Separation and stratification of the orbits of the representation of $\mathrm{SO}_3(\mathbb{R})$ on piezoelectricity tensors

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We compute a set of invariant separating the orbits of some representations of $\mathrm{SO}_3(\mathbb{R})$, and stratify the orbit space. We proceed by decomposing the representation $\mathcal{V}$ in a sequence of Seshadri slices.

Separating invariants

Let $\mathcal{F}$ be a set of invariant polynomials defined each on a G-stable subvariety in $\mathcal{V}$. We say that $\mathcal{F}$ separates the orbits in $\mathcal{V}$ if for any two points $x, y \in \mathcal{V}$ which are not in the same orbit, there is some $P \in \mathcal{F}$ defined at x and y such that $P(x) \neq P(y)$.

Piezoelectricity tensors

$\mathbb{P}\mathrm{iez}$ is the space of three order tensors verifying $\forall i, j, k \in \{1, 2, 3\}$, $P_{ijk} = P_{ikj}$.

$\dim(\mathbb{P}\mathrm{iez}) = 18$.
[Oli14] provides a generating set of polynomials of cardinal 495.
[Che+19] deduces a minimal separating set of cardinality 260.
It is difficult to stratify the orbit space with such a cardinality.

A first example: on symmetric traceless tensors of order three $\mathcal{H}_3$

A separating set on $\mathcal{H}_3$ [SB97]

$$\left\{ \begin{array}{lll} K_2 = \sum_{i,j,k} A_{ijk}^2 & K_4 = \sum_{i,j} B_{ij}^2 & K_6 = \sum_{i} C_i^2 \\ K_{10} = \sum_{i,j,k} A_{ijk} C_i C_j C_k & & K_{15} = \sum_{i,j,k,p,q} \varepsilon_{ijk} C_i B_{jp} C_p A_{kqr} C_q C_r \end{array} \right\}$$

The stratification of the orbit space is then obtained by computing the ideal of relations on subvarieties:

Example: the strata of points of isotropy class C_2

The strata $\overline{\Sigma_{\mathrm{C}_2}}$ is defined by the system

$$\left\{ \begin{array}{l} -2K_2^6 + 14K_2^4 K_4 - 6K_2^3 K_6 - 32K_2^2 K_4^2 + 12K_2 K_4 K_6 + 24K_4^3 + 9K_6^2 = 0 \\ -K_2^5 + 5K_2^3 K_4 - 6K_2^2 K_6 - 6K_2 K_4^2 + 9K_4 K_6 + 9K_{10} = 0 \end{array} \right.$$

1. Separating orbits with the Seshadri slice lemma
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3. Stratification of the orbit space

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Sechadri slice Lemma [CS05]

Let G be a real algebraic group acting on a geometrically irreducible variety $\mathcal{V}$. Take $\mathcal{S} \subset \mathcal{V}$ a geometrically irreducible subvariety and $\mathrm{N} < \mathrm{G}$ its normalizer. Assume that

- The closure of $\mathrm{G} \times \mathcal{S}$ is $\mathcal{V}$.
- There is an non empty Zariski open subset $\mathcal{Z} \subset \mathcal{S}$ such that for all $g \in \hat{\mathrm{G}}$ and $s \in \hat{\mathcal{Z}}$ satisfying $g(s) \in \hat{\mathcal{Z}}$, then there exists $h \in \hat{\mathrm{N}}$ such that $h(s) = g(s)$.

Then, the restriction of invariants to $\mathcal{S}$ gives a field isomorphism

$$\mathbb{R}(\mathcal{V})^{\mathrm{G}} \cong \mathbb{R}(\mathcal{S})^{\mathrm{N}}$$

Definition

The pair $(\mathcal{S}, \mathrm{N})$ is named a Seshadri slice.

