Un point de vue de mécanicien sur la structure des équations des poutres : différentes formulations du problème

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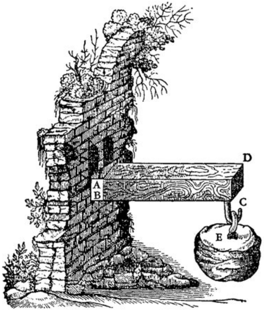
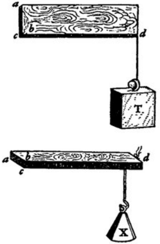
Principe of beams

⇒ Asymptotical model for **elongated** body $L \gg D$





Leonardo da Vinci, *Codex Madrid*, 1493.

Galileo, *Discorsi e Dimonstrazioni matematiche intorno a due scienze attenanti alla mecanica ed i movimenti locali*, 1638.

- **One dimensional** body $(s, t) \in [0, L] \times \mathbb{R}$
- **Immersion** of a 1D material manifold in a 3D Euclidean space
- Two methodologies
  - Asymptotical approach from a 3 dimensional analysis
  - Kinematical hypothesis: **rigid section**

Kinematical configuration of a Timoshenko beam-like structure

Material coordinates

$s \in [0, L]$ (ξ_1, ξ_2) local chart in $\mathcal{S}$

Configuration in the ambiant space

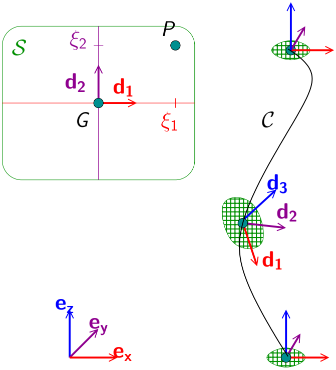
$$\varphi(s,t) = \mathbf{OG}$$

$$\mathbb{Q}(s,t) \in SO(3) \mid \mathbf{d_i}(s,t) = \mathbb{Q}\mathbf{e_i}$$

Place of a point P of the beam

$$(\xi_1, \xi_2, s, t) \rightarrow \mathbf{OP} = \mathbf{OG} + \mathbf{GP}$$

$$\mathbf{GP} = \xi_1 \mathbf{d_1} + \xi_2 \mathbf{d_2}$$



E.Cosserat, F.Cosserat, *Théorie des Corps déformables*, 1909
S.P.Timoshenko, J.C.Gere, *Theory of Elastic Stability*, 1961

$\mathcal{S}$ is **rigidly** transformed

$\mathbf{d_3}$ is **not** tangent to $\mathcal{C}$

G is the center of mass of $\mathcal{S}$

$\{\mathbf{d_i}\}$ is an inertial principal basis

Rigid transformation of the section $\mathcal{S}$

Curvature $\boldsymbol{\kappa}$

$$\frac{\partial \mathbf{d_i}}{\partial s} = \boldsymbol{\kappa} \wedge \mathbf{d_i}$$

$$\boldsymbol{\kappa} = \mathrm{axial}(\mathbb{Q}^T \frac{\partial \mathbb{Q}}{\partial s})$$

$$\mathbb{Q}^T \frac{\partial \mathbb{Q}}{\partial s} \in T_{Id} SO(3)$$

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Spin $\boldsymbol{\omega}$

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No commutation

$$\frac{\partial \boldsymbol{\kappa}}{\partial t} - \frac{\partial \boldsymbol{\omega}}{\partial s} = \boldsymbol{\omega} \wedge \boldsymbol{\kappa}$$

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Application for (time) derivation of a vector

$$\mathbf{u}(t) = u_i(t)\, \mathbf{d_i}(t)$$

$$\begin{aligned}
\frac{\partial \mathbf{u}}{\partial t} &= \frac{\partial u_i}{\partial t}\mathbf{d_i} + u_i \frac{\partial \mathbf{d_i}}{\partial t} \\
&= \frac{\partial u_i}{\partial t}\mathbf{d_i} + u_i\left(\boldsymbol{\omega} \wedge \mathbf{d_i}\right) \\
&= \frac{\partial u_i}{\partial t}\mathbf{d_i} + \boldsymbol{\omega} \wedge \left(u_i \mathbf{d_i}\right)
\end{aligned}$$

$$\frac{\partial \mathbf{u}}{\partial t} = \frac{\partial u_i}{\partial t}\mathbf{d_i} + \boldsymbol{\omega} \wedge \mathbf{u}$$

$$\frac{\partial \mathbf{u}}{\partial t} \neq \frac{\partial}{\partial t}\begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{bmatrix}$$

