Électrique ? ... Alors c'est géométrique !

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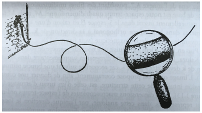
Symmetry groups

- **A geometry, it is a group,**
  acting onto objects and conserving invariants
  - **Affine group** : the ratio, the midpoint
  - **Euclid group** : also the distances
  - **Galilei group** : also the durations, the inertial motion

  Find the group ; you can develop a Physics by geometrization
- **Electric ?**

... Then it is geometric !

Kaluza-Klein theory



- [Kaluza 1921] to unify gravitation and electromagnetism, he considers a 5D Universe $\hat{\mathcal{U}}$ equipped with a $1+4$ metric $\hat{G} = \mathrm{diag}(1, -1, -1, -1, -1)$
- [Klein 1926] the 5^{th} dimension is curled up and overwhelmingly small

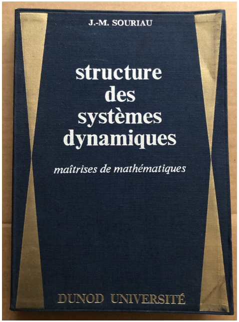
$$l_K \cong 10^{-31} \text{ cm !}$$

Conjecture : **the electric charge q is the linear momentum along the 5^{th} dimension**

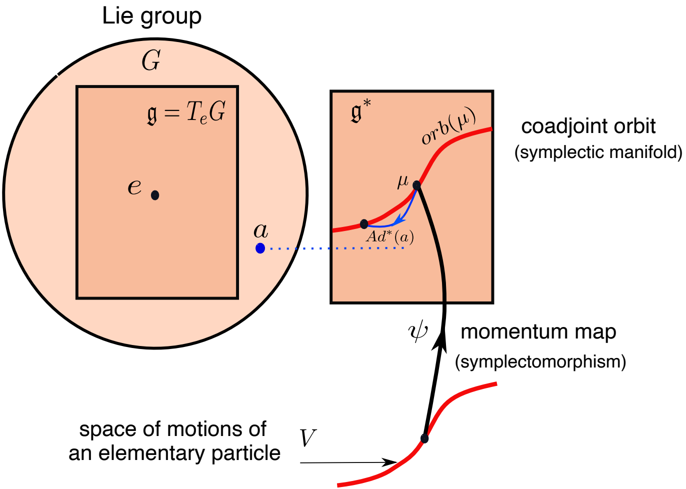
Issue : **as expected, is the electric charge of an elementary particle frame-invariant ?**

Answer : **using the coadjoint orbit method** [Souriau SSD 1970]

The coadjoint orbit method



The coadjoint orbit method

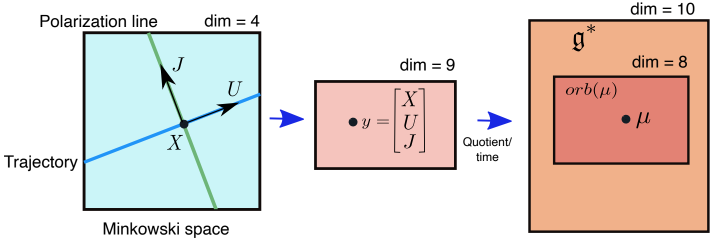


classifying the coadjoint orbits,
you classify the elementary particles of physicists

Example : Poincaré's group

- **Poincaré's group :** $\mathbb{G} = \mathbb{R}^4 \rtimes \mathbb{SO}(1,3) \ni a = (C, P) \qquad P^* P = Id$
  It is a Lie group of dimension 10
- **Infinitesimal generators :** $\mathfrak{g} \ni \delta a = (\delta C, \delta P) \qquad \delta P$ skew-adjoint
- $\mu \in \mathfrak{g}^* \qquad \Leftrightarrow \qquad \mu = (\Pi, M) \qquad M$ skew-adjoint

Orbit : particle with spin
Linear 4-momentum Π, Polarization $W = (*M)\,\Pi$



2 independent invariants : rest mass $\quad m_0 = \sqrt{\Pi^* \Pi}$

spin $\quad s = \dfrac{\sqrt{-W^* W}}{\sqrt{\Pi^* \Pi}}$

Other orbits : spinless particles, massless particles

Charged elementary particles in Kaluza-Klein relativity (group $\hat{\mathbb{G}}_1$)

