# Lie symmetry and Nœther's theorem in fluid mechanics

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# Outline

- Lie symmetry
- Analytical solution of the compressible Navier-Stokes equations
- Nœther's theorem
- Non-local conservation laws of the Navier-Stokes equations

# Symmetry: simple examples

- Riccati's equation: $\frac{\mathrm{d}u}{\mathrm{d}x} + u^2 - \frac{2}{x^2} = 0$ $E(x, u, u') = 0$
- Simple symmetry: $(x, u) \longmapsto (\epsilon x, \epsilon^{-1} u)$
- New variables $(\xi, v)$ to make the equation autonomous ? $F(v, v') = 0$
- Autonomous $\longleftrightarrow (\xi, v) \longmapsto (\xi + \varepsilon, v)$ is a symmetry

- $(\xi = \ln x, v = xu) \longrightarrow \frac{\mathrm{d}v}{\mathrm{d}\xi} + v^2 - v - 2 = 0$ Sep. variables

- Heat equation: $\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial x^2}$
- Simple symmetry: $(t, x, u) \longmapsto (\epsilon^2 t, \varepsilon x, u)$
- Invariant: $\xi = \frac{x}{\sqrt{t}}$
- Reduction: $u(t, x) = v(\xi) \longrightarrow -\frac{1}{2}\xi \frac{\mathrm{d}v}{\mathrm{d}\xi} = \frac{\mathrm{d}^2 v}{\mathrm{d}\xi^2}$ erf

# Another example

- Anisothermal laminar thin shear layer flows

$$\begin{cases} \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0 \\ u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} - \nu\dfrac{\partial^2 u}{\partial y^2} + \beta g\theta = 0 \\ u\dfrac{\partial \theta}{\partial x} + v\dfrac{\partial \theta}{\partial y} - \kappa\dfrac{\partial^2 \theta}{\partial y^2} = 0 \end{cases}$$

- Symmetry:

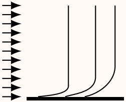
$$g_\epsilon : (x, y, u, v, \theta) \mapsto (\epsilon x, \epsilon^{\alpha} y, \epsilon^{1-2\alpha} u, \epsilon^{-\alpha} v, \epsilon^{1-4\alpha}\theta)$$

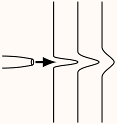
- Reduction into ODE:

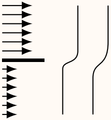
$$\xi = \frac{y}{x^{\alpha}}, \quad U(\xi) = \frac{u}{x^{1-\alpha}}, \quad V(\xi) = \frac{v}{\epsilon^{-\alpha}}, \quad \Theta(\xi) = \frac{\theta}{x^{1-4\alpha}}$$

$$\begin{cases} (1-2\alpha)U - \alpha\xi\dot{U} + \dot{V} = 0, \\ U[(1-\alpha)U - \alpha\xi\dot{U}] + \dot{U}V = \nu\ddot{U} \\ U[(1-4\alpha)\Theta - \alpha\xi\dot{\Theta}] + V\dot{\Theta} = \kappa\ddot{\Theta} \end{cases}$$

- Boundary condition $\longrightarrow$ value of $\alpha$ (Blasius $\alpha = 1/2, \dots$)

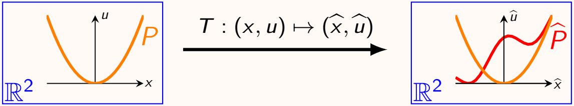

Boundary layer


Jet/Plume

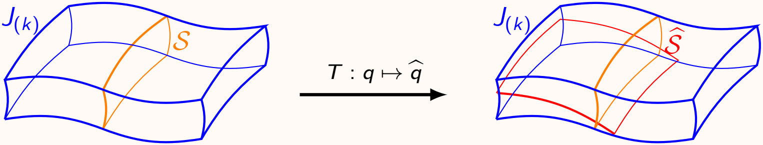

Mixing layer

# (Dynamical) Symmetry of an equation

- Eg: Parabola $P$ (of equation $u = x^2$) in $\mathbb{R}^2$



  - Symmetry if $\widehat{u} = \widehat{x}^2$
  - Eg: $G = \{g_\epsilon : (x, u) \longmapsto (e^\epsilon x, e^{2\epsilon} u)\}$ is a symmetry group of $P$
- Diff. Eq.: $E(x, u, u_{(1)}, \cdots, u_{(k)}) = 0$ $u = u(x)$
  - Background space: Jet space $J_{(k)} = \{(x, u, u_{(1)}, \cdots, u_{(k)})\}$
  - Solution manifold $\mathcal{S}$
  - Transformation $T : q = (x, u) \longrightarrow \widehat{q} = (\widehat{x}, \widehat{u})$
  - Symmetry if $E