Two-Scale Geometric Modelling for Defective Media

an approach using fibre bundles

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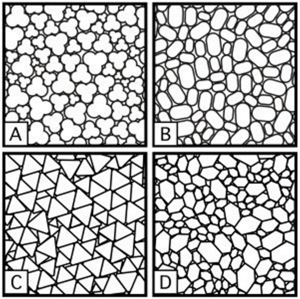
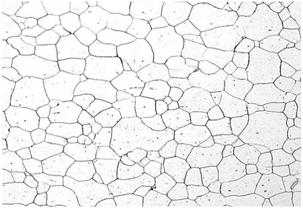
Introduction

transversal slice of wood with rings and cracks



isotropic tiling of the plane



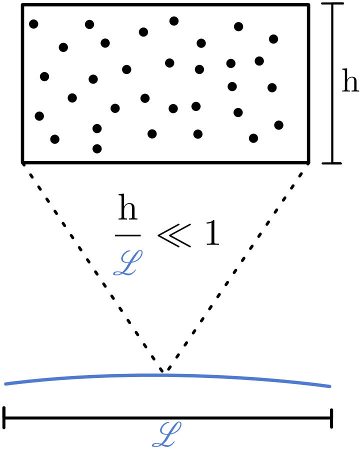
polycrystalline silicon sample

E. Cosserat and F. Cosserat, *Sur la théorie des corps déformables*, 1909

E. Cartan, *Sur une généralisation de la notion de courbure de Riemann et les espaces à torsion*, 1922

V. H. Nguyen, G. Casale, L. Le Marrec, *On tangent geometry and generalized continuum with defects*, 2021

Classical continuum mechanics



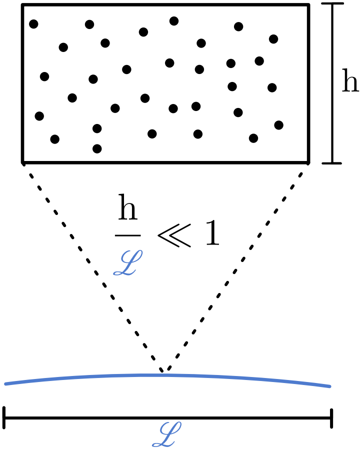
Continuous approximation of a macroscopic material

- $\mathsf{F} = \mathrm{T}\varphi : \mathrm{T}\mathbb{B} \to \mathrm{T}\mathbb{E}$: space gradient of the placement map
- $\mathrm{g} \equiv \mathrm{Id}$: Euclidean metric in $\mathbb{E}$.
- $\mathfrak{G} = \mathsf{F}^{\mathrm{t}} \cdot \mathrm{g} \cdot \mathsf{F}$: Cauchy-Green tensor in $\mathbb{B}$

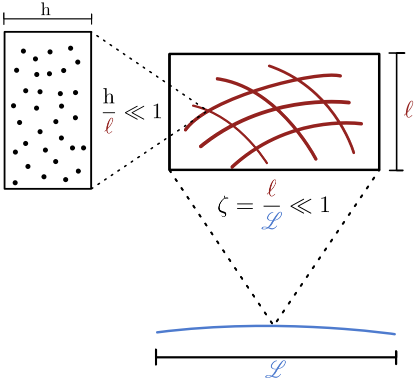
Objectivity

$\alpha : \mathsf{F} \longmapsto \alpha(\mathsf{F})$ is frame indifferent iff one has $\tilde{\alpha} : \mathfrak{G} \longmapsto \tilde{\alpha}(\mathfrak{G})$ such that $\alpha(\mathsf{F}) = \tilde{\alpha}(\mathfrak{G})$.

Generalised continuum mechanics



Continuous approximation of a macroscopic material



Continuous approximation of a micro-structured material

A natural fibre bundle structure

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Fibre bundle

$\mathcal{B} \xrightarrow{\pi_{\mathcal{B}}} \mathbb{B}$ is a fibre bundle. Meaning, it is

- a disjoint union:

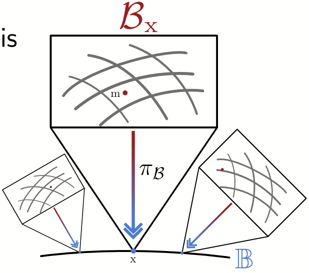
$$\mathcal{B} \equiv \bigsqcup_{\mathrm{X} \in \mathbb{B}} \mathcal{B}_{\mathrm{X}}$$