- **Kaluza-Klein group :** $\hat{\mathbb{G}}_1 = \mathbb{R}^5 \rtimes \mathbb{SO}(1,4) \ni \hat{a} = (\hat{C}, \hat{P}) \qquad \hat{P}^* \hat{P} = Id$

  These elements are such that :

  $$\hat{P} = \begin{bmatrix} P & \beta P^{*-1} b \\ b^* & \beta \end{bmatrix} \qquad \begin{array}{l} b \in \mathbb{R}^4 \quad \beta = \sqrt{1 + b^* b} \\ P = P_L B, \quad P_L \text{ Lorentz}, \quad B = Id + \frac{1}{\beta+1} b\, b^* \end{array}$$

  It is a Lie group of dimension 15

- $\hat{\mu} \in \hat{\mathfrak{g}}_1^* \qquad \Leftrightarrow \qquad \hat{\mu} = (\hat{\Pi}, \hat{M}) \qquad \hat{M}$ skew-adjoint
- **Orbit : charged particle with spin**

  $$\hat{\Pi} = \begin{bmatrix} \Pi \\ q \end{bmatrix} \quad \begin{array}{l} \text{linear 4-momentum} \\ \text{electric charge} \end{array}$$

  Coadjoint representation : $\Pi = P\,\Pi' + q'\,\beta\,P^{*-1} b, \qquad q = b^* \Pi + \beta\, q'$

   **The charge is frame dependent**

$\hat{\mathbb{G}}_1$ is not the symmetry group of the Universe **today** as we know it.

Nonetheless it must not be *a priori* rejected.

We have only to find the Physics that could admit it as symmetry group.

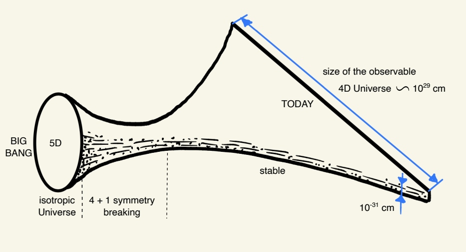
Our goal now is to find a more appropriate symmetry group for the Physics today.

Charged elementary particles today (group $\hat{\mathbb{G}}_0$)

- As the scale of the Universe along $\hat{X}^5$ is overwhelmingly small (10^{-31} cm) we zoom in : $\hat{X}'^5 = \hat{X}^5/\omega$. When $\omega \to 0$, the metric degenerates
$$\hat{G}' = \begin{bmatrix} G & 0 \\ 0 & -\omega^2 \end{bmatrix} \to \begin{bmatrix} G & 0 \\ 0 & 0 \end{bmatrix}, \qquad \hat{G}'^{-1} \to -\omega^{-2} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
- The set of affine transformations $\hat{a} = (\hat{C}, \hat{P})$ of $\mathbb{R}^5$ that conserve the components $\begin{bmatrix} G & 0 \\ 0 & 0 \end{bmatrix}$ of $\hat{\boldsymbol{G}}_0$ and the components $\hat{\Omega}_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ of $\hat{\boldsymbol{\Omega}}_0$ is a Lie group $\hat{\mathbb{G}}_0$ of dimension 15
Its elements $\hat{a}$ are such that $\hat{P} = \begin{bmatrix} P_L & 0 \\ b^* & 1 \end{bmatrix}$ where P_L is Lorentz
- $\hat{\mu} \in \hat{\mathfrak{g}}_0^* \quad \Leftrightarrow \quad \hat{\mu} = ((\Pi, q), (M, Q)) \quad M$ skew-adjoint wrt $G \quad Q \in \mathbb{R}^4$
- **Orbit : charged particle with spin :** coadjoint representation
$q = q', \quad \Pi = P\,(\Pi' - q'\,b), \quad Q = P\,Q' + q'\,C$
$$M = P\,M'P^* + C\,(P\,(\Pi' - q\,b))^* - (P\,(\Pi' - q\,b))\,C^* + (P\,b)(P\,Q')^* - (P\,Q')(P\,b)^*$$
- **3 independent invariants :** rest mass m_0
spin s
charge q

A cosmological scenario for the evolution of elementary particle structure

Inspired from Kaluza-Klein cosmology [Sahdev 1984, Matzner et al 1985, Copeland et al 1985, Okada 1986, Giorgini & Kerner 1988]



- the elementary particles of the isotropic early 5D Universe are classified from the momenta of the group $\hat{\mathbb{G}}_1$
- next the three former space dimensions inflate quickly while the last one shrinks
- leading to the 4D era in which as today the particles are characterized by the momenta of the group $\hat{\mathbb{G}}_0$.

