Yamabe problem and its first gap invariant

Un invariant d'écart entre la première et la deuxième constante de Yambe.

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EIGSI La Rochelle

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- $\mathbb{S}^n$ is the standard sphere $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$
- The conformal classe of g is $[g] := \left\{ \tilde{g} := u^{N-2} g, u \in C^\infty(M), u > 0 \right\}$

- In a local coordinate system, if we denote the componants of the Reimann tensor by R_{ijl}^{m} then the components of the Riemann Christoffel curvature are defined by

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- The scalar curvature is the contraction of Ric, i.e. $S_g = g^{ij} R_{ij}$.

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- So one can infer that $\tilde{g} := u^{N-2} g$ has constant scalar curvature λ if and only if u satisfies the **Yamabe equation**

$$L_g(u) = \lambda u^{N-1}. \tag{1.1}$$

The variational formulation of the problem

Finding a metric $\tilde{g}$ in $[g]$ with constant scalar curvature.

$\Longleftrightarrow$

Finding a solution for the PDE

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$$Y(u) = \frac{\int_M c_n |\nabla u|^2 + S_g u^2 \, dv_g}{\left(\int_M |u|^N \, dv_g\right)^{\frac{2}{N}}}.$$

This observation motivates the definition and study of the following quantity, known as the Yamabe constant or invariant:

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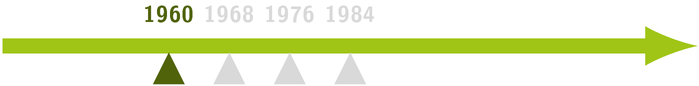
Note that $|\mu(M)| < \infty$ by the Sobolev inequality, and $\mu(M)$ is conformally invariant. In fact, by (1.3), it can be rewritten as

$$\mu(M, g) = \inf \left\{ \frac{\int_M S_{\tilde{g}} \, dv_{\tilde{g}}}{\mathrm{vol}_{\tilde{g}}(M)^{2/N}} : \tilde{g} \text{ conformal to } g \right\}$$

History of Yamabe problem



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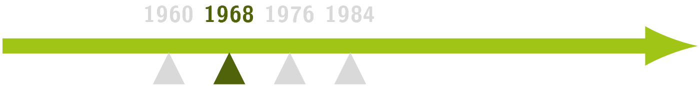

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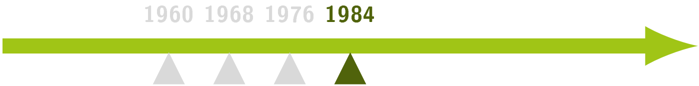

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- 1984: Schoen proved the remaining cases.

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Richard Schoen
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Resolution of Yamabe Probleme

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Theorem 1.2 (Aubin).

If M *has dimension* $n \geq 6$ *and is not locally conformally flat at some point* $p \in M$, *then* $\mu(M) < \mu(\mathbb{S}^n)$.

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Theorem 1.3 (Schoen).

If M has dimension 3,4 , or 5 or if M is locally conformally flat at some point $p \in M$, then $\mu(M) < \mu(\mathbb{S}^n)$ unless M is conformal to $\mathbb{S}^n$.

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Analytically speaking, the direct method falls short because the embedding of $W^{1,2}(M)$ into $L^N(M)$ is not compact.

$$\mu(M) = \inf_{u\neq 0,\, u\in C_+^\infty(M)} \frac{\int_M c_n|\nabla u|^2 + S_g u^2 \, dv_g}{\left(\int_M |u|^N \, dv_g\right)^{\frac{2}{N}}}.$$

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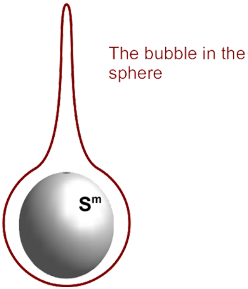
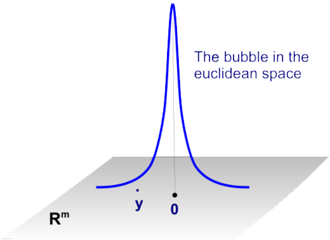
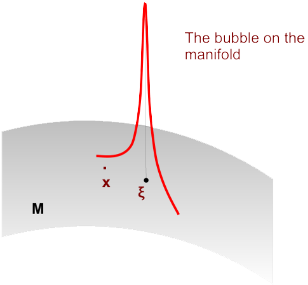


Theorem 1.1 represents the analytic portion of the problem. The idea behind Theorem 1.1 is the following:

A minimizing sequence for $\mu(M)$ must either converge in $W^{1,2}(M)$ to a minimizer, or else it must concentrate at a point $p \in M$. A concentration (or "bubble") contributes $\mu(S^n)$ to the energy, so if $\mu(M) < \mu(S^n)$, this possibility cannot occur.