$\boldsymbol{\omega}$ plays the role of a linear connection

Application for (time) derivation of a quadratic form

Derivation of a vector

$$\frac{\partial \mathbf{u}}{\partial t} = \frac{\partial u_i}{\partial t}\mathbf{d_i} + \boldsymbol{\omega} \wedge \mathbf{u}$$

$$f = \frac{1}{2}\mathbf{u}\mathbb{X}\mathbf{u}, \qquad \mathbb{X}^T = \mathbb{X}$$

$$\frac{\partial f}{\partial t} = \frac{\partial u_i}{\partial t}\mathbb{X}_{ij}u_j$$

$$= \left(\frac{\partial \mathbf{u}}{\partial t} - \boldsymbol{\omega} \wedge \mathbf{u}\right) \cdot (\mathbb{X}\mathbf{u})$$

$$\frac{\partial f}{\partial t} = \frac{\partial \mathbf{u}}{\partial t}\mathbb{X}\mathbf{u} - (\mathbf{u} \wedge (\mathbb{X}\mathbf{u})) \cdot \boldsymbol{\omega}$$

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\mathbf{u}\mathbb{X}\mathbf{u}\right) \neq \frac{\partial \mathbf{u}}{\partial t}\mathbb{X}\mathbf{u}$$

$\frac{\partial \mathbf{u}}{\partial t} - \boldsymbol{\omega} \wedge \mathbf{u}$ is a corotational derivative

Energy and associated moments

Kinetic energy density

$$T(\mathbf{v},\boldsymbol{\omega}) = \frac{1}{2}\left(\mathbf{v}\mathbb{A}\mathbf{v} + \boldsymbol{\omega}\mathbb{J}\boldsymbol{\omega}\right)$$

$$\mathbb{A} = \begin{bmatrix} \rho A & 0 & 0 \\ 0 & \rho A & 0 \\ 0 & 0 & \rho A \end{bmatrix} \quad \mathbb{J} = \begin{bmatrix} \rho I_1 & 0 & 0 \\ 0 & \rho I_2 & 0 \\ 0 & 0 & \rho I_3 \end{bmatrix}$$

Deformation energy density

$$U(\boldsymbol{\varepsilon},\boldsymbol{\kappa}) = \frac{1}{2}\left(\boldsymbol{\varepsilon}\mathbb{G}\boldsymbol{\varepsilon} + \boldsymbol{\kappa}\mathbb{H}\boldsymbol{\kappa}\right)$$

$$\mathbb{G} = \begin{bmatrix} GA & 0 & 0 \\ 0 & GA & 0 \\ 0 & 0 & EA \end{bmatrix} \quad \mathbb{H} = \begin{bmatrix} EI_1 & 0 & 0 \\ 0 & EI_2 & 0 \\ 0 & 0 & GI_3 \end{bmatrix}$$

Momentums

$$\mathbb{A}\mathbf{v} := \frac{\partial T}{\partial \mathbf{v}} \qquad \mathbb{J}\boldsymbol{\omega} := \frac{\partial T}{\partial \boldsymbol{\omega}}$$

Forces and torques

$$\mathrm{N} := \frac{\partial U}{\partial \boldsymbol{\varepsilon}} = \mathbb{G}\boldsymbol{\varepsilon} \qquad \mathrm{M} := \frac{\partial U}{\partial \boldsymbol{\kappa}} = \mathbb{H}\boldsymbol{\kappa}$$

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$$\mathbf{v} := \frac{\partial \varphi}{\partial t} \qquad \boldsymbol{\omega} := \mathrm{axial}(\mathbb{Q}^T\frac{\partial \mathbb{Q}}{\partial t}) \qquad \boldsymbol{\varepsilon} := \frac{\partial \varphi}{\partial s} - \mathbf{d}_3 \qquad \boldsymbol{\kappa} := \mathrm{axial}(\mathbb{Q}^T\frac{\partial \mathbb{Q}}{\partial s})$$