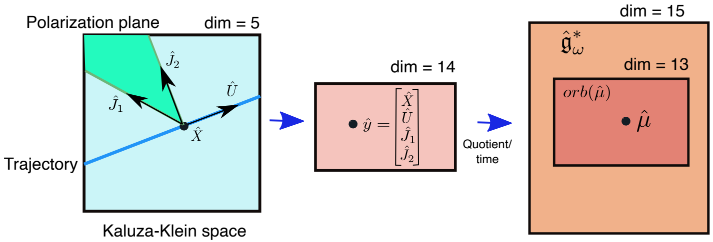
By this mechanism, the elementary particles can acquire electric charge as a by-product of the $4+1$ symmetry breaking of the Universe.

4 + 1 symmetry breaking (group $\hat{\mathbb{G}}_\omega$)

- The set of affine transformations $\hat{a} = (\hat{C}, \hat{P})$ of $\mathbb{R}^5$ that conserve $\hat{G}_\omega = \begin{bmatrix} G & 0 \\ 0 & -\omega^2 \end{bmatrix}$ is the Lie group $\hat{\mathbb{G}}_\omega$ of which the elements are such as

$$\hat{P} = \begin{bmatrix} P & \omega^2 \beta\, P^{*-1} b \\ b^* & \beta \end{bmatrix} \qquad \begin{array}{l} b \in \mathbb{R}^4 \quad \beta = \sqrt{1 + \omega^2 b^* b} \\ P = P_L B, \quad P_L \text{ Lorentz}, \quad B = Id + \frac{\omega^2}{\beta+1}\, b\, b^* \end{array}$$

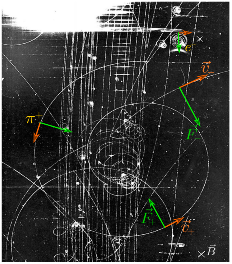
- Extreme cases : $\omega = 1$ Kaluza-Klein group, $\omega = 0$ today (singular)



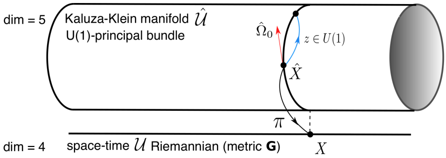
2 independent invariants : rest mass $m_0 = \sqrt{\hat{\Pi}^* \hat{\Pi}}$

spin $s = \dfrac{\sqrt{-(\text{pol}_{\hat{\mu}}(\hat{J}_1))^* \text{pol}_{\hat{\mu}}(\hat{J}_1)}}{\sqrt{\hat{\Pi}^* \hat{\Pi}}}$

The pullback connection



$\hat{\mathbb{G}}_0$-connection



- **Purpose :** to construct an Ehresmann $\hat{\mathbb{G}}_0$-connection
- **[H1]** The space-time $\mathcal{U}$, endowed with the metric $\boldsymbol{G}$, is Riemaniann
- **[H2]** The Kaluza-Klein manifold $\hat{\mathcal{U}}$ is a principal $U(1)$-bundle of base $\mathcal{U}$, projection π, equipped with
  - the pullback of the metric $\hat{\boldsymbol{G}}_0 = \pi^* \boldsymbol{G}$ (which is not a metric !)
  - the vector field $\hat{\boldsymbol{\Omega}}_0$ of which the integral curves are the fibers

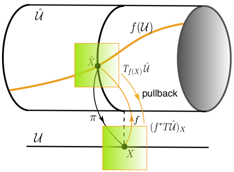
  A $\hat{\mathbb{G}}_0$-basis $(\hat{\boldsymbol{e}}'_\alpha)$, is a basis in which $\hat{\boldsymbol{G}}_0$ and $\hat{\boldsymbol{\Omega}}_0$ are represented by

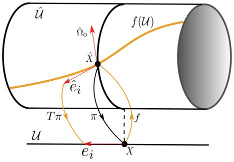
  $$\hat{G}'_0 = \begin{bmatrix} G'_0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad \hat{\Omega}'_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \text{Minkowski metric } G'_0$$

