lambda}}u_{\kappa}u_{\lambda}u^{\nu}\right) \tag{31}$$

Time-covariant Clausius-Duhem Equation

We have defined the entropy current

$$\tilde{S}^{\nu} = (\beta\tilde{s} - \mathcal{U}_T)\frac{u^{\nu}}{\beta} + T^{\mu\nu}\frac{u_{\mu}}{\beta} \tag{32}$$

or

$$\tilde{S}^{\nu} = \frac{\partial A^M}{\partial\beta}u^{\nu} + \frac{2}{\beta}\left(\frac{\partial A^M}{\partial g_{\mu\nu}}u_{\mu} - \frac{\partial A^M}{\partial g_{\kappa\lambda}}u_{\kappa}u_{\lambda}u^{\nu}\right) \tag{33}$$

The divergence of this current is :

$$Div\tilde{S} = \boldsymbol{T} : \boldsymbol{D}_{\frac{u}{\beta}} - \mathscr{L}_{\frac{u}{\beta}}(\mathcal{U}_T - \beta\tilde{s}) \tag{34}$$

or :

$$\nabla_{\nu}\tilde{S}^{\nu} = \nabla_{\nu}\left(\frac{\partial A^M}{\partial\beta}u^{\nu} + \frac{2}{\beta}\left(\frac{\partial A^M}{\partial g_{\mu\nu}}u_{\mu} - \frac{\partial A^M}{\partial g_{\kappa\lambda}}u_{\kappa}u_{\lambda}u^{\nu}\right)\right) \tag{35}$$

Conclusion for time-corvariant continua

The energy functional $\mathscr{E}_\Omega(\boldsymbol{g}, \beta, \alpha_X) = \int_\Omega \left(\frac{k}{2} R + A^M(\boldsymbol{g}, \beta, \alpha_X) \right) \sqrt{g}\; dx^4$ is time-covariant if

- $\nabla \boldsymbol{g} = 0$
- $\mathscr{L}_{\beta u}(\alpha_X) = 0$
- $k\boldsymbol{G} = \boldsymbol{T}$
  with $\boldsymbol{T} = 2\dfrac{\partial A^M}{\partial \boldsymbol{g}} + A^M \boldsymbol{g}^{-1} + (\beta \tilde{s}) u \otimes u$ and $\tilde{s} = \frac{\partial A^M}{\partial \beta}$

This implies

$$Div \tilde{S} = \boldsymbol{T} : \boldsymbol{D}_{\frac{u}{\beta}} - \mathscr{L}_{\frac{u}{\beta}}(\mathcal{U}_T - \beta \tilde{s}) \tag{36}$$

with

$$\nabla_\nu \tilde{S}^\nu = \nabla_\nu \left(\tilde{s} u^\nu + \frac{2}{\beta} \left(\frac{\partial A^M}{\partial g_{\mu\nu}} u_\mu - \frac{\partial A^M}{\partial g_{\kappa\lambda}} u_\kappa u_\lambda u^\nu \right) \right) \tag{37}$$

Plan

Anisotropic hyperelastic matter

The hypotheses for the material contribution to the energy density are :

- the transformations are reversible,
- A^M depends only on the mass density ρ_c, the metric tensor $\boldsymbol{g}$, the deformation tensor $\boldsymbol{b}$ and a tensor $\boldsymbol{\mathcal{C}}$ containing the characteristics of the material,
- the existence of a neutral state : A^M reduces to the mass energy density when the deformation $\boldsymbol{b}$ is equal to the ambient metric.