- homogeneous:

$$\forall (\mathrm{X}, \mathrm{Y}) \in \mathbb{B}^2 \qquad \mathcal{B}_{\mathrm{X}} \simeq \mathcal{B}_{\mathrm{Y}}$$

- locally trivial (i.e. $\mathcal{B} \simeq_{\text{loc.}} \mathbb{B} \times$ micro):

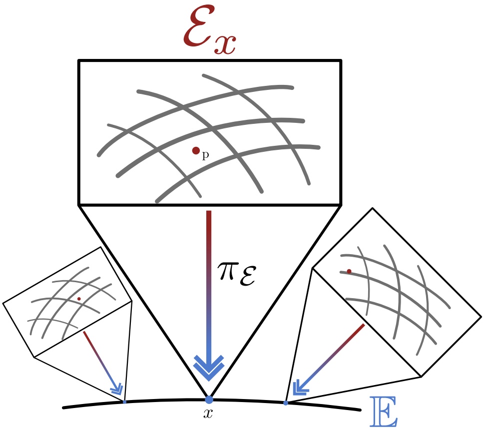
$\forall\, \mathrm{X} \in \mathbb{B}, \exists\, \mathbb{U}_X \subset \mathbb{B}$ neigh. of X, s.t. $\pi_{\mathcal{B}}^{-1}(\mathcal{U}_{\mathrm{X}}) \simeq \mathcal{U}_{\mathrm{X}} \times \mathcal{B}_{\mathrm{X}}$



M. Epstein, *The Geometrical Language of Continuum Mechanics*, 2014

Ambiant Space

The ambient space $\mathcal{E} \xrightarrow{\pi_{\mathcal{E}}} \mathbb{E}$



$\mathcal{E}$ is holonomic, meaning $\mathcal{E} \simeq_{\text{loc.}} \mathbb{E} \times \mathbb{E} := \mathrm{T}\mathbb{E}$ and $\mathcal{E}_x \simeq \mathrm{T}_x\mathbb{E}$

Ehresmann connection

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Ehresmann connection

An (Ehresmann) connection on $\mathcal{E} \xrightarrow{\pi_{\mathcal{E}}} \mathbb{E}$ is the data at each point $(\bar{\mathsf{x}}, \delta\mathsf{x}) \in \mathcal{E}_{\bar{\mathsf{x}}}$ of a linear injection

$$\gamma_{(\bar{\mathsf{x}},\, \delta\mathsf{x})} : \mathrm{T}_{\bar{\mathsf{x}}}\mathbb{E} \hookrightarrow \mathrm{T}_{(\bar{\mathsf{x}},\, \delta\mathsf{x})}\mathcal{E}$$

which is horizontal (i.e. macroscopic). In other words:

$$\forall (\bar{\mathsf{x}}, \delta\mathsf{x}) \in \mathcal{E}_{\bar{\mathsf{x}}} \qquad \mathrm{T}\pi_{\mathcal{E}} \circ \gamma_{(\bar{\mathsf{x}},\, \delta\mathsf{x})} = \mathrm{Id}_{\mathrm{T}_{\bar{\mathsf{x}}}\mathbb{E}}$$

C. Ehresmann, *Les connexions infinitésimales dans un espace fibré différentiable*, 1950

Covariant derivative (Koszul con.)

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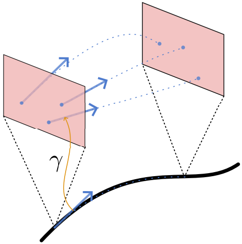

When γ is linear in the fibre's coordinates:

$$\gamma|_{(\bar{x}, y^i \mathrm{y}_i)}\left(\bar{u}^j \frac{\partial}{\partial \bar{x}^j}\right) := \bar{u}^j \left[\frac{\partial}{\partial \mathrm{x}^j} + \gamma^k_{ij}(\bar{x})\ \ y^i \frac{\partial}{\partial \mathrm{y}^k}\right]$$