- **[H3]** A $\hat{\mathbb{G}}_0$-**connection** $\hat{\nabla}$ is a 1-form field $(\hat{\boldsymbol{e}}'_\alpha) \mapsto \hat{\Gamma}' \in \hat{\mathfrak{g}}_0$

  $$\hat{\Gamma}' = \delta \begin{bmatrix} P & 0 \\ b^* & 1 \end{bmatrix} = \begin{bmatrix} \delta P & 0 \\ \delta b^* & 0 \end{bmatrix} = \begin{bmatrix} \Gamma' & 0 \\ \Gamma'^5 & 0 \end{bmatrix}, \qquad \Gamma' \text{ skew-adjoint}$$

The pullback connection





- **Zoom out :** $\hat{X}^5 = \omega \hat{X}'^5$ is singular when $\omega \to 0$, that **leads to a deadlock**
- **To break the deadlock**, we consider a **section** f of the principal U(1)-bundle $\hat{\mathcal{U}}$ and the **pullback bundle** $f^* T\hat{\mathcal{U}}$ on the spacetime
- **[H4]** The spacetime $\mathcal{U}$, endowed with the **pullback connection**
  $(f^*\hat{\nabla})_{\boldsymbol{U}}(f^*\hat{\boldsymbol{W}}) = f^*(\hat{\nabla}_{(Tf)\boldsymbol{U}}\hat{\boldsymbol{W}})$ $\boldsymbol{U}$ tangent to $\mathcal{U}$, $\hat{\boldsymbol{W}}$ tangent to $\hat{\mathcal{U}}$
  is **torsion free** $\hat{\nabla}_{(Tf)\,\boldsymbol{U}}(Tf)\boldsymbol{V} - \hat{\nabla}_{(Tf)\,\boldsymbol{V}}(Tf)\boldsymbol{U} - (Tf)\,[\boldsymbol{U},\boldsymbol{V}] = 0$
- **Convention :** Latin indices from 1 to 4, Greek indices form 1 to 5
  In a basis of $\hat{\mathcal{U}}$ **adapted to the section** f : $(\hat{\boldsymbol{e}}_1, \cdot\cdot, \hat{\boldsymbol{e}}_4, \hat{\boldsymbol{\Omega}}_0)$ s.t. $\hat{\boldsymbol{e}}_i$ is tangent to $f(\hat{\mathcal{U}})$ and in the projected basis $(\boldsymbol{e}_1, \cdot\cdot, \boldsymbol{e}_4)$ of $\mathcal{U}$ s.t. $\boldsymbol{e}_i = (T\pi)\,\hat{\boldsymbol{e}}_i$
  the torsion free condition leads to the 50 scalar relations $\Gamma^{\mu}_{ij} - \Gamma^{\mu}_{ji} - \delta^{\mu}_{k} c^{k}_{ij} = 0$

Equation of motion of a charged particle

- **[H5]** The **linear** 5-**momentum** is a **1-form** $\hat{\Pi} \in T^*\hat{\mathcal{U}}$
  The motion of a particle $s \mapsto X(s)$ and its charge evolution $s \mapsto q(s)$ are s.t.
  $\hat{\Pi}$ is **paralled-transported** : $(f^*\hat{\nabla})_U(f^*\hat{\mathbf{\Pi}}) = 0$

- **Integrability consideration**
  $(\hat{e}'_\alpha) = (\hat{e}'_i, \hat{\Omega}_0)$ $\hat{\mathbb{G}}_0$-basis $\quad \xrightarrow{T\pi} \quad (e'_i) = ((T\pi)\hat{e}'_i)$ not integrable
  $\downarrow \hat{P} = \begin{bmatrix} P & 0 \\ -2\,A^* & 1 \end{bmatrix}$ where $A \in \mathbb{R}^4$ is the **electromagnetic potential**
  $(\hat{e}_\alpha) = (\hat{e}_i, \hat{\Omega}_0) \quad \xrightarrow{T\pi} \quad (\partial_i) = ((T\pi)\hat{e}_i)$ associated to coordinates (X^i)
  in which the connection is $\hat{\Gamma} = \begin{bmatrix} \Gamma & 0 \\ \Gamma^5 & 0 \end{bmatrix}$