Anisotropic hyperelastic matter

The energy functional for such material is :

$$\mathscr{E}_\Omega(\boldsymbol{g}, \rho_c, \boldsymbol{\mathcal{C}}, \boldsymbol{b}) = \int_\Omega \left(\frac{k}{2} R + A^M(\boldsymbol{g}, \rho_c, \boldsymbol{\mathcal{C}}, \boldsymbol{b}) \right) \sqrt{g}\, dx^4 \tag{38}$$

where the material contribution takes the form :

$$A^M(\boldsymbol{g}, \rho_c, \boldsymbol{\mathcal{C}}, \boldsymbol{b}) = \rho_c(c^2 + W(\boldsymbol{g}, \boldsymbol{\mathcal{C}}, \boldsymbol{b})) \tag{39}$$

with :

$$W(\boldsymbol{g}, \boldsymbol{\mathcal{C}}, \boldsymbol{g}) = 0 \tag{40}$$

For example :

$$W(\boldsymbol{g}, \boldsymbol{\mathcal{C}}, \boldsymbol{g}) = (\boldsymbol{g} - \boldsymbol{b})\boldsymbol{\mathcal{C}}(\boldsymbol{g} - \boldsymbol{b}) \tag{41}$$

Anisotropic hyperelastic matter

Evaluate $\mathscr{L}_X A^M$:

$$\mathscr{L}_X A^M = \frac{\partial A^M}{\partial \boldsymbol{g}} : \mathscr{L}_X \boldsymbol{g} + \frac{\partial A^M}{\partial \rho_c} : \mathscr{L}_X \rho_c + \frac{\partial A^M}{\partial \boldsymbol{C}} : \mathscr{L}_X \boldsymbol{C} + \frac{\partial A^M}{\partial \boldsymbol{b}} : \mathscr{L}_X \boldsymbol{b} \quad (42)$$

Remember that $\mathscr{L}_X \rho_c = 0$ and $\mathscr{L}_X \boldsymbol{b} = 0$.
The energy $\mathscr{E}_\Omega$ is then time-covariant if

$$\begin{aligned} \mathscr{L}_X \boldsymbol{C} &= 0 \\ k\boldsymbol{G} &= \boldsymbol{T} \end{aligned} \quad (43)$$

with

$$T^{\mu\nu} = 2\frac{\partial W}{\partial g_{\mu\nu}} + (\rho_c c^2 + W) g^{\mu\nu} \quad (44)$$

Anisotropic hyperelastic matter

To ensure that there is no dissipation we impose

$$q = 0 \tag{45}$$

$$\frac{\partial W}{\partial g_{\kappa\lambda}} u_\kappa u_\lambda = 0 \tag{46}$$

leading to :

$$T^{\mu\nu} = \mathcal{U}_T u^\mu u^\nu + \underline{\underline{T}}^{\mu\nu} \tag{47}$$

where :

$$\mathcal{U}_T = (\rho_c c^2 + W) = A^M \tag{48}$$

$$\underline{\underline{T}}^{\mu\nu} = 2\frac{\partial W}{\partial g_{\mu\nu}} + (\rho_c c^2 + W)\underline{\underline{g}}^{\mu\nu} \tag{49}$$

Plan

Introduction

Space-time kinematics

Time-covariant functionnals

Covariant thermodynamics

Anisotropic hyperelasticity

Conclusion

Definitions for deformation tensor fields

Conclusion

Time-covariance enables to construct :

- Covariant thermodynamics
- Covariant anisotropic hyper-elasticity

Next step : model irreversible phenomena (covariant plasticity...)

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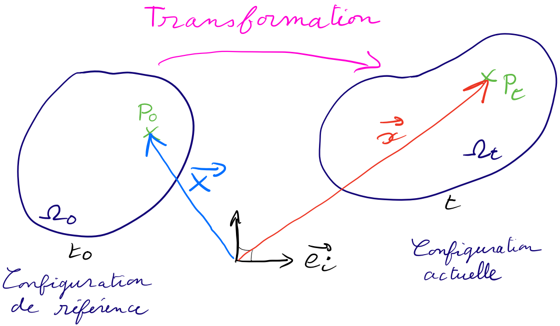
Définitions 3D, en mécanique classique



- Espace euclidien, système de coord. orthonormé.
- Transformation : $\vec{x}(t,\vec{X}) = \vec{X} + \vec{u}(t,\vec{X})$
- Gradient de la transformation : $F_{ij} = \frac{\partial x_i}{\partial X_j}$