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Theorems 1.2 and 1.3 represent the contributions on the geometry side of the problem.

To show that an infimum $\mu(M)$ is strictly less than a certain number $\mu(S^n)$, one must construct a test function u with $Y(u) < \mu(M)$. For both theorems, the test functions involve suitable modifications of minimizers for $\mu(\mathbb{S}^n)$, though we crucially must first choose the right conformal representative and coordinate system.

- The Yamabe invariant $\mu(M, g)$ is associated with the first eigenvalue of the conformal Laplacian L_g:

$$\mu(M, g) = \inf_{\tilde{g} \in [g]} \lambda_1(\tilde{g}) \mathrm{Vol}(M, \tilde{g})^{\frac{2}{n}}.$$

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- Ammann and Humbert studied the **second Yamabe invariant**, $\mu_2(M,g)$, defined by taking the infimum over the second eigenvalue.
- To ensure the existence of minimizers for μ_2, the class of metrics is extended to "generalized metrics," where $\tilde{g} = u^{N-2}g$ and u is no longer necessarily positive and smooth, but $u \in L^N_+(M) := \left\{ f \in L^N(M),\ f \geq 0, \text{ and } f \not\equiv 0 \right\}$.

Transition from the second invariant to the gap invariant

Theorem 2.1 (A.H).

Let (M, g) be an n-dimensional compact Riemannian manifold with $\mu_1(M, g) \geq 0$. Then,

$$\mu_2(M, g) \leq \left(\mu_1(M, g)^{\frac{n}{2}} + \mu_1(\mathbb{S}^n)^{\frac{n}{2}}\right)^{\frac{2}{n}}. \tag{2.1}$$

Furthermore, if the inequality is strict, the infimum in $\mu_2(M, g)$ is achieved.

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Motivation for our work

We want to quantify the "jump" when passing from the first to the second Yamabe invariant. We introduce a new invariant to study the gap between λ_1 and λ_2 in the conformal class.

Defining the Yamabe Gap Invariant

We introduce a new quantity to capture the relationship between the first and second eigenvalues.

Definition 3.1 (The Yamabe Gap Invariant).

The **first Yamabe gap invariant**, denoted $\alpha(M, g)$, is defined by:

$$\alpha(M, g) := \inf_{\tilde{g} \in [g]} \sqrt{\lambda_1(\tilde{g})\lambda_2(\tilde{g})} \operatorname{Vol}(M, \tilde{g})^{\frac{2}{n}}$$

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- This invariant measures the smallest possible geometric mean of the first two eigenvalues of the conformal Laplacian within the conformal class.
- Understanding this gap provides new insights into the stability of conformal metrics and their scalar curvature properties.

Main Theorem 1: An Upper Bound for the Gap Invariant

Similar to Aubin's bound for μ_1, we establish a universal upper bound for $\alpha(M, g)$.

Theorem 4.1.

Let (M, g) be a compact Riemannian manifold with $\mu_1(M, g) \geq 0$. Then,

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Proof Idea: Our invariant in an explicit way has this form

$$\boxed{\alpha(M, g) := \inf_{\substack{u \in L_+^N(M) \\ V \in \mathrm{Gr}_2^u(H^1(M))}} \inf_{v \in V \setminus \{0\}} F(u, v)^{\frac{1}{2}} \sup_{v \in V \setminus \{0\}} F(u, v)^{\frac{1}{2}}.}$$

with $F(u, v) = \dfrac{\int_M c_n |\nabla v|^2 + S_g v^2 \, dv_g}{\int_M v^2 u^{N-2} \, dv_g} \left(\int_M u^N \, dv_g\right)^{\frac{2}{n}}.$

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- We use a carefully constructed test function $u_\epsilon = v + v_\epsilon$, where:
  - v is a smooth positive minimizer for the Yamabe functional Y on M.
  - v_ϵ is a function that concentrates at a point $x_0 \in M$ and whose energy approaches $\mu(\mathbb{S}^n)$ as $\epsilon \to 0$.
- This test function models the manifold "splitting off" a sphere. By evaluating the functional with this specific choice, we derive the inequality. The value is attained at the data $(u_\epsilon, v, v_\epsilon)$.