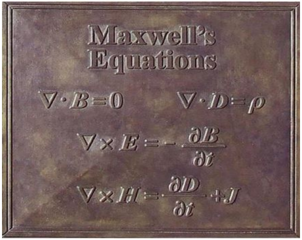
- $\hat{\nabla}$ torsion free $\quad \Rightarrow \quad \Gamma^k_{ij} = \Gamma^k_{ji} \quad \Rightarrow \quad \Gamma$ is **Levi-Civita**
  5-momentum $\hat{\Pi}'^* = [\Pi'^*, q']$ (free particle) $\Rightarrow \hat{\Pi}^* = \hat{\Pi}'^*\hat{P} = [\Pi^* - 2\,q\,A^*, q]$

  Using **[H5]**, we prove that m_0 **and** q **are integrals of the motion**
  and $\Gamma^5_j = U^k F_{kj} - 2\,\nabla_U A_j$

  $\hat{\nabla}$ torsion free $\quad \Rightarrow \quad \Gamma^5_{ij} = \Gamma^5_{ji} \quad \Rightarrow \quad F_{ij} = \partial_i A_j - \partial_j A_i$ **electromagnetic field**
  Then the **equation of motion** is $\boxed{m_0 U^k \nabla_k U^i = -q\,F^i_j U^j}$ (**Lorentz force**)

- To each change of section f corresponds a gauge transformation

The variational relativity



Stationary action principle

We use the variational approach of [Palatini 1919]

It is extended to involve the electromagnetic field

(P1) To every **physical phenomenon** corresponds a field z and a Lagrangian $L_{(z)}(z, \partial_i z, G, A^*)$

(P2) The **action** $S = \int_{\mathcal{D}} L\, vol = \sum_z \int_{\mathcal{D}} L_{(z)}\, vol$ on $\mathcal{D} \subset \mathcal{U}$ spacetime

is stationary for every variation of G, A^* and z

$$T_{(z)} = 2\,\partial_G L_{(z)} + L_{(z)} G^{-1}, \quad \tilde{T}_{(z)} = \partial_{A^*} L_{(z)}, \quad W_{(z)} = \partial_z L_{(z)} - \partial_i(\partial_{\partial_i z} L_{(z)})$$

$$\boxed{T = \sum_z T_{(z)} = 0}, \quad \boxed{\tilde{T} = \sum_z \tilde{T}_{(z)} = 0}, \quad \boxed{W_{(z)} = 0}$$

Einstein Euler-Lagrange

The two former conditions are called **field equations**

- **Laws of conservation :**

$$\boxed{\text{div}\, T_{(z)} + \tilde{T}_{(z)} \cdot F = 0}$$

$$\boxed{\text{div}\, \tilde{T}_{(z)} = 0}$$

The matter and its motion

- A particle is identified by its **Lagrangean coordinates** $a = \pi_A(X)$ (equation of its trajectory)
- The Lagrangians $L_{(z)}$ must be invariant by every diffeo of the spacetime
- $L_M = L_M\left(a, \frac{\partial a}{\partial X}, G, A^*\right) = L_M\left(a, h, G, A^*\right)$
  where $h = -\det H$, $\quad H = \frac{\partial a}{\partial X}\, G^{-1} \frac{\partial a}{\partial X}^T$ conformation, $\quad c = 1$ (speed of light)
- **mass density** $\rho_m = \rho_{m0}(a)\,\sqrt{h}$
  electric charge density $\rho_e = \rho_{e0}(a)\,\sqrt{h}$
- **Lagrangian of the matter** $L_M = \kappa\,[L_m(a, h, G) + L_e(a, h, A^*)]$ κ coupling C^{te}
  - $L_m(a, h, G) = \rho_{m0}(a)\,[\sqrt{h} + \psi(h)]$ $\quad \psi$ internal energy
  - $Ł_e(a, h, A^*) = -\rho_{e0}(a)\,\sqrt{h}\,A^*\,U$
- **density of energy** $\rho = L_M/\kappa$, $\qquad$ **pressure** $p = \rho_{m0}\,(2\,h\,\partial_h\psi - \psi)$
- **Laws of conservation**

  of the linear momentum $\boxed{\nabla_i[(\rho + p)\,U^i\,U_j - p\,\delta^i_j] - \rho_e\,U^k\,F_{kj} = 0}$

  of the electric current $\boxed{\nabla_i(\rho_e\,U^i) = 0}$

The gravitation and electromagnetism

- **Curvature tensor** $\hat{\boldsymbol{R}}$ of the space-time, s.t.