Définitions 3D, en mécanique classique

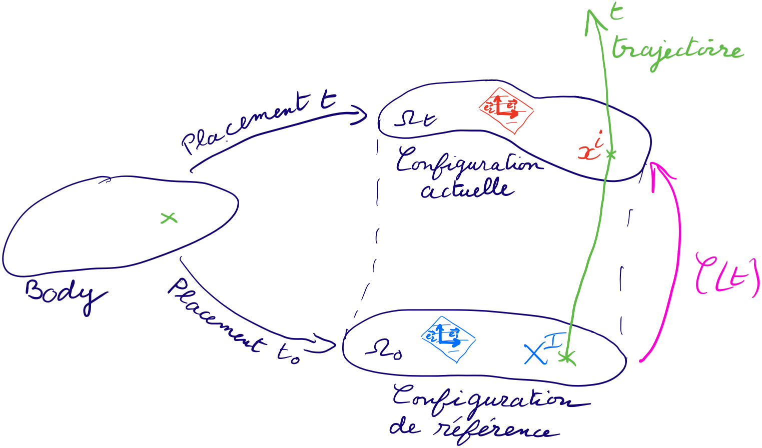
- La décomposition polaire de F permet d'obtenir les définitions des tenseurs des dilatations en multipliant F par F^T.

Entité	Configuration de référence	Configuration actuelle
Gradient	$F = RU$	$F = VR$
Déformation	$F^T F = U^2$	$FF^T = V^2$
Déformation	$C = F^T F$	
Déformation	$C^{-1} = F^{-1} F^{-T}$	
Déformation		$B = FF^T$
Déformation		$b = B^{-1} = F^{-T} F^{-1}$

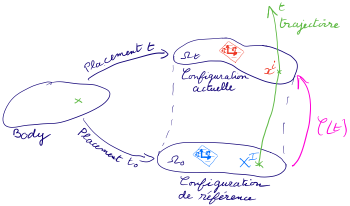
$$ds^2 = I_{ij} dx_i dx_j = I_{ij} \frac{\partial x_i}{\partial X_K} \frac{\partial x_j}{\partial X_L} dX_K dX_L$$

Définitions 3D, approche géométrique

Référence : TP Aussois Mecamat 2023.



Définitions 3D, approche géométrique



- Espace euclidien.
- Transformation : $\varphi : \Omega_0 \to \Omega, \varphi : (t, X^I) \mapsto x^i = \varphi^i(t, X^I)$ coord. sur la config. actuelle à t d'un point de coord. X^I sur la config. de référence.
- Soit $F := T\varphi : T\Omega_0 \to T\Omega$ l'application linéaire tangente de φ ;
  $F : \left(\frac{\partial x^i}{\partial X^J}\right)$
  φ et $T\varphi$ sont inversibles.

Passer de la configuration de référence à la configuration déformée et inversement

- Mécanique classique : le transport convectif
  On définit des règles de transport pour chaque entité : densité, vecteur, tenseur d'ordre 2...
- Approche géométrique
  Transformation φ
  Passage de la config. de référence vers la config. actuelle : push forward
  Passage de la config. actuelle vers la config. de référence : pull back

Tenseur	Config. de référence	Config. actuelle
Cov. ordre 2	K	$k = \varphi_\star K = F^{-*}(K \circ \varphi^{-1})F^{-1}$
	$K = \varphi^\star k = F^*(k \circ \varphi)F$	k
Contra. ordre 2	T	$t = \varphi_\star T = F(T \circ \varphi^{-1})F^*$
	$T = \varphi^\star t = F^{-1}(t \circ \varphi)F^{-*}$	t

Définitions 3D, approche géométrique

- Soit le tenseur métrique q.

$$q_{ij} dx^i dx^j = ds^2$$

Entité	Configuration de référence	Configuration actuelle
Déformation	$C = \varphi^\star q$	q
Déformation	C^{-1}	
Déformation	q^{-1}	$B = \varphi_\star q^{-1}$
Déformation		$b = B^{-1}$

Les tenseurs des dilatations sont des métriques.