Main Theorem 2: Existence of a Minimizer

The upper bound is not just a curiosity; it provides a condition for the existence of a minimizer.

Theorem 4.2.

Assume that the inequality is strict:

$$\alpha(M,g) < \left(\mu(M,g) \cdot 2^{\frac{2}{n}} \cdot \mu(\mathbb{S}^n)\right)^{\frac{1}{2}}$$

Then the infimum in the definition of $\alpha(M,g)$ is achieved by a generalized metric $\tilde{g} = u^{N-2}g$.

Significance: This is a **compactness result**. The strict inequality prevents a minimizing sequence from "losing energy" through concentration phenomena (or "bubbling"), guaranteeing that a limit exists.

Proof Strategy for Theorem 2

The proof is by contradiction, using the **concentration-compactness** method.

1. **Subcritical Approximation:** We take a sequence of minimizers (u_q, v_q, w_q) for a sub-critical problem α_q where the exponent $q < N$. Minimizers are known to exist in this setting.

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1. **Subcritical Approximation:** We take a sequence of minimizers (u_q, v_q, w_q) for a sub-critical problem α_q where the exponent $q < N$. Minimizers are known to exist in this setting.
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The proof proceeds by analyzing all possible behaviors of the weak limit (u, v, w).

Proof Analysis: The Four Cases

Case 1: $uv \not\equiv 0$ and $uw \not\equiv 0$ (The "Good" Case)

- The weak limit is non-trivial and well-behaved.
- We prove that the limiting eigenvalues are distinct ($\hat{\mu}_1 < \hat{\mu}_2$), which implies the limiting functions v and w are orthogonal in the correct sense.
- This limit (u, v, w) directly achieves the infimum, so $\alpha(M, g)$ is attained.

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Cases 2 & 3: One limit is zero (e.g., $v \equiv 0$, $w \not\equiv 0$)

- The energy of the sequence v_q is entirely lost to concentration at points in the set A_v.
- We use a sharp Sobolev inequality to show the energy of this "bubble" is related to $\mu(\mathbb{S}^n)$.
- By balancing the energy of the bubble with the energy of the non-vanishing part (w), we derive a contradiction with the strict inequality assumption.

Proof Analysis: The Critical Case

Case 4: $uv \equiv 0$ and $uw \equiv 0$ (The "Full Blow-up" Case)

This is the most complex scenario, where all energy from both sequences v_q and w_q concentrates. The proof proceeds in steps:

1. **Uniqueness of Concentration Point:** We first prove that if blow-up occurs, it must be at a single, common point for both sequences ($A_v = A_w = \{x_0\}$). Multiple concentration points would violate the strict inequality.

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3. **Analysis of the Bubble:** The rescaled sequence converges to a "bubble solution" on $\mathbb{R}^n$.
   - If this bubble solution is non-trivial, it can be mapped to a solution on the sphere $\mathbb{S}^n$, which means $\alpha(M, g)$ is attained.
   - If the bubble solution degenerates, we again derive a contradiction using energy-splitting arguments similar to Cases 2 & 3.

Completing the Picture: The Equality Case

Our work establishes the fundamental properties of $\alpha(M, g)$, but one crucial question remains, analogous to the final step in the classical Yamabe problem.

The Next Major Question

Which manifolds (M, g) are responsible for the equality case? That is, for which manifolds does

$$\alpha(M, g) = \left(\mu(M, g) \cdot 2^{\frac{2}{n}} \cdot \mu(\mathbb{S}^n)\right)^{\frac{1}{2}}$$

hold?

- In the classical Yamabe problem, equality $\mu(M, g) = \mu(\mathbb{S}^n)$ holds if and only if (M, g) is conformally equivalent to the sphere. This is a profound geometric rigidity result.
- We expect a similar rigidity result for the Yamabe gap invariant.

A Path to Proving Strict Inequality

A key step is to show that "most" manifolds satisfy the strict inequality. The classical approach provides a template.

Classical Result (Aubin, Schoen)

For a test function u concentrating at a point p, the Yamabe functional has an expansion:

$$Y(u) \approx \mu(\mathbb{S}^n) - C(n)|W(p)|^2\alpha^4 + \dots \quad (\text{for } n \geq 6)$$

where W is the Weyl tensor.