  $$\hat{\nabla}_{(Tf)\,\boldsymbol{U}}\,(\hat{\nabla}_{(Tf)\,\boldsymbol{V}}\,\hat{\boldsymbol{W}}) - \hat{\nabla}_{(Tf)\,\boldsymbol{V}}\,(\hat{\nabla}_{(Tf)\,\boldsymbol{U}}\,\hat{\boldsymbol{W}}) - \hat{\nabla}_{(Tf)\,[\boldsymbol{U},\boldsymbol{V}]}\,\hat{\boldsymbol{W}} = \hat{\boldsymbol{R}}(\boldsymbol{U},\boldsymbol{V})\,\hat{\boldsymbol{W}}$$

  The non null curvature components are

  $$R^p_{ijk} = \hat{R}^p_{ijk} = \Gamma^p_{i\mu}\Gamma^\mu_{jk} - \Gamma^p_{j\mu}\Gamma^\mu_{ik} + \partial_i\Gamma^p_{jk} - \partial_j\Gamma^p_{ik}$$

  $$\tilde{R}_{ijk} = \hat{R}^5_{ijk} = \nabla_k F_{ji} + 2A_q R^q_{ijk}$$

- **Lagrangian** $L_G = L_G(\hat{\Gamma}, \partial_i\hat{\Gamma}, G, A^*) = L_G(\hat{\Gamma}, \hat{R}, G, A^*)$

  $$L_G = L_g(R,G) + L_{em}(\tilde{R}, G, A^*) = -\Lambda \;+\; \frac{1}{2}\,G^{ij}R_{ij} \;-\; \tilde{k}\,A_r\,G^{ri}\,G^{jk}\,\tilde{R}_{ijk}$$

  R_{ij} Ricci $\quad$ $\tilde{k}$ coupling C^{te}

- 14 **field equations** for 14 unknown potentials G_{ij}, A_i

$$R^{ij} - \frac{1}{2}\,R\,G^{ij} + \Lambda\,G^{ij} - \tilde{k}\,[A^{(i}\,\tilde{R}^{j)}{}_k{}^k + A_r\tilde{R}^{r(ij)} - \frac{1}{2}\,\tilde{R}\,G^{ij}] = \kappa\,[(\rho+p)U^iU^j - p\,G^{ij}]$$

$$-\tilde{k}\,[\nabla_j F^{ji} + 2A_q R^{qij}{}_j] = \kappa\,\rho_e\,U^i$$

Classical limit

Reminder :

$$\heartsuit \quad R^{ij} - \frac{1}{2}\, R\, G^{ij} + \Lambda\, G^{ij} - \tilde{k}\, [A^{(i}\, \tilde{R}^{j)}{}_k{}^k + A_r \tilde{R}^{r(ij)} - \frac{1}{2}\, \tilde{R}\, G^{ij}] = \kappa\, [(\rho + p) U^i U^j - p\, G^{ij}]$$

$$\spadesuit \quad -\tilde{k}\, [\nabla_j F^{ji} + 2 A_q R^{qij}{}_j] = \kappa\, \rho_e U^i$$

- In absence of electromagnetism, at the Newtonian approximation, **identification to Newton's universal attraction law :** $\kappa = 8\,\pi\, G_N$
  Galilei connection : $\Gamma^a_{tt} = -g^a$ gravity $\quad \Gamma^a_{tb} = \Gamma^a_{bt} = \Omega^a_b$ Coriolis
  Ω^a_b and F^{ij} are skew-symmetric $\quad \Rightarrow \quad \nabla_j F^{ji} = \partial_j F^{ji}$
  In the lab, $g^a, \Omega^a_b \cong C^{te} \Rightarrow$ **identification to Coulomb's law :** $\tilde{k} = 8\,\pi\, G_N\, \epsilon_0$
  $\spadesuit \quad \Rightarrow \quad \partial_j F^{ji} = -\frac{1}{\epsilon_0}\, \rho_e U^i$ (**2nd group of Maxwell equations**)