For $\boldsymbol{\sigma} : \mathbb{E} \to \mathcal{E}$ a section of $\mathcal{E}$ $(\pi_{\mathcal{E}} \cdot \boldsymbol{\sigma} = \mathrm{Id}_{\mathbb{E}})$ and $\bar{\mathrm{u}} \in \mathrm{T}_{\bar{\mathrm{x}}}\mathbb{E}$ we have:

$$\nabla_{\bar{\mathrm{u}}}\, \boldsymbol{\sigma} := \mathrm{v}_{\boldsymbol{\gamma}} \cdot \mathrm{T}_{\bar{\mathrm{x}}}\, \boldsymbol{\sigma} \cdot \boldsymbol{\gamma}_{\boldsymbol{\sigma}(\bar{\mathrm{x}})} \cdot \bar{\mathrm{u}} \quad = \left[\frac{\partial \sigma^k}{\partial \mathrm{x}^j} - \gamma^k_{ij}(\bar{\mathrm{x}})\sigma^i\right] \bar{u}^j \frac{\partial}{\partial \mathrm{y}^k}$$

with $\mathrm{v}_{\boldsymbol{\gamma}} = \mathrm{Id} - \boldsymbol{\gamma} \cdot \mathrm{T}\pi_{\mathcal{E}}$

Solder form

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Solder form

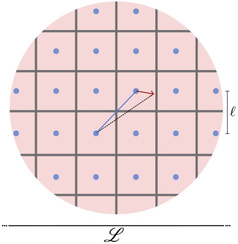
A solder form on $\mathcal{E} \xrightarrow{\pi_{\mathcal{E}}} \mathbb{E}$ is the data at each point $(\bar{\mathsf{x}}, \delta\mathsf{x}) \in \mathcal{E}_{\bar{\mathsf{x}}}$ of a linear injection

$$\boldsymbol{\vartheta}_{(\bar{\mathsf{x}},\, \delta\mathsf{x})} : \mathrm{T}_{\bar{\mathsf{x}}}\mathbb{E} \hookrightarrow \mathrm{T}_{(\bar{\mathsf{x}},\, \delta\mathsf{x})}\mathcal{E}$$

which is vertical (i.e. microscopic). In other words:

$$\forall (\bar{\mathsf{x}}, \delta\mathsf{x}) \in \mathcal{E}_{\bar{\mathsf{x}}} \qquad \mathrm{T}\pi_{\mathcal{E}} \circ \boldsymbol{\vartheta}_{(\bar{\mathsf{x}},\, \delta\mathsf{x})} = 0_{\mathrm{T}_{\bar{\mathsf{x}}}\mathbb{E}}$$

The pseudo-metric

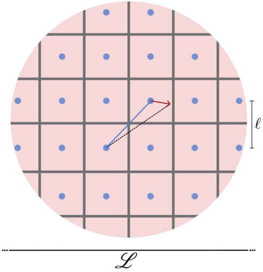


$$(\gamma)^{-1} = \mathrm{T}\pi_{\mathcal{E}} : \mathrm{Im}\,(\gamma) \longrightarrow \mathrm{T}\mathbb{E} \qquad (\boldsymbol{\vartheta})^{-1} : \mathrm{Im}\,(\boldsymbol{\vartheta}) \longrightarrow \mathrm{T}\mathbb{E}$$

$$\begin{aligned} \left\| \dashrightarrow \right\|_{\mathfrak{g}} &:= \left\| \rightsquigarrow + \rightsquigarrow \right\|_{\mathrm{g}} \\ &= \left\| \boldsymbol{\gamma}^{-1} \cdot \rightsquigarrow + \boldsymbol{\vartheta}^{-1} \cdot \rightsquigarrow \right\|_{\mathrm{g}} \\ &= \left\| \left(\mathrm{T}\pi_{\mathcal{E}} + \boldsymbol{\vartheta}^{-1} \cdot \mathrm{v}_{\boldsymbol{\gamma}} \right) \cdot \dashrightarrow \right\|_{\mathrm{g}} \\ &= \left\| \text{interpretation}\,(\dashrightarrow) \right\|_{\mathrm{g}} \end{aligned}$$

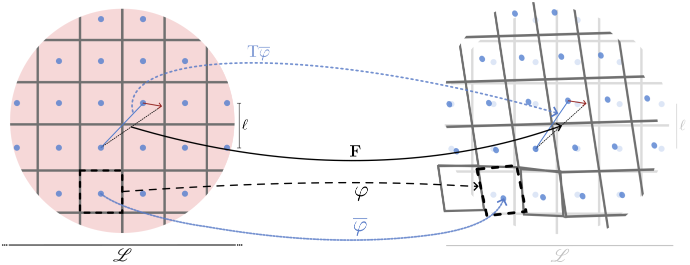
$$\mathfrak{g} := \text{interpretation}^{\mathrm{t}} \cdot \mathrm{g} \cdot \text{interpretation}$$