- If the manifold is **not locally conformally flat** ($|W(p)| > 0$) and $n \geq 6$, this expansion proves that $\mu(M, g) < \mu(\mathbb{S}^n)$.
- **Our Goal:** A similar analysis should be possible for our gap functional $G(u, V)$.
- The presence of the Weyl tensor should again "lower the energy" away from the maximum possible value, proving the strict inequality for $\alpha(M, g)$ on a large class of manifolds.

Conjecture and Research Direction

Based on the structure of the problem, we propose the following conjecture.

Conjecture

The equality $\alpha(M,g) = \left(\mu(M,g) \cdot 2^{\frac{2}{n}} \cdot \mu(\mathbb{S}^n)\right)^{\frac{1}{2}}$ holds if and only if (M,g) is conformally equivalent to the standard sphere $\mathbb{S}^n$.

Roadmap for the proof:

- The "if" direction involves calculating $\alpha(\mathbb{S}^n)$ directly.
- The "only if" direction is much deeper. It would require analyzing a minimizing sequence for a manifold assumed to satisfy the equality.
- This rigidity often requires powerful tools from geometric analysis, such as the **Positive Mass Theorem**, to show that the underlying manifold must be conformally flat and, ultimately, the sphere.

Proving this would provide a complete geometric understanding of the Yamabe gap invariant.

Merci pour votre attention

Part I

Appendix

Omitted formalism : Generalized metrics

Assume now that the Yamabe constant $\mu(M, g) \geq 0$. It is easy to check that

$$\mu(M, g) = \inf_{\tilde{g} \in [g]} \lambda_1(\tilde{g}) \mathrm{Vol}(M, \tilde{g})^{\frac{2}{n}}.$$

[1] **B. Ammann, E. Humbert** The second Yamabe Invariant. Journal of functional analysis, 235(2), 377-412.

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$$\mu(M,g) = \inf_{\tilde{g} \in [g]} \lambda_1(\tilde{g}) \mathrm{Vol}(M,\tilde{g})^{\frac{2}{n}}.$$

We can then naturally extend this definition as follows. Let $k \in \mathbb{N}^*$. Then, the k^{th} Yamabe constant is defined by

$$\mu_k(M,g) = \inf_{\tilde{g} \in [g]} \lambda_k(\tilde{g}) \mathrm{Vol}(M,\tilde{g})^{\frac{2}{n}}.$$

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To address achievability for $k=2$, i.e., the existence of minimizers for $\mu_2(M,g)$, the authors in [A.H][1] extended the conformal class $[g]$ to what is referred to as the class of *generalized metrics* conformal to g. This extension was necessary because otherwise, there would be no minimizers once M is connected

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A generalized metric is a "metric" of the form

$$\tilde{g} = u^{N-2} g$$

u is no longer necessarily positive and smooth, but $u \in L^N_+(M) := \left\{ f \in L^N(M),\ f \geq 0, \text{ and } f \not\equiv 0 \right\}$.

Then, it is shown in [A.H] that a coherent generalization can be given as follows.

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- $\lambda_k(\tilde{g}) = \inf_{V \in \mathrm{Gr}^u_k(H^2_1(M))} \sup_{v \in V \setminus \{0\}} \dfrac{\int_M v L_g v \, dv_g}{\int_M v^2 u^{N-2} \, dv_g}$,
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On the other hand, it is convenient to rewrite the functional F with some additional variable $v \in V$. For given functions u and v, we define the following functional:

$$F(u, v) = \frac{\int_M c_n |\nabla v|^2 + S_g v^2 \, dv_g}{\int_M v^2 u^{N-2} \, dv_g} \left(\int_M u^N \, dv_g \right)^{\frac{2}{n}} . \tag{6.1}$$

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Based on the above, we arrive at the following refined definition of the k-th Yamabe invariant:

Definition 6.1.

Let $k \in \mathbb{N}^*$. Then, the k^{th} Yamabe invariant for a generalized metric $\tilde{g} = u^{N-2} g$ is defined by

$$\mu_k(M, g) = \inf_{\substack{u \in L^N_+(M) \\ V \in \mathrm{Gr}^u_k(H^2_1(M))}} \sup_{v \in V \setminus \{0\}} F(u, v). \tag{6.2}$$

Definition 6.2 (first Yamabe gap invariant).