Composantes

- Calcul dans un repère orthonormé avec un tenseur métrique noté I

$$C = F^T F \to C_{IJ} = \frac{\partial x^k}{\partial X^I}\frac{\partial x^l}{\partial X^J} I_{kl}$$

$$C^{-1} = F^{-1} F^{-T} \to C^{IJ} = \frac{\partial X^I}{\partial x^k}\frac{\partial X^J}{\partial x^l} I^{kl}$$

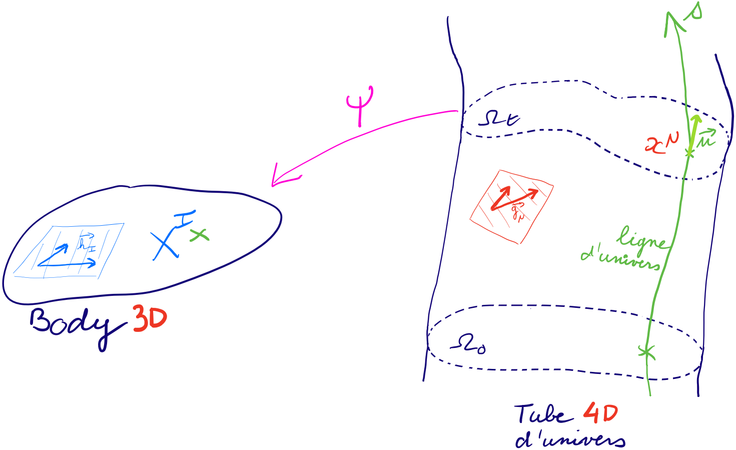
$$b = F^{-T} F^{-1} \to b_{ij} = \frac{\partial X^K}{\partial x^i}\frac{\partial X^L}{\partial x^j} I_{KL}$$

$$B = b^{-1} = F F^T \to B^{ij} = \frac{\partial x^i}{\partial X^K}\frac{\partial x^j}{\partial X^L} I^{KL}$$

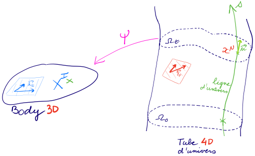
- En plus de ceux ci-dessus, on peut définir une multitude de tenseurs des dilatations.

Définitions 4D, approche Toupin, Souriau, Eringen

Référence : cours de B. Kolev et article B. Kolev et R. Desmorat



Définitions 4D, approche Toupin, Souriau, Eringen



- DEUX Variétés
  1. le body $\mathscr{B}$ 3D
  2. le tube de courant $\mathscr{T}$ dans $\mathscr{M}$ 4D, Espace Riemannien, tenseur (pseudo-)métrique g de signature $(1, -1, -1, -1)$
- Transformation : champ de matière $\psi : \mathscr{T} \to \mathscr{B}$, $\psi : x^\mu \mapsto X^I = \psi(x^\mu)$ coord. 3D sur le body d'un point de coord. 4D x^μ sur le tube.
- Soit $T\psi$ l'application linéaire tangente de ψ : $T\psi : \left(\frac{\partial X^I}{\partial x^\mu}\right)$
  ⚠(3x4) pas de $\frac{\partial X^0}{\partial x^\mu}$
  ψ et $T\psi$ ne sont PAS inversibles.