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$$\text{Lagrangian} \quad \mathcal{L}(\mathbf{v},\boldsymbol{\omega},\boldsymbol{\varepsilon},\boldsymbol{\kappa}) := \int_0^L \frac{1}{2}\mathbf{v}\mathbb{A}\mathbf{v} + \frac{1}{2}\boldsymbol{\omega}\mathbb{J}\boldsymbol{\omega} - \frac{1}{2}\boldsymbol{\varepsilon}\mathbb{G}\boldsymbol{\varepsilon} - \frac{1}{2}\boldsymbol{\kappa}\mathbb{H}\boldsymbol{\kappa} \ \ \mathrm{d}s$$

$$\text{Hamiltonian} \quad \mathcal{H}(\mathbf{v},\boldsymbol{\omega},\boldsymbol{\varepsilon},\boldsymbol{\kappa}) := \int_0^L \frac{1}{2}\mathbf{v}\mathbb{A}\mathbf{v} + \frac{1}{2}\boldsymbol{\omega}\mathbb{J}\boldsymbol{\omega} + \frac{1}{2}\boldsymbol{\varepsilon}\mathbb{G}\boldsymbol{\varepsilon} + \frac{1}{2}\boldsymbol{\kappa}\mathbb{H}\boldsymbol{\kappa} \ \ \mathrm{d}s$$

Equilibrium

$$\frac{\partial \mathsf{N}}{\partial s} = \frac{\partial\, \mathbb{A}\mathbf{v}}{\partial t} \qquad (1)$$

$$\frac{\partial \mathsf{M}}{\partial s} + \frac{\partial \boldsymbol{\varphi}}{\partial s} \wedge \mathsf{N} = \frac{\partial\, \mathbb{J}\boldsymbol{\omega}}{\partial t} \qquad (2)$$

$$\mathsf{N} := \mathbb{G}\boldsymbol{\varepsilon}$$

$$\mathsf{M} := \mathbb{H}\boldsymbol{\kappa}$$

$$\mathbf{v} := \frac{\partial \boldsymbol{\varphi}}{\partial t} \qquad \boldsymbol{\omega} := \mathrm{axial}(\mathbb{Q}^T \frac{\partial \mathbb{Q}}{\partial t}) \qquad \boldsymbol{\varepsilon} := \frac{\partial \boldsymbol{\varphi}}{\partial s} - \mathbf{d}_3 \qquad \boldsymbol{\kappa} := \mathrm{axial}(\mathbb{Q}^T \frac{\partial \mathbb{Q}}{\partial s})$$

Equilibrium

$$\frac{\partial\ \mathbb{G}\boldsymbol{\varepsilon}}{\partial s} = \frac{\partial\ \mathbb{A}\mathbf{v}}{\partial t} \qquad (1)$$

$$\frac{\partial\ \mathbb{H}\boldsymbol{\kappa}}{\partial s} + (\boldsymbol{\varepsilon} + \mathbf{d}_3) \wedge \mathbb{G}\boldsymbol{\varepsilon} = \frac{\partial\ \mathbb{J}\boldsymbol{\omega}}{\partial t} \qquad (2)$$

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- $\boldsymbol{\varphi}(s,t)$ and $\mathbb{Q}(s,t)$ are not used as explicit unknowns
- The unknowns are 4 vectors $\mathbf{v}$, $\boldsymbol{\omega}$, $\boldsymbol{\kappa}$, $\boldsymbol{\varepsilon}$.
- Ill-posed problem with 12 unknown components and 6 scalar equations
- first order, non-linear, partial differential equations.

Equilibrium

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Additional *closure* relations

$$\frac{\partial \boldsymbol{\varepsilon}}{\partial t} = \frac{\partial \mathbf{v}}{\partial s} - \boldsymbol{\omega} \wedge \mathbf{d}_3 \qquad \frac{\partial \boldsymbol{\kappa}}{\partial t} = \frac{\partial \boldsymbol{\omega}}{\partial s} + \boldsymbol{\omega} \wedge \boldsymbol{\kappa}$$

obtained from

$$\frac{\partial}{\partial t}\frac{\partial \boldsymbol{\varphi}}{\partial s} = \frac{\partial}{\partial s}\frac{\partial \boldsymbol{\varphi}}{\partial t} \qquad \frac{\partial}{\partial t}\frac{\partial \mathbf{d}_i}{\partial s} = \frac{\partial}{\partial s}\frac{\partial \mathbf{d}_i}{\partial t}$$