Separating orbits with the Seshadri slice lemma

Proposition

Note $\mathcal{Z}$ the stable open subset satisfying the second condition of the Seshadri slice lemma and $\mathcal{F} \subset \mathbb{R}[\mathcal{S}]^{\mathrm{N}}$ separating the orbits in $\mathcal{Z}$. Then, $\tilde{\mathcal{F}} \subset \mathbb{R}(\mathcal{V})^{\mathrm{G}}$ separates orbits in $\tilde{\mathcal{Z}} = \mathrm{G} \times \mathcal{Z}$.

It remains a subvariety $\mathcal{Q} = \mathrm{G} \times (\mathcal{S} \setminus \mathcal{Z})$ where the orbits are not generated.

Strategy

$$
\begin{array}{ccccccccccccc}
\mathcal{V} & = & \mathcal{Q}_0 & \supset & \mathcal{Q}_1 & \supset & \dots & \supset & \mathcal{Q}_{n-1} & \supset & \mathcal{Q}_n \\
 & & | & \nearrow & | & \nearrow & & \nearrow & | & \nearrow & \\
 & & (\mathcal{S}_1, \mathrm{N}_1) & & (\mathcal{S}_2, \mathrm{N}_2) & & \dots & & (\mathcal{S}_n, \mathrm{N}_n) & & \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \\
\mathcal{V} & = & \tilde{\mathcal{Z}}_1 & \sqcup & \tilde{\mathcal{Z}}_2 & \sqcup & \dots & \sqcup & \tilde{\mathcal{Z}}_n & \sqcup & \mathcal{Q}_n
\end{array}
$$

where for each $1 \leq i \leq n$, $\mathcal{Q}_i$ is the variety non separated by the previous slice, namely $\mathcal{Q}_i = \mathcal{Q}_{i-1} \setminus \underbrace{(\mathrm{G} \times \mathcal{Z}_i)}_{\tilde{\mathcal{Z}}_i}$. For each $1 \leq i \leq n$, consider $\mathcal{F}_i$ a set of polynomials in $\mathbb{R}[\mathcal{S}_i]^{\mathrm{N}_i}$ separating orbits in $\mathcal{Z}_i$. Consider also a last set $\mathcal{F}_{n+1} \subset \mathbb{R}[\mathcal{Q}_n]^{\mathrm{G}}$ separating orbits in $\mathcal{Q}_n$. The union $\tilde{\mathcal{F}} = \bigcup_{i=1}^{n+1} \tilde{\mathcal{F}}_i$ is a set of rational functions separating orbits in the disjoint union $\mathcal{V} = \tilde{\mathcal{Z}}_1 \sqcup \dots \sqcup \tilde{\mathcal{Z}}_n \sqcup \mathcal{Q}_n$.

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Harmonic polynomials

Let d be an integer. The space of homogeneous harmonic polynomials of degree d is

$$\mathcal{H}_d := \left\{ P \in \mathbb{R}[U,V,W]_d \,\middle|\, \nabla(P) = \frac{\partial^2 P}{\partial U^2} + \frac{\partial^2 P}{\partial V^2} + \frac{\partial^2 P}{\partial W^2} = 0 \right\}$$

The spaces $\mathcal{H}_d$ are endowed with the action $\rho_d : \left\{ \begin{array}{ccc} \mathrm{SO}_3(\mathbb{R}) & \longrightarrow & \mathrm{GL}(\mathcal{H}_d) \\ g & \longmapsto & \{P \to P \circ g^{-1} \end{array} \right.$

Irreducible representations of $\mathrm{SO}_3(\mathbb{R})$

The irreducible representations of $\mathrm{SO}_3(\mathbb{R})$ are $\{(\rho_d, \mathcal{H}_d), d \in \mathbb{N}\}$.