Kernel of the interpretation



$$
\begin{aligned}
\ker \mathfrak{g} &= \ker(\text{interpretation}) \\
&= \left\{ \mathrm{u} \in \mathrm{T}\mathcal{E} \;\middle|\; \mathrm{T}\pi_{\mathcal{E}}(\mathrm{u}) + \vartheta^{-1}\left(\mathrm{v}_{\gamma}(\mathrm{u})\right) = 0 \right\} \\
&= \left\{ \mathrm{u} \in \mathrm{T}\mathcal{E} \;\middle|\; \mathrm{v}_{\gamma}(\mathrm{u}) = -\vartheta\left(\mathrm{T}\pi_{\mathcal{E}}(\mathrm{u})\right) \right\} \\
&= \left\{ \gamma(\overline{\mathrm{u}}) - \boldsymbol{\vartheta}(\overline{\mathrm{u}}) \;\middle|\; \overline{\mathrm{u}} \in \mathrm{T}\mathbb{E} \right\} \\
&= \mathrm{Im}(\gamma - \boldsymbol{\vartheta})
\end{aligned}
$$

Placement map



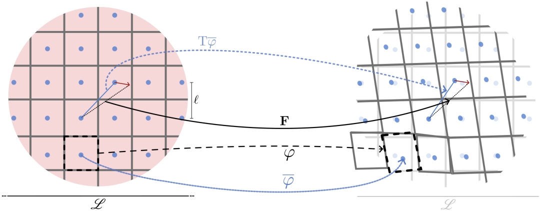
$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{\varphi} & \mathcal{E} \\ \pi_{\mathcal{B}} \downarrow & & \downarrow \pi_{\mathcal{E}} \\ \mathbb{B} & \xrightarrow{\overline{\varphi}} & \mathbb{E} \end{array}$$

$$\overline{\varphi} \circ \pi_{\mathcal{B}} = \pi_{\mathcal{E}} \circ \varphi$$

using overlines for projections:

$$\forall X \in \mathcal{B} \qquad \overline{\varphi}\left(\overline{X}\right) = \overline{\varphi(X)}$$

Pull-backs



$$\begin{array}{ccc} \mathrm{T}^\star\mathcal{B} & \xleftarrow{\quad \mathsf{F}^{\mathrm{t}} \quad} & \mathrm{T}^\star\mathcal{E} \\ \mathfrak{G} \uparrow & & \uparrow \mathfrak{g} \\ \mathrm{T}\mathcal{B} & \xrightarrow{\quad \mathsf{F} \quad} & \mathrm{T}\mathcal{E} \end{array}$$

Material pseudo-metric

$$\begin{array}{ccc} \mathrm{Im}\left(\Theta\right) & \xrightarrow{\quad \mathsf{F} \quad} & \mathrm{Im}\left(\boldsymbol{\vartheta}\right) \\ \Theta \uparrow & & \boldsymbol{\vartheta}^{-1} \downarrow \uparrow \boldsymbol{\vartheta} \\ \mathrm{T}\mathbb{B} & \xrightarrow{\quad \mathrm{T}\overline{\varphi} \quad} & \mathrm{T}\mathbb{E} \end{array}$$

Material solder form

$$\begin{array}{ccc} \mathrm{T}\mathcal{B} & \xrightarrow{\quad \mathsf{F} \quad} & \mathrm{T}\mathcal{E} \\ \Gamma \uparrow \downarrow \mathrm{T}\pi_{\mathcal{B}} & & \mathrm{T}\pi_{\mathcal{E}} \downarrow \uparrow \gamma \\ \mathrm{T}\mathbb{B} & \xrightarrow{\quad \mathrm{T}\overline{\varphi} \quad} & \mathrm{T}\mathbb{E} \end{array}$$

Material connection

Generalised Galilean invariants

Orbital invariants

$\mathsf{F} \mapsto \alpha(\mathsf{F})$ is frame indifferent iff there exists $\tilde{\alpha}$ such that $\alpha(\mathsf{F}) = \tilde{\alpha}\,(\mathfrak{G})$