Two systems

First system: unknowns $\mathbf{v}$, $\boldsymbol{\omega}$, $\boldsymbol{\varepsilon}$, $\boldsymbol{\kappa}$

$$\frac{\partial\, \mathbb{G}\boldsymbol{\varepsilon}}{\partial s} = \frac{\partial\, \mathbb{A}\mathbf{v}}{\partial t} \tag{3}$$

$$\frac{\partial\, \mathbb{H}\boldsymbol{\kappa}}{\partial s} + (\boldsymbol{\varepsilon} + \mathbf{d}_3) \wedge (\mathbb{G}\boldsymbol{\varepsilon}) = \frac{\partial\, \mathbb{J}\boldsymbol{\omega}}{\partial t} \tag{4}$$

$$\frac{\partial\, \mathbf{v}}{\partial s} - \boldsymbol{\omega} \wedge \mathbf{d}_3 = \frac{\partial\, \boldsymbol{\varepsilon}}{\partial t} \tag{5}$$

$$\frac{\partial\, \boldsymbol{\omega}}{\partial s} + \boldsymbol{\omega} \wedge \boldsymbol{\kappa} = \frac{\partial\, \boldsymbol{\kappa}}{\partial t} \tag{6}$$

Second system: unknowns $\mathbb{Q}$, $\mathbf{d}_i$ and φ

$$\boldsymbol{\omega} := \mathrm{axial}(\mathbb{Q}^T \frac{\partial \mathbb{Q}}{\partial t}) \qquad \boldsymbol{\kappa} := \mathrm{axial}(\mathbb{Q}^T \frac{\partial \mathbb{Q}}{\partial s}) \quad \| \quad \boldsymbol{\varepsilon} := \frac{\partial \varphi}{\partial s} - \mathbf{d}_3 \qquad \mathbf{v} := \frac{\partial \varphi}{\partial t}$$

- One second order PDE $\Rightarrow$ Two first order PDE

!! Initial and boundary conditions

Numerical formulation: an algebraic point of view

$$\underbrace{\begin{bmatrix} 0 & 0 & \mathbb{G} & 0 \\ 0 & 0 & 0 & \mathbb{H} \\ \mathbb{G} & 0 & 0 & 0 \\ 0 & \mathbb{H} & 0 & 0 \end{bmatrix}}\begin{bmatrix} v' \\ \omega' \\ \varepsilon' \\ \kappa' \end{bmatrix} + \underbrace{\begin{bmatrix} -\mathbb{W}\mathbb{A} & 0 & \mathbb{K}\mathbb{G} & 0 \\ 0 & -\mathbb{W}\mathbb{J} & \mathbb{E}\mathbb{G} & \mathbb{K}\mathbb{H} \\ \mathbb{G}\mathbb{K} & \mathbb{G}\mathbb{E} & -\mathbb{G}\mathbb{W} & 0 \\ 0 & \mathbb{H}\mathbb{K} & 0 & 0 \end{bmatrix}}\begin{bmatrix} v \\ \omega \\ \varepsilon \\ \kappa \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbb{A} & 0 & 0 & 0 \\ 0 & \mathbb{J} & 0 & 0 \\ 0 & 0 & \mathbb{G} & 0 \\ 0 & 0 & 0 & \mathbb{H} \end{bmatrix}}\begin{bmatrix} \dot{v} \\ \dot{\omega} \\ \dot{\varepsilon} \\ \dot{\kappa} \end{bmatrix}$$

skew-matrices (induce non-linearity)

diagonal matrices

$$\mathbb{W} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

$$\mathbb{K} = \begin{bmatrix} 0 & -\kappa_3 & \kappa_2 \\ \kappa_3 & 0 & -\kappa_1 \\ -\kappa_2 & \kappa_1 & 0 \end{bmatrix}$$

$$\mathbb{E} = \begin{bmatrix} 0 & -(\varepsilon_3 + 1) & \varepsilon_2 \\ \varepsilon_3 + 1 & 0 & -\varepsilon_1 \\ -\varepsilon_2 & \varepsilon_1 & 0 \end{bmatrix}$$

$$\mathbb{A} = \begin{bmatrix} \rho A & 0 & 0 \\ 0 & \rho A & 0 \\ 0 & 0 & \rho A \end{bmatrix} \qquad \mathbb{J} = \begin{bmatrix} \rho I_1 & 0 & 0 \\ 0 & \rho I_2 & 0 \\ 0 & 0 & \rho I_3 \end{bmatrix}$$