Piezoelectricity tensors

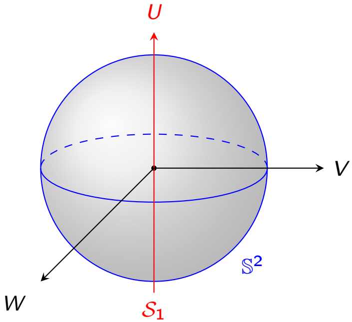
$$\begin{array}{ccc} \mathbb{P}\mathrm{iez} & \cong & \mathcal{H}_1 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3 \\ \rho & \sim & \rho_1 \oplus \rho_1 \oplus \rho_2 \oplus \rho_3 \end{array}$$

A slice with $\mathcal{H}_1$

$$\mathcal{S}_1 := \mathrm{Vect}(U) \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3 \subset \mathbb{P}\mathrm{iez}$$

Then, $(\mathcal{S}_1, \mathrm{N}_1)$ is a Seshadri slice with normalizer $\mathrm{N}_1 \cong \mathrm{O}_2(\mathbb{R})$:

$$\mathrm{N}_1 = \left\{ \left(\begin{array}{c|cc} \det(g) & 0 & 0 \\ \hline 0 & & \\ 0 & \multicolumn{2}{c}{g} \end{array} \right), \; g \in \mathrm{O}_2(\mathbb{R}) \right\}$$



The separated set

Here, we have $\mathcal{Z}_1 = \{s \in \mathcal{S}_1 \mid d_1 \neq 0\} \subset \mathcal{S}_1$.
The remaining not separated variety is thus $\mathcal{Q}_1 = \mathbb{P}\mathrm{iez} \setminus \mathrm{SO}_3(\mathbb{R}) \times \mathcal{Z}_1$.

Irreducible representations of $\mathrm{O}_2(\mathbb{R})$

The two dimensional irreducible representations of $\mathrm{O}_2(\mathbb{R})$ are $\rho_j : \mathrm{O}_2(\mathbb{R}) \longmapsto \mathrm{GL}(\mathcal{V}_j)$, $j \in \mathbb{N}^*$ with

$$\rho_j(g^+_{e^{i\theta}}) = \begin{pmatrix} \cos(j\theta) & -\sin(j\theta) \\ \sin(j\theta) & \cos(j\theta) \end{pmatrix} \quad \text{and} \quad \rho_j(g^-_{e^{i\theta}}) = \begin{pmatrix} \cos(j\theta) & \sin(j\theta) \\ \sin(j\theta) & -\cos(j\theta) \end{pmatrix}$$

The representation of N_1 on $\mathcal{S}_1$

The representation of $\mathrm{N}_1 \cong \mathrm{O}_2(\mathbb{R})$ on $\mathcal{S}_1$ is isomorphic to $\mathcal{V}^3_{-1} \oplus \mathcal{V}_0 \oplus \mathcal{V}^3_1 \oplus \mathcal{V}^2_2 \oplus \mathcal{V}_3$.

The spherical harmonic basis

For $1 \leq m \leq d$ note P^m_d the $(d,m)^{th}$ associated Legendre polynomial. The following set of functions of (r, θ, φ) is a polynomial basis for $\hat{\mathcal{H}}_d$:

$$\hat{\mathcal{B}}_d := \left\{ \begin{array}{rcll} Y^m_d & = & \mathrm{i}^{m+d}\, r^d\, e^{\mathrm{i}m\varphi}\, P^m_d(\cos(\theta)), & 1 \leq m \leq d \\ Y^{-m}_d & = & \mathrm{i}^{-m-d}\, r^d\, e^{-\mathrm{i}m\varphi}\, P^m_d(\cos(\theta)), & 1 \leq m \leq d \\ Y^0_d & = & r^d\, P^0_d(\cos(\theta)) & \end{array} \right\}$$

Then, the subspace generated by Y^m_d and Y^{-m}_d is stable by N_1, and it results:

The first separating set

Let $(d_1, d_2, d_3, t_1, y_1, y_{-1}, \dots, y_6, y_{-6})$ be coordinates on

$\mathcal{S}_1 \cong \mathcal{V}_{-1}^3 \oplus \mathcal{V}_0 \oplus \mathcal{V}_1^3 \oplus \mathcal{V}_2^2 \oplus \mathcal{V}_3$. For $1 \leq i \leq j \leq 6$, we note $p_{-ij} = y_{-i}^{\frac{a_i \vee a_j}{a_i}} y_j^{\frac{a_i \vee a_j}{a_j}}$.