The first Yamabe gap invariant is defined as follows:

$$\alpha(M,g) := \inf_{\substack{u\in L_+^N(M) \\ V\in \mathrm{Gr}_2^u(H^1(M))}} G(u,V), \tag{6.3}$$

where the quantity $G(u,V)$ is given by:

$$G(u,V) = \sqrt{\left(\inf_{v\in V\setminus\{0\}} F(u,v)\right)\left(\sup_{v\in V\setminus\{0\}} F(u,v)\right)}. \tag{6.4}$$

Notice that the first Yamabe gap invariant can be alternatively expressed as follows:

$$\alpha(M,g) = \inf_{\tilde{g}\in[g]} \sqrt{\lambda_1(\tilde{g})\lambda_2(\tilde{g})}\mathrm{Vol}(M,\tilde{g})^{\frac{2}{n}}.$$

Anatomy of the "Bubble" Test Function

The "bubble" functions are explicit solutions to the Yamabe equation on the sphere.

The Yamabe Equation on $\mathbb{S}^n$

The equation for a conformal factor u on $(\mathbb{S}^n, g_0)$ that yields a metric of constant scalar curvature is:

$$\frac{4(n-1)}{n-2}\Delta_{g_0} u + n(n-1)u = \mu(\mathbb{S}^n)u^{\frac{n+2}{n-2}}$$

- Via stereographic projection, this is equivalent to $-\Delta U = U^{\frac{n+2}{n-2}}$ on $\mathbb{R}^n$.
- The solutions are the well-known "standard bubbles":

$$U_{\delta,y}(x) := \delta^{-\frac{n-2}{2}}\left(\frac{\alpha_n}{1+|\frac{x-y}{\delta}|^2}\right)^{\frac{n-2}{2}}$$

  where y is the center and δ controls the concentration.
- Our test function v_ϵ is a localized version of these functions transplanted onto the manifold M.

Subcritical problem

Let us define the sub-critical functional associated to F Let $q \in]2, N]$ and for $u, v \in L^q(M)$, $u \geq 0$, $v \in H^1(M)$ such that $uv \neq 0$, consider the functional

$$F_q(u,v) = \frac{\int_M c_n |dv|^2 + S_g v^2 \, dv_g}{\int_M v^2 u^{q-2} \, dv_g} \left(\int_M u^q \, dv_g \right)^{\frac{q-2}{q}}.$$

For such u, let Ω_u be the set of 2-dimensional subspaces V of $H^1(M)$ such that uV is still of dimension 2. For any $V \in \Omega_u$ we define

$$G_q(u,V) = \sqrt{\left(\inf_{v \in V} F_q(u,v) \right) \left(\sup_{v \in V} F_q(u,v) \right)}.$$

Finally let us define the sub-critical Yamabe gap invariant

$$\alpha_q(M,g) = \inf_u \inf_{V \in \Omega_u} G_q(u,V).$$

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Finally let us define the sub-critical Yamabe gap invariant

$$\alpha_q(M,g) = \inf_u \inf_{V \in \Omega_u} G_q(u,V).$$

$$\lim_{q \to N} G_q = G_N = G \quad \text{and} \quad \lim_{q \to N} \alpha_q = \alpha_N = \alpha$$

$$S_{g_0} = n(n-1)$$

$$\Downarrow$$

$$-\Delta_{g_0} u + \frac{n(n-2)}{4} u = u^{\frac{n+2}{n-2}}, u > 0, \text{ in } (\mathbb{S}^n, g_0)$$

which is equivalent (via the stereographic projection) to

$$-\Delta U = U^{\frac{n+2}{n-2}}, U > 0, \text{ in } \mathbb{R}^n$$

The solutions are the standard bubbles

$$U_{\delta,y}(x) := \delta^{-\frac{n-2}{2}} U\left(\frac{x-y}{\delta}\right), x, y \in \mathbb{R}^n, \delta > 0,$$

- $U(x) := \alpha_n \dfrac{1}{(1+|x|^2)^{\frac{n-2}{2}}}$
- y is the center of the bubble
- δ is the weight of the bubble

$$
Y(u) \leq \begin{cases} \mu\left(\mathbb{S}^n\right) - c|W(p)|^2\alpha^4 + O\left(\alpha^{n-2}\right) & \text{if } n < 6 \\ \mu\left(\mathbb{S}^n\right) - c|W(p)|^2\alpha^4 \log(1/\alpha) + O\left(\alpha^{n-2}\right) & \text{if } n = 6 \\ \mu\left(\mathbb{S}^n\right) + O\left(\alpha^{n-2}\right) & \text{if } n > 6 \end{cases}
$$

In particular, since $|W(p)| > 0$ and $n \geq 6$, we see that

$$
\mu(M) \leq Y(u) < \mu\left(\mathbb{S}^n\right).
$$

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