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remark $(\mathbb{K}\mathbb{G})^T = -\mathbb{G}\mathbb{K}$ $\quad (\mathbb{E}\mathbb{G})^T = -\mathbb{G}\mathbb{E}$ $\quad (\mathbb{K}\mathbb{H})^T = -\mathbb{H}\mathbb{K}$

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$$\underbrace{\begin{bmatrix} 0 & 0 & \mathbb{G} & 0 \\ 0 & 0 & 0 & \mathbb{H} \\ \mathbb{G} & 0 & 0 & 0 \\ 0 & \mathbb{H} & 0 & 0 \end{bmatrix}}_{symmetric} \begin{bmatrix} v' \\ \omega' \\ \varepsilon' \\ \kappa' \end{bmatrix} + \underbrace{\begin{bmatrix} -\mathbb{W}\mathbb{A} & 0 & \mathbb{K}\mathbb{G} & 0 \\ 0 & -\mathbb{W}\mathbb{J} & \mathbb{E}\mathbb{G} & \mathbb{K}\mathbb{H} \\ \mathbb{G}\mathbb{K} & \mathbb{G}\mathbb{E} & -\mathbb{G}\mathbb{W} & 0 \\ 0 & \mathbb{H}\mathbb{K} & 0 & 0 \end{bmatrix}}_{skew-symmetric} \begin{bmatrix} v \\ \omega \\ \varepsilon \\ \kappa \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbb{A} & 0 & 0 & 0 \\ 0 & \mathbb{J} & 0 & 0 \\ 0 & 0 & \mathbb{G} & 0 \\ 0 & 0 & 0 & \mathbb{H} \end{bmatrix}}_{diagonal} \begin{bmatrix} \dot{v} \\ \dot{\omega} \\ \dot{\varepsilon} \\ \dot{\kappa} \end{bmatrix}$$

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remark $(\mathbb{K}\mathbb{G})^T = -\mathbb{G}\mathbb{K}$ $(\mathbb{E}\mathbb{G})^T = -\mathbb{G}\mathbb{E}$ $(\mathbb{K}\mathbb{H})^T = -\mathbb{H}\mathbb{K}$

Obstruction

$$\begin{bmatrix} 0 & 0 & \mathbb{G} & 0 \\ 0 & 0 & 0 & \mathbb{H} \\ \mathbb{G} & 0 & 0 & 0 \\ 0 & \mathbb{H} & 0 & 0 \end{bmatrix} \begin{bmatrix} v' \\ \omega' \\ \varepsilon' \\ \kappa' \end{bmatrix} + \begin{bmatrix} -\mathbb{W}\mathbb{A} & 0 & \mathbb{K}\mathbb{G} & 0 \\ 0 & -\mathbb{W}\mathbb{J} & \mathbb{E}\mathbb{G} & \mathbb{K}\mathbb{H} \\ \mathbb{G}\mathbb{K} & \mathbb{G}\mathbb{E} & -\mathbb{G}\mathbb{W} & 0 \\ 0 & \mathbb{H}\mathbb{K} & 0 & 0 \end{bmatrix} \begin{bmatrix} v \\ \omega \\ \varepsilon \\ \kappa \end{bmatrix} = \begin{bmatrix} \mathbb{A} & 0 & 0 & 0 \\ 0 & \mathbb{J} & 0 & 0 \\ 0 & 0 & \mathbb{G} & 0 \\ 0 & 0 & 0 & \mathbb{H} \end{bmatrix} \begin{bmatrix} \dot{v} \\ \dot{\omega} \\ \dot{\varepsilon} \\ \dot{\kappa} \end{bmatrix}$$