A set separating orbits in $\mathcal{Z}_1$ [HJ24]

The following set in $\mathbb{R}[\mathcal{S}_1]^{\mathrm{N}_1}$ separates the orbits in $\mathcal{Z}_1$:

$$\mathcal{F}_1 = \left\{ \begin{array}{rcll} t_1 & & & \\ D_{1j} & = & d_1 d_j, & 1 \leq j \leq 3 \\ P_{ij} & = & \frac{1}{2}(p_{-ij} + p_{-ji}) & 1 \leq i \leq j \leq 6 \\ S_{ij1} & = & \mathrm{i}\left(p_{-ij} - p_{-ji}\right) d_1, & 1 \leq i < j \leq 6 \end{array} \right\}$$

NB: $\#\mathcal{F}_1 = 6^2 + 4 = 40$.

It remains to separate orbits in the variety $\mathcal{Q}_1 = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3$.

A second slice of same type

The set $\check{\mathcal{F}}_1$ separate orbits everywhere but in $\mathcal{Q}_1 = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3$. We identify the slice $(\mathcal{S}_1', \mathrm{N}_1)$ by the same method.

A set separating orbits in $\mathcal{Z}_1'$

The following set in $\mathbb{R}[\mathcal{S}_1']^{\mathrm{N}_1}$ separates the orbits in $\mathcal{Z}_1'$:

$$\mathcal{F}_1' = \left\{ \begin{array}{rcll} t_1' & & & \\ D_{2j}' & = & d_1 d_j, & 2 \leq j \leq 3 \\ P_{ij}' & = & \frac{1}{2}(p_{-ij} + p_{-ji}) & 2 \leq i \leq j \leq 6 \\ S_{ij1}' & = & \mathrm{i}\left(p_{-ij} - p_{-ji}\right) d_1, & 2 \leq i < j \leq 6 \end{array} \right\}$$

$\#\mathcal{F}_1' = 28$.

It remains to separate orbits in the variety $\mathcal{Q}_1' = \mathcal{H}_2 \oplus \mathcal{H}_3$.

Recall the isomorphism $\left\{\begin{array}{rcl} \mathrm{S}_3(\mathbb{R}) & \longrightarrow & \mathbb{R}_2[U,V,W] \\ S & \mapsto & \{x \mapsto x^t S x\} \end{array}\right.$ mapping $\mathcal{H}_2 \subset \mathbb{R}_2[U,V,W]$ to traceless matrices $\mathcal{A}$.

Proposition

Note $\mathcal{D} \subset \mathrm{S}_3(\mathbb{R})$ the subspace of diagonal matrices and $\mathcal{S}_2 = \mathcal{D} \oplus \mathcal{H}_3$. Its normalizer is $\mathrm{N}_2 = \mathrm{B}_3 \cap \mathrm{SO}_3(\mathbb{R})$. The pair $(\mathcal{S}_2, \mathrm{N}_2)$ is a Seshadri slice.

Then, $\mathcal{Z}_2 = (\mathcal{H}_2 \setminus \mathcal{D}^o) \oplus \mathcal{H}_3$, where $\mathcal{D}^o$ is the subspace of diagonal matrices with two identic coefficients.

The non separated variety

The remaining varitey is $\mathcal{Q}_2 = (\mathrm{SO}_3(\mathbb{R}) \times \mathcal{D}^o) \oplus \mathcal{H}_3 = \mathcal{A}^* \oplus \mathcal{H}_3$, where $\mathcal{A}^*$ is the space of traceless matrices with two identic eigenvalues.

The separating set on $\mathcal{Z}_2$

We endow $\mathcal{H}_3$ with the cubic harmonic basis $(a_1, a_2, a_3, b_1, b_2, b_3, c)$.
Note $[\lambda] = (\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)$.