Définitions 4D, approche Toupin, Souriau, Eringen

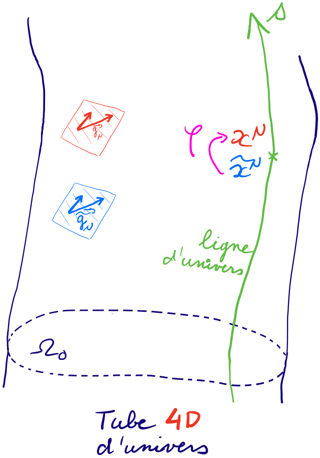
- Soit le tenseur métrique g.
- Soit la projection sur l'espace de g et de g^{-1} :

$$(h) \quad h_{\mu\nu} = g_{\mu\nu} - u_\mu u_\nu$$
$$(h^\#) \quad h^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu$$

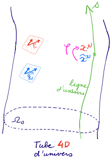
Entité	Sur le body	Sur le tube
Déformation H	$H = C^{-1} = T\psi h^\# (T\psi)^*$	$(h^\#) : h^{\mu\nu}$
Déformation $C = H^{-1}$	C	$h = (T\psi)^* C T\psi$

$H = C^{-1}$ est appelé conformation par Souriau
⚠ Les déformations sont des tenseurs 3x3

Déformations 4D



Déformations 4D



- Une seule variété 4D $\mathscr{M}$, Espace Riemannien, tenseur (pseudo-)métrique g de signature $(1, -1, -1, -1)$
- Soit le système de coordonnées courant x^μ et le système de coordonnées propre $\hat{x}^\mu$ tel que le champ de vitesse dans soit $\hat{u}^\mu(1, 0, 0, 0)$.
- Transformation : champ de matière $\varphi : \mathscr{M} \to \mathscr{M}$, $\varphi : \hat{x}^\mu \mapsto x^\mu = \varphi(\hat{x}^\mu)$.
- Soit $T\varphi$ l'application linéaire tangente de φ : F : $\left(\frac{\partial x^\mu}{\partial \hat{x}^\mu}\right)$, matrice jacobienne du changement de coord. de propre vers courant
  φ et $T\varphi$ inversibles.

Déformations 4D

- Soit le tenseur métrique g.
- Les déformations sont des tenseurs 4x4.

Sur la variété $\mathscr{M}$

Entité	Syst. de coord. propre	Syst. de coord. courant
Vitesse u	$\hat{u}^\mu(1,0,0,0)$	u^μ
Métrique g	$\hat{g}_{\mu\nu} d\hat{x}^\mu \hat{x}^\nu = ds^2$	$g_{\mu\nu} dx^\mu x^\nu = ds^2$
Métrique inverse g^{-1}	$\hat{g}^{\mu\nu}$	$g^{\mu\nu}$
Déformation	$C_{\mu\nu} = \hat{g}_{\mu\nu}$	$g_{\mu\nu}$
Déformation C^{-1}	$C^{\mu\nu} = \hat{g}^{\mu\nu}$	$g^{\mu\nu}$
Déformation	$\hat{b}_{\mu\nu} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$	$b_{\mu\nu}$
Déformation $B = b^{-1}$	$\hat{B}^{\mu\nu} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$	$B^{\mu\nu}$

Définitions 4D : comparaison des approches

- Calcul de la conformation :

$$H^{IJ} = \frac{\partial X^I}{\partial x^\alpha}\frac{\partial X^J}{\partial x^\beta}\overbrace{(g^{\alpha\beta} - u^\alpha u^\beta)}^{h^{\alpha\beta}}$$

- Calcul de C^{-1} dans l'approche 4D :

$$C^{\mu\nu} = \hat{g}^{\mu\nu} = \frac{\partial \hat{x}^\mu}{\partial x^\alpha}\frac{\partial \hat{x}^\mu}{\partial x^\beta} g^{\alpha\beta}$$

Projection de la métrique sur l'espace dans le système propre et courant :

$$\hat{g}^{\mu\nu} - \hat{u}^\mu \hat{u}^\nu = \frac{\partial \hat{x}^\mu}{\partial x^\alpha}\frac{\partial \hat{x}^\mu}{\partial x^\beta}(g^{\alpha\beta} - u^\alpha u^\beta)$$

- Si on calcule :

$$\frac{\partial \hat{x}^I}{\partial x^\alpha}\frac{\partial \hat{x}^J}{\partial x^\beta}(g^{\alpha\beta} - u^\alpha u^\beta) = H^{IJ}$$

On retrouve la conformation.