Obstruction :

$$\mathbb{W}\mathbb{A} = \begin{bmatrix} 0 & -\rho A\omega_3 & \rho A\omega_2 \\ \rho A\omega_3 & 0 & -\rho A\omega_1 \\ -\rho A\omega_2 & \rho A\omega_1 & 0 \end{bmatrix} \quad \textit{ok}$$

$$\mathbb{W}\mathbb{J} = \begin{bmatrix} 0 & -\rho I_2\omega_3 & \rho I_3\omega_2 \\ \rho I_1\omega_3 & 0 & -\rho I_3\omega_1 \\ -\rho I_1\omega_2 & \rho I_2\omega_1 & 0 \end{bmatrix} \quad \textit{not ok}$$

$$\mathbb{G}\mathbb{W} = \begin{bmatrix} 0 & -EI_1\omega_3 & EI_1\omega_2 \\ EI_2\omega_3 & 0 & -EI_2\omega_1 \\ -GI_3\omega_2 & GI_3\omega_1 & 0 \end{bmatrix} \quad \textit{not ok}$$

Dissolution

- Static problem
- Plane motion
- Infinitesimal vibrations ($\|\boldsymbol{\omega}\| \ll 1$)

Difficulties

- 3D-dynamical problems

Limits

$$\begin{bmatrix} 0 & 0 & \mathbb{G} & 0 \\ 0 & 0 & 0 & \mathbb{H} \\ \mathbb{G} & 0 & 0 & 0 \\ 0 & \mathbb{H} & 0 & 0 \end{bmatrix} \begin{bmatrix} v' \\ \omega' \\ \varepsilon' \\ \kappa' \end{bmatrix} + \begin{bmatrix} -\mathbb{W}\mathbb{A} & 0 & \mathbb{K}\mathbb{G} & 0 \\ 0 & -\mathbb{W}\mathbb{J} & \mathbb{E}\mathbb{G} & \mathbb{K}\mathbb{H} \\ \mathbb{G}\mathbb{K} & \mathbb{G}\mathbb{E} & -\mathbb{G}\mathbb{W} & 0 \\ 0 & \mathbb{H}\mathbb{K} & 0 & 0 \end{bmatrix} \begin{bmatrix} v \\ \omega \\ \varepsilon \\ \kappa \end{bmatrix} = \begin{bmatrix} \mathbb{A} & 0 & 0 & 0 \\ 0 & \mathbb{J} & 0 & 0 \\ 0 & 0 & \mathbb{G} & 0 \\ 0 & 0 & 0 & \mathbb{H} \end{bmatrix} \begin{bmatrix} \dot{v} \\ \dot{\omega} \\ \dot{\varepsilon} \\ \dot{\kappa} \end{bmatrix}$$

- Problem have a more complex structure for 3D-dynamical configurations
- This problem determines *components* of vectors $\{v_i, \omega_i, \varepsilon_i, \kappa_i\}$ in the moving frame $\{\mathbf{d}_i(s,t)\}$. However the directors $\{\mathbf{d}_i(s,t)\}$ are still unknowns.
- Two alternatives to determine φ (in fact $\{\varphi_i\}$)

$$\boldsymbol{\varepsilon} := \frac{\partial \boldsymbol{\varphi}}{\partial s} - \mathbf{d}_3 \qquad \mathbf{v} := \frac{\partial \boldsymbol{\varphi}}{\partial t}$$

- Two alternatives to determine $\mathbb{Q}$ (and then $\mathbf{d_i}$)

$$\boldsymbol{\kappa} := \mathrm{axial}(\mathbb{Q}^T \frac{\partial \mathbb{Q}}{\partial s}) \qquad \boldsymbol{\omega} := \mathrm{axial}(\mathbb{Q}^T \frac{\partial \mathbb{Q}}{\partial t})$$

Space or time integration ?