A set separating orbits in $\mathcal{Z}_2$

$R_c = [\lambda]c$, $R_0 = b_1 b_2 b_3$, $R_1 = [\lambda](a_1 b_2 b_3 + a_2 b_1 b_3 + a_3 b_1 b_2)$,
$R_2 = [\lambda]^2 (b_1 a_2 a_3 + b_2 a_1 a_3 + b_3 a_1 a_2)$, $R_3 = [\lambda]^3 a_1 a_2 a_3$ and the twelve entries of the following matrix separates the orbits of N_2 in $\mathcal{Z}_2$.

$$R = \begin{pmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{pmatrix} \begin{pmatrix} \lambda_1 & a_1^2 & b_1^2 & [\lambda] a_1 b_1 \\ \lambda_2 & a_2^2 & b_2^2 & [\lambda] a_2 b_2 \\ \lambda_3 & a_3^2 & b_3^2 & [\lambda] a_3 b_3 \end{pmatrix}$$

NB: $\#\mathcal{F}_2 = 17$.

The non separated variety

The remaining varitey is $\mathcal{Q}_2 = (\mathrm{SO}_3(\mathbb{R}) \times \mathcal{D}^o) \oplus \mathcal{H}_3 = \mathcal{A}^* \oplus \mathcal{H}_3$, where $\mathcal{A}^*$ is the space of traceless matrices with two identic eigenvalues.

Proposition

Consider the matrix $M := \mathrm{Diag}(2, -1, -1)$, and $\mathcal{U}_3 \subset \mathcal{A}^*$ the vector line $\{t_1 M, t_1 \in \mathbb{R}\}$. Complete it by $\mathcal{S}_3 = \mathcal{U}_3 \oplus \mathcal{H}_3$. Then, $(\mathcal{S}_3, \mathrm{N}_3)$ is a Seshadri slice with normalizer $\mathrm{N}_3 \cong \mathrm{O}_2(\mathbb{R})$ given by

$$\mathrm{N}_3 = \left\{ \left(\begin{array}{c|cc} \det(g) & 0 & 0 \\ \hline 0 & & \\ 0 & & g \end{array} \right), \; g \in \mathrm{O}_2(\mathbb{R}) \right\}$$

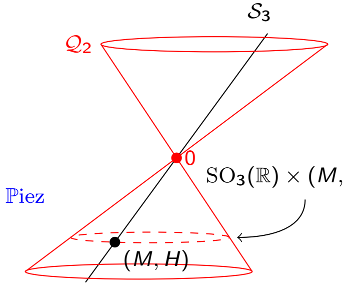


Figure: The Seshadri slice in $\mathcal{Q}_2$.

$$\mathcal{Z}_3 = \{t_1 \neq 0\} \text{ and } \mathcal{Q}_3 = \mathcal{H}_3$$

The separating set [HJ24]

The set $\mathcal{F}_3 \in \mathbb{R}[\mathcal{S}_3]^{\mathrm{N}_3}$ of cardinal 12 separates orbits in $\mathcal{Z}_3$:

$$\mathcal{F}_3 = \left\{ \begin{array}{c} t_1 \quad d_3^2 \quad S_{353} \quad S_{363} \quad S_{563} \\ P_{33} \quad P_{35} \quad P_{36} \quad P_{55} \quad P_{56} \quad P_{66} \\ T_{561} = (y_5 y_{-3}^2 - y_{-5} y_3^2)(y_6 y_{-3}^3 - y_{-6} y_3^3) \end{array} \right\}$$

At this step, orbits are separated everywhere but in $\mathcal{Q}_3 = \mathcal{H}_3$.
Here a separating set is provided in the literature:

A separating set on $\mathcal{Q}_3 = \mathcal{H}_3$ [SB97]

The following set of polynomials separates the orbits of $\mathrm{SO}_3(\mathbb{R})$ in the space of symmetric traceless tensors:

$$\mathcal{F}_4 = \left\{ \begin{array}{lll} K_2 = \sum_{i,j,k} A_{ijk}^2 & K_4 = \sum_{i,j} B_{ij}^2 & K_6 = \sum_{i} C_i^2 \\ K_{10} = \sum_{i,j,k} A_{ijk} C_i C_j C_k & & K_{15} = \sum_{i,j,k,p,q} \varepsilon_{ijk} C_i B_{jp} C_p A_{kqr} C_q C_r \end{array} \right\}$$

The final set

The union $\check{\mathcal{F}} = \check{\mathcal{F}}_1 \cup \check{\mathcal{F}}_1' \cup \check{\mathcal{F}}_2 \cup \check{\mathcal{F}}_3 \cup \check{\mathcal{F}}_4$ separates the orbits of the representation of $\mathrm{SO}_3(\mathbb{R})$ on $\mathbb{P}\mathrm{iez}$.