- $F = \mathrm{T}\varphi : \mathrm{T}\mathcal{B} \longrightarrow \mathrm{T}\mathcal{E}$ — first order placement map
- $\mathfrak{G} = \mathsf{F}^{\mathrm{t}} \cdot \mathfrak{g} \cdot \mathsf{F}$ — material pseudo-metric

Generalised Galilean invariants

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Orbital invariants

$\mathsf{F} \mapsto \alpha(\mathsf{F})$ is frame indifferent iff there exists $\tilde{\alpha}$ such that

~~$\alpha(\mathsf{F}) = \tilde{\alpha}\,(\mathfrak{G})$~~ (only if micro-linear)

$\alpha(\mathsf{F}) = \tilde{\alpha}\,(\mathfrak{G}^{\mathrm{v}}_{\mathrm{v}}, \Theta, \Gamma)$

- $F = \mathrm{T}\varphi : \mathrm{T}\mathcal{B} \longrightarrow \mathrm{T}\mathcal{E}$ first order placement map
- $\mathfrak{G} = \mathsf{F}^{\mathrm{t}} \cdot \mathfrak{g} \cdot \mathsf{F}$ material pseudo-metric
- $\mathfrak{G}^{\mathrm{v}}_{\mathrm{v}}$ material micro-metric (*i.e.* microscopic Cauchy-Green)
- $\Theta = \mathsf{F}^{*}\boldsymbol{\vartheta} : \mathrm{T}\mathbb{B} \longrightarrow \mathrm{V}\mathcal{B}$ material solder form
- $\Gamma = \mathsf{F}^{*}\boldsymbol{\gamma} : \mathrm{T}\mathbb{B} \longrightarrow \mathrm{H}^{\Gamma}\mathcal{B}$ material connection

Generalised Galilean invariants

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Orbital invariants

F $\mapsto \alpha(\mathsf{F})$ is frame indifferent iff there exists $\tilde{\alpha}$ such that

~~$\alpha(\mathsf{F}) = \tilde{\alpha}\,(\mathfrak{G})$~~ (only if micro-linear)

~~$\alpha(\mathsf{F}) = \tilde{\alpha}\,(\mathfrak{G}^{\mathrm{v}}_{\mathrm{v}}, \Theta, \Gamma)$~~ (only if $\mathsf{F} = \mathrm{T}\varphi$)

$\alpha(\mathsf{F}) = \tilde{\alpha}\,(\mathfrak{G}^{\mathrm{v}}_{\mathrm{v}}, \Theta, \Gamma, \Gamma_{\mathrm{holo}})$

- $F : \mathrm{T}\mathcal{B} \longrightarrow \mathrm{T}\mathcal{E}$ **generalised** first order placement map
- $\mathfrak{G} = \mathsf{F}^{\mathrm{t}} \cdot \mathfrak{g} \cdot \mathsf{F}$ material pseudo-metric
- $\mathfrak{G}^{\mathrm{v}}_{\mathrm{v}}$ material micro-metric (*i.e.* microscopic Cauchy-Green)
- $\Theta = \mathsf{F}^{*}\boldsymbol{\vartheta} : \mathrm{T}\mathbb{B} \longrightarrow \mathrm{V}\mathcal{B}$ material solder form
- $\Gamma = \mathsf{F}^{*}\boldsymbol{\gamma} : \mathrm{T}\mathbb{B} \longrightarrow \mathrm{H}^{\Gamma}\mathcal{B}$ material connection
- $\Gamma_{\mathrm{holo}} = (\mathrm{T}\varphi)^{*}\,\boldsymbol{\gamma} : \mathrm{T}\mathbb{B} \longrightarrow \mathrm{H}^{\Gamma_{\mathrm{holo}}}\mathcal{B}$ holonomic material connection

Conclusion.