Weak and strong formulation

Lagrangian density $\ell(\mathbf{v}, \boldsymbol{\omega}, \boldsymbol{\varepsilon}, \boldsymbol{\kappa}) := \frac{1}{2}\mathbf{v}\mathbb{A}\mathbf{v} + \frac{1}{2}\boldsymbol{\omega}\mathbb{J}\boldsymbol{\omega} - \frac{1}{2}\boldsymbol{\varepsilon}\mathbb{G}\boldsymbol{\varepsilon} - \frac{1}{2}\boldsymbol{\kappa}\mathbb{H}\boldsymbol{\kappa}$

Action $\mathcal{S}$ $\quad \mathcal{S} = \int_{t_1}^{t_2}\int_0^L \ell \; \mathrm{d}s \; \mathrm{d}t$

Least action $\delta\mathcal{S} = 0$ $\quad \int_{t_1}^{t_2}\int_0^L \delta\ell \; \mathrm{d}s \; \mathrm{d}t = 0$

Weak and strong formulation

Lagrangian density $\ell(\mathbf{v}, \boldsymbol{\omega}, \boldsymbol{\varepsilon}, \boldsymbol{\kappa}) := \frac{1}{2}\mathbf{v}\mathbb{A}\mathbf{v} + \frac{1}{2}\boldsymbol{\omega}\mathbb{J}\boldsymbol{\omega} - \frac{1}{2}\boldsymbol{\varepsilon}\mathbb{G}\boldsymbol{\varepsilon} - \frac{1}{2}\boldsymbol{\kappa}\mathbb{H}\boldsymbol{\kappa}$

Action $\mathcal{S}$: $\mathcal{S} = \int_{t_1}^{t_2} \int_0^L \ell \; \mathrm{d}s \; \mathrm{d}t$

Least action $\delta\mathcal{S} = 0$: $\int_{t_1}^{t_2} \int_0^L \delta\ell \; \mathrm{d}s \; \mathrm{d}t = 0$

Tools

Independent: $\delta \frac{\partial \mathbf{u}}{\partial t} = \frac{\partial \, \delta\mathbf{u}}{\partial t}$ $\quad \delta \frac{\partial \mathbf{u}}{\partial s} = \frac{\partial \, \delta\mathbf{u}}{\partial s}$

Compatible: $\delta\boldsymbol{\omega} = \frac{\partial \, \delta\boldsymbol{\theta}}{\partial t} + \delta\boldsymbol{\theta} \wedge \boldsymbol{\omega}$ $\quad \delta\boldsymbol{\kappa} = \frac{\partial \, \delta\boldsymbol{\theta}}{\partial s} + \delta\boldsymbol{\theta} \wedge \boldsymbol{\kappa}$

Corotational: $\delta\left(\frac{1}{2}\mathbf{u}\mathbb{X}\mathbf{u}\right) = (\delta\mathbf{u} - \delta\boldsymbol{\theta} \wedge \mathbf{u})\,\mathbb{X}\mathbf{u}$ $\quad \delta\boldsymbol{\theta} = \mathrm{axial}(\mathbb{Q}^T \delta\mathbb{Q})$

Weak and strong formulation

Lagrangian density $\ell(\mathbf{v}, \boldsymbol{\omega}, \boldsymbol{\varepsilon}, \boldsymbol{\kappa}) := \frac{1}{2}\mathbf{v}\mathbb{A}\mathbf{v} + \frac{1}{2}\boldsymbol{\omega}\mathbb{J}\boldsymbol{\omega} - \frac{1}{2}\boldsymbol{\varepsilon}\mathbb{G}\boldsymbol{\varepsilon} - \frac{1}{2}\boldsymbol{\kappa}\mathbb{H}\boldsymbol{\kappa}$

Action $\mathcal{S}$ $\quad \mathcal{S} = \int_{t_1}^{t_2}\int_0^L \ell \ \mathsf{d}s\ \mathsf{d}t$

Least action $\delta\mathcal{S} = 0$ $\quad \int_{t_1}^{t_2}\int_0^L \delta\ell \ \mathsf{d}s\ \mathsf{d}t = 0$

Tools

Independent $\quad \delta\frac{\partial \mathbf{u}}{\partial t} = \frac{\partial\,\delta\mathbf{u}}{\partial t} \qquad \delta\frac{\partial \mathbf{u}}{\partial s} = \frac{\partial\,\delta\mathbf{u}}{\partial s}$

Compatible $\quad \delta\boldsymbol{\omega} = \frac{\partial\,\delta\boldsymbol{\theta}}{\partial t} + \delta\boldsymbol{\theta}\wedge\boldsymbol{\omega} \qquad \delta\boldsymbol{\kappa} = \frac{\partial\,\delta\boldsymbol{\theta}}{\partial s} + \delta\boldsymbol{\theta}\wedge\boldsymbol{\kappa}$