NB: We obtain the final cardinal $\#\check{\mathcal{F}}_4 = 40 + 28 + 17 + 12 + 5 = 102$.

To compare

[Oli14] provides a generating set of $\mathbb{R}[\mathbb{P}\mathrm{iez}]^{\mathrm{SO}_3(\mathbb{R})}$ of cardinality 495.
[Che+19] deduces, by another method, a minimal separating set of cardinality 260.

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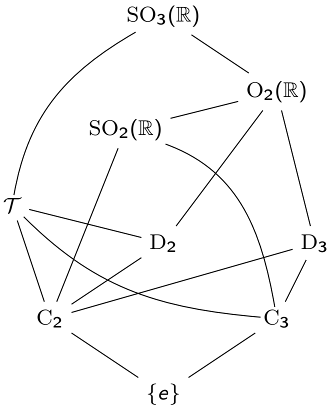
We aim to give polynomial equalities defining the strata of the orbit space $\mathbb{P}\mathrm{iez}/\mathrm{SO}_3(\mathbb{R})$. The set of isotropy classes is provided by clip operations [Azz23]. The induced decomposition of $\mathbb{P}\mathrm{iez}$ in disjoint subsets provides an efficient strategy to determine the isotropy group of a vector h:

Strategy

- The evaluation of some specific polynomials in $\check{\mathcal{F}}$ allows to determine which subset $\tilde{\mathcal{Z}}_i$ contains h.
- Then the set $\check{\mathcal{F}}_i$ determines the isotropy group in N_i.

Figure: Poset of isotropy classes for $\mathbb{P}\mathrm{iez}$



First case: $h \in \mathcal{Z}_1 \sqcup \mathcal{Z}_1'$

Proposition [HJ24]

If $D_{11} \in \mathcal{F}_1$ does not vanish at h, then $h \in \tilde{\mathcal{Z}}_1 = \mathbb{P}\mathrm{iez} \setminus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3$. In that case we have the following stratification:

- $h \in \overline{\Sigma_{\mathrm{SO}_2(\mathbb{R})}} \Leftrightarrow \forall 1 \leq i \leq 6,\ P_{ii}(h) = 0.$
- $h \in \overline{\Sigma_{\mathrm{C}_2}} \Leftrightarrow \forall i = 1, 2, 3, 6,\ P_{ii}(h) = 0.$
- $h \in \overline{\Sigma_{\mathrm{C}_3}} \Leftrightarrow \forall i = 1, 2, 3, 4, 5,\ P_{ii}(h) = 0.$

Suppose now that $D_{11}(h) = 0$. That is $h \notin \tilde{\mathcal{Z}}_1$ and:

Proposition [HJ24]

If $D'_{22} \in \mathcal{F}'_1$ does not vanish at h, then $h \in \tilde{\mathcal{Z}}'_1$. In that case we have the following stratification:

- $h \in \overline{\Sigma_{\mathrm{SO}_2(\mathbb{R})}} \Leftrightarrow \forall 2 \leq i \leq 6,\ P'_{ii}(h) = 0.$
- $h \in \overline{\Sigma_{\mathrm{C}_2}} \Leftrightarrow \forall i = 2, 3, 6,\ P'_{ii}(h) = 0.$
- $h \in \overline{\Sigma_{\mathrm{C}_3}} \Leftrightarrow \forall i = 2, 3, 4, 5,\ P'_{ii}(h) = 0.$