Thank you
four your attention

Results

- Mechanical construction: no thermodynamics, viscosity, etc.
- Generic vector bundles: dislocations, disclinations, meta-materials, etc.
- Obtained from general principles (Frame-indifference, interpretation preservation, etc.)
- Degenerated lengths: *pseudo*-metrics

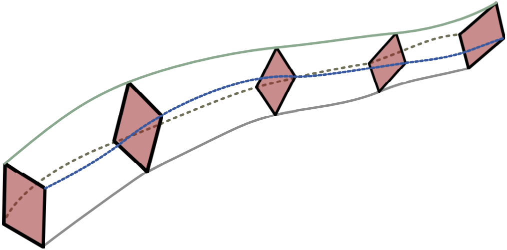
Results not shown

- Reduction to micromorphic media (with $\dim(\mathbb{B}) = 3$ and $\mathsf{F} = \mathrm{T}\varphi$)
- Reduction to beam models & measures of deformation
- Beam theory energy functionals

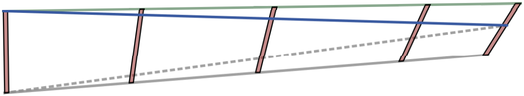
Future goals

- Static equilibrium equations (W.I.P)
- Dynamical equations
- Reduction to other models (Cosserat, etc.)

Example: The beam



Isometric projection of a micro-structured configuration of a beam (a 1×3 fibre bundle).



Cross-section "along" the macroscopic axis of the same micro-structured configuration.

Reduction to other models

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Inexctensible beams' standard measures of deformation:

$$\kappa : \mathrm{T}\mathbb{B} \longrightarrow \mathrm{Im}(\Theta) = \Gamma - \Gamma_{|t=0}$$

$$\varepsilon : \mathrm{T}\mathbb{B} \longrightarrow \mathrm{Im}(\Theta) = \Theta - \Theta_{|t=0}$$

$$\mathsf{G}_{\mathrm{v}}^{\mathrm{v}} : \mathrm{Im}(\Theta) \longrightarrow \mathrm{Im}^{\star}(\Theta) = \mathsf{F}^{\mathrm{t}} \cdot \mathsf{g}_{\mathrm{v}}^{\mathrm{v}} \cdot \mathsf{F}_{|\mathrm{V}\mathcal{B}} = \mathsf{F}^{\mathrm{t}} \cdot \vartheta^{-1^{\mathrm{t}}} \cdot \mathsf{g}_{\mathsf{eucl}} \cdot \vartheta^{-1} \cdot \mathsf{F}_{|\mathrm{V}\mathcal{B}}$$

Timoshenko–Ehrenfest

- **slices rigidity:** $\dot{\mathsf{G}}_{\mathrm{v}}^{\mathrm{v}} = 0$
- **no disclinaison:** $\mathsf{F} = \mathrm{T}\varphi$

$$\Longrightarrow \mathsf{F} = \begin{bmatrix} \widehat{\mathsf{F}}_{\mathrm{h}}^{\mathrm{h}} & 0 \\ \widehat{\mathsf{F}}_{\mathrm{h}}^{\mathrm{v}} & \vartheta \cdot \mathsf{Q} \cdot \vartheta^{-1} \end{bmatrix} \cdot \mathsf{F}_{|t=0}$$

with $\mathsf{Q}(\mathsf{x}) \in \mathrm{O}(\mathsf{g}_{\mathsf{eucl}}, \mathbb{E})$ for $\mathsf{x} \in \mathbb{E}$

Euler-Bernoulli

- **slices orthogonality:** $\varepsilon = 0$

+ **Timoshenko assumptions**

$$\Longrightarrow \mathsf{F} = \begin{bmatrix} \gamma \cdot \mathsf{Q} \cdot \mathrm{T}\pi_{\mathcal{B}} & 0 \\ \widehat{\mathsf{F}}_{\mathrm{h}}^{\mathrm{v}} & \vartheta \cdot \mathsf{Q} \cdot \vartheta^{-1} \end{bmatrix} \cdot \mathsf{F}_{|t=0}$$

with $\mathsf{Q}(\mathsf{x}) \in \mathrm{O}(\mathsf{g}_{\mathsf{eucl}}, \mathbb{E})$ for $\mathsf{x} \in \mathbb{E}$

Rigid bar

- **slices parallelism:** $\kappa = 0$

+ **Euler-Bernoulli assumptions**

$$\Longrightarrow \mathsf{F} = \begin{bmatrix} \gamma \cdot \mathrm{Q} \cdot \mathrm{T}\pi_{\mathcal{B}} & 0 \\ 0 & \vartheta \cdot \mathrm{Q} \cdot \vartheta^{-1} \end{bmatrix} \cdot \mathsf{F}_{|t=0}$$

with $\mathrm{Q} \in \mathrm{O}(\mathsf{g}_{\mathsf{eucl}}, \mathbb{E})$ *constant*

Conclusion.

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