Corotational $\quad \delta\left(\frac{1}{2}\mathbf{u}\mathbb{X}\mathbf{u}\right) = (\delta\mathbf{u} - \delta\boldsymbol{\theta}\wedge\mathbf{u})\,\mathbb{X}\mathbf{u} \qquad \delta\boldsymbol{\theta} = \mathrm{axial}(\mathbb{Q}^T\delta\mathbb{Q})$

$$\int_0^L \delta\boldsymbol{\varphi}\cdot\left(\frac{\partial\,\mathbb{G}\boldsymbol{\varepsilon}}{\partial s} - \frac{\partial\,\mathbb{A}\mathbf{v}}{\partial t}\right) + \delta\boldsymbol{\theta}\cdot\left(\frac{\partial\,\mathbb{H}\boldsymbol{\kappa}}{\partial s} + \frac{\partial\boldsymbol{\varphi}}{\partial s}\wedge(\mathbb{G}\boldsymbol{\varepsilon}) - \frac{\partial\,\mathbb{J}\boldsymbol{\omega}}{\partial t}\right)\ \mathsf{d}s = \Big[\delta\boldsymbol{\varphi}\cdot(\mathbb{G}\boldsymbol{\varepsilon}) + \delta\boldsymbol{\theta}\cdot(\mathbb{H}\boldsymbol{\kappa})\Big]_0^L$$

Something wrong ?

$$\text{Lagrangian} \quad \mathcal{L}(\mathbf{v},\boldsymbol{\omega},\boldsymbol{\varepsilon},\boldsymbol{\kappa}) := \int_0^L \frac{1}{2}\mathbf{v}\mathbb{A}\mathbf{v} + \frac{1}{2}\boldsymbol{\omega}\mathbb{J}\boldsymbol{\omega} - \frac{1}{2}\boldsymbol{\varepsilon}\mathbb{G}\boldsymbol{\varepsilon} - \frac{1}{2}\boldsymbol{\kappa}\mathbb{H}\boldsymbol{\kappa} \ \mathrm{d}s$$

$$\text{Hamiltonian} \quad \mathcal{H}(\mathbf{v},\boldsymbol{\omega},\boldsymbol{\varepsilon},\boldsymbol{\kappa}) := \int_0^L \frac{1}{2}\mathbf{v}\mathbb{A}\mathbf{v} + \frac{1}{2}\boldsymbol{\omega}\mathbb{J}\boldsymbol{\omega} + \frac{1}{2}\boldsymbol{\varepsilon}\mathbb{G}\boldsymbol{\varepsilon} + \frac{1}{2}\boldsymbol{\kappa}\mathbb{H}\boldsymbol{\kappa} \ \mathrm{d}s$$

$$\Downarrow$$

$$\int_0^L \delta\boldsymbol{\varphi}\cdot\left(\frac{\partial\,\mathbb{G}\boldsymbol{\varepsilon}}{\partial s} - \frac{\partial\,\mathbb{A}\mathbf{v}}{\partial t}\right) + \delta\boldsymbol{\theta}\cdot\left(\frac{\partial\,\mathbb{H}\boldsymbol{\kappa}}{\partial s} + \frac{\partial\boldsymbol{\varphi}}{\partial s}\wedge(\mathbb{G}\boldsymbol{\varepsilon}) - \frac{\partial\,\mathbb{J}\boldsymbol{\omega}}{\partial t}\right) \ \mathrm{d}s = \Big[\delta\boldsymbol{\varphi}\cdot(\mathbb{G}\boldsymbol{\varepsilon}) + \delta\boldsymbol{\theta}\cdot(\mathbb{H}\boldsymbol{\kappa})\Big]_0^L$$

- $\{\mathbf{v},\boldsymbol{\omega},\boldsymbol{\varepsilon},\boldsymbol{\kappa}\}$ are not the natural variable $\Rightarrow$ $\{\boldsymbol{\varphi},\mathbb{Q}\}$
- Wich formulation ? $\mathbf{u}$ or $\{u_i\}$?

We need to clarify the methods
We have to expose advantage and drawback of each formulation

Conclusion

- Travail en cours
- … et en collaboration avec Oscar Cosserat