Assume that $D_{11}(h) = D'_{22}(h) = 0$. That is, $h \notin \tilde{\mathcal{Z}}_1 \sqcup \tilde{\mathcal{Z}}'_1$ and:

Proposition

If $[\lambda]^2$ does not vanish at h, then $h \in \tilde{\mathcal{Z}}_2$. In that case we have the following stratification:

- $h \in \overline{\Sigma_{\mathrm{D}_2}} \Leftrightarrow \begin{cases} R_{1,1}(h) = a_1^2 + a_2^2 + a_3^2 = 0 \\ R_{1,2}(h) = b_1^2 + b_2^2 + b_3^2 = 0 \end{cases}$
- $h \in \overline{\Sigma_{\mathrm{C}_2}} \Leftrightarrow \begin{cases} \forall\, 0 \leq j \leq 3,\ R_j(h) = 0 \\ \left(R_{1,1}R_{3,1} - R_{2,1}^2\right)(h) = 0 \\ \left(R_{1,2}R_{3,2} - R_{2,2}^2\right)(h) = 0 \\ \left([\lambda]^2 R_{1,1}R_{1,2} - R_{1,3}^2\right)(h) = 0 \end{cases}$

Third case: $h \in \mathcal{Z}_3$

Assume that $D_{11}(h) = D'_{22}(h) = [\lambda]^2(h) = 0$. That is, $h \notin \tilde{\mathcal{Z}}_1 \sqcup \tilde{\mathcal{Z}}'_1 \sqcup \tilde{\mathcal{Z}}_2$ and:

Proposition [HJ24]

If $t_1 \in \mathcal{F}_3$ does not vanish at h, then $h \in \tilde{\mathcal{Z}}_3$. In that case we have the following stratification:

- $h \in \overline{\Sigma_{\mathrm{C}_2}} \Leftrightarrow P_{33}(h) = P_{66}(h) = 0$.
- $h \in \overline{\Sigma_{\mathrm{C}_3}} \Leftrightarrow P_{33}(h) = P_{55}(h) = 0$.
- $h \in \overline{\Sigma_{\mathrm{D}_2}} \Leftrightarrow \begin{cases} P_{33}(h) = P_{66}(h) = 0 \\ D_{33}(h) = 0 \end{cases}$
- $h \in \overline{\Sigma_{\mathrm{D}_3}} \Leftrightarrow \begin{cases} P_{33}(h) = P_{55}(h) = 0 \\ D_{33}(h) = 0 \end{cases}$
- $h \in \overline{\Sigma_{\mathrm{SO}_2(\mathbb{R})}} \Leftrightarrow P_{33}(h) = P_{55}(h) = P_{66}(h) = 0$.
- $h \in \overline{\Sigma_{\mathrm{O}_2(\mathbb{R})}} \Leftrightarrow \begin{cases} P_{33}(h) = P_{55}(h) = P_{66}(h) = 0 \\ D_{33}(h) = 0 \end{cases}$

Assume that $D_{11}(h) = D'_{22}(h) = [\lambda]^2(h) = t_1(h) = 0$. That is, $h \in \mathcal{H}_3$. the representation on $\mathcal{H}_3$ is simple enough to compute directly the stratification:

Proposition

- The strata $\overline{\Sigma_{\mathrm{C}_2}}$ is defined by the system

$$\begin{cases} -2K_2^6 + 14K_2^4K_4 - 6K_2^3K_6 - 32K_2^2K_4^2 + 12K_2K_4K_6 + 24K_4^3 + 9K_6^2 = 0 \\ -K_2^5 + 5K_2^3K_4 - 6K_2^2K_6 - 6K_2K_4^2 + 9K_4K_6 + 9K_{10} = 0 \end{cases}$$

- The strata $\overline{\Sigma_{\mathrm{C}_3}}$ is defined by the system

$$\begin{cases} K_2^6 - 8K_2^4K_4 + 6K_2^3K_6 + 21K_2^2K_4^2 - 18K_2K_4K_6 - 18K_4^3 + 27K_6^2 = 0 \\ -K_2^5 + 5K_2^3K_4 + 3K_2^2K_6 - 6K_2K_4^2 - 18K_4K_6 + 27K_{10} = 0 \end{cases}$$