- In $\heartsuit$, the coupling term is of the order of the stored electric density, very negligible with respect to the energy density ρ ($= \rho\, c^2$ in SI units)

- The very weak coupling between gravitation and electromagnetism can be related to Dirac's Large Number Hypothesis (LNH)
  $\tilde{k} = \frac{2\, e^2}{m_e m_p}\, \frac{F_g}{F_e}$ where $\frac{F_g}{F_e} \cong 10^{-40}$
  Note that $\frac{F_g}{F_e} = \left(\frac{l_K}{2\sqrt{\lambda_C \lambda_{C,p}}} \right)^2$ where $l_K = 8\,\pi^{3/2}\, \frac{h\sqrt{G_N \epsilon_0}}{e\, c} = 2.38 \times 10^{-31}$ cm

Conclusions and perspectives

Conclusions

- We proposed a symmetry group $\hat{\mathbb{G}}_0$ for which the electric charge of a particle is an **invariant**,
  a $\hat{\mathbb{G}}_0$-connection allowing to recover the expected equation of motion of a particle today, including **Lorentz force**,
  and a 5D extension of the relativity allowing to recover at the classical limit **Maxwell equations**
- $\hat{\mathbb{G}}_0$ **is the symmetry group of the electrodynamics compatible with the observations today**
- the symmetry group $\hat{\mathbb{G}}_1$ of the Kaluza-Klein theory leads to **an unified theory merging the gravitational and electromagnetic force, relevant in the early Universe**

Perspectives

1. Application of Kostant-Souriau geometric quantization to 5D
2. Extension to a non-abelian gauge group
3. Revisit of works on the early universe cosmology
   They could provide a better understanding for Dirac's hypothesis [Chodos 1980] and to offer a resolution to the horizon problem [Sahdev 1984].

La Fée Électricité (The Electricity Fairy)

Raoul Dufy (1937)

"Les savants, ô Monseigneur, et les astronomes en particulier, ne suivent pas les usages de tout le monde. C'est pourquoi les aventures qui leur arrivent ne sont pas celles de tout le monde."

Contes des mille et une nuits

Thanks to **Richard Kerner** and **Leonid Ryvkin** for their valuable advise

Thank you for your attention !

Polarization map

Momentum $\hat{\mu} = (\hat{\Pi}, \hat{M})$, $\hat{M}$ skew-adjoint

$$\hat{M} \xrightarrow{\hat{G}} A_{\hat{M}} \xrightarrow[Hodge]{\star} \star A_{\hat{M}} \longrightarrow \star A_{\hat{M}}(\hat{\Pi}, \hat{J}) \xrightarrow{G^{-1}} \mathrm{pol}_{\hat{\mu}}(\hat{J})$$

$$\Omega^2(\hat{\mathcal{U}}) \qquad \Omega^3(\hat{\mathcal{U}}) \qquad \Omega^1(\hat{\mathcal{U}}) \qquad \mathrm{Vect}(\hat{\mathcal{U}})$$

Coadjoint representation

- A Lie group G naturally acts on its Lie algebra $\mathfrak{g}$ by the **adjoint representation**
  $Ad(a)\,Z = a\,Z\,a^{-1}$
- G acts on the dual $\mathfrak{g}^*$ by the induced action (**coadjoint representation**)
  $(Ad^*(a)\,\mu)\,(Z) = \mu\,(Ad(a^{-1})\,Z)$

Electromagnetic field

- The 4-potential $\boldsymbol{A}$ and its adjoint $\boldsymbol{A}^*$ are represented respectively by

$$A = \begin{bmatrix} \phi \\ \mathrm{A} \end{bmatrix}, \qquad A^* = [\phi, -\mathrm{A}^T]$$

- The electromagnetic field $\boldsymbol{F}$ is represented by

$$F = \begin{bmatrix} 0 & -E^T \\ E & -j(B) \end{bmatrix}$$

where E and B are respectively the electric and magnetic components

$$E = -grad\,\phi - \partial_t \mathrm{A}, \qquad B = curl\,\mathrm{A}$$

and $j(B)\,v = B \times v$

- The Lorentz force $\boldsymbol{f}$ is represented by

$$f = \gamma\, q \begin{bmatrix} E \cdot v \\ E + v \times B \end{bmatrix}$$