- The strata $\overline{\Sigma_{\mathcal{T}}}$ is defined by the system $\begin{cases} -K_2^2 + 3K_4 = 0 \\ K_6 = 0 \\ K_{10} = 0 \end{cases}$
- The strata $\overline{\Sigma_{\mathrm{D}_3}}$ is defined by the system $\begin{cases} -K_2^2 + 2K_4 = 0 \\ K_6 = 0 \\ K_{10} = 0 \end{cases}$
- The strata $\overline{\Sigma_{\mathrm{SO}_2(\mathbb{R})}}$ is defined by the system $\begin{cases} -11K_2^2 + 25K_4 = 0 \\ -8K_2^3 + 125K_6 = 0 \\ -32K_2^5 + 3125K_{10} = 0 \end{cases}$

To go further

- Provides inequalities defining each strata has a semialgebraic set.
  For the same reason, the sequencing seems to helps efficiently.
- Make the same on the Elasticity tensors: $\mathbb{E}\mathrm{la} = \mathcal{H}_0 \oplus \mathcal{H}_0 \oplus \mathcal{H}_2 \oplus \mathcal{H}_2 \oplus \mathcal{H}_4$.
- Extension to the action of $\mathrm{O}_3(\mathbb{C})$ on $\mathbb{P}\mathrm{iez}$?

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Polynomial or rational invariants?

- For $1 \leq i \leq n$, $\mathcal{F}_i$ is a set of polynomials defined on $\mathcal{S}_i$:

$$\mathcal{F}_i \subset \mathbb{R}[\mathcal{S}_i]^{\mathrm{N}_i}$$

- They correspond to rational functions on $\mathcal{Q}_{i-1}$ with singularities on $\mathcal{Q}_i$:

$$\tilde{\mathcal{F}}_i \subset \mathbb{R}(\mathcal{Q}_{i-1})^{\mathrm{G}}$$

- In our examples, $\mathcal{Q}_i$ is an irreducible subvariety in $\mathcal{Q}_{i-1}$, defined by the polynomial Q_i. Then, the denominators of functions $f \in \tilde{\mathcal{F}}_i$ is a power of Q_i.

Polynomial separating set

Note $\tilde{\mathcal{F}}_i = \left\{ \frac{P_1}{Q_i^{a_1}}, \dots, \frac{P_n}{Q_i^{a_n}} \right\} \subset \mathbb{R}(\mathcal{Q}_{i-1})^{\mathrm{G}}$. Then, the set composed with Q_i and the numerators $\check{\mathcal{F}}_i = \{Q_i, P_1, \dots, P_n\} \subset \mathbb{R}[\mathcal{Q}_{i-1}]^{\mathrm{G}}$ still separate orbits in $\tilde{\mathcal{Z}}_i$.

$\mathcal{Q}_2$ is not a vector space. Hence, polynomials of $\check{\mathcal{F}}_2 \subset \mathbb{R}[\mathcal{Q}_2]^{\mathrm{SO}_3(\mathbb{R})}$ cannot be extended algebraically on $\mathbb{P}\mathrm{iez}$.

A separating set

Let $\mathcal{F}$ be a set of invariant polynomials defined each on a G-stable subvariety in $\mathcal{V}$. We say that $\mathcal{F}$ separates the orbits in $\mathcal{V}$ if for any two points $x, y \in \mathcal{V}$ which are not in the same orbit, there is some $P \in \mathcal{F}$ defined at x and y such that $P(x) \neq P(y)$.

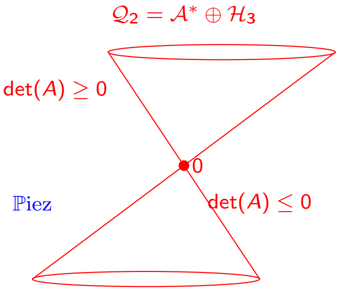

Figure: The variety $\mathcal{Q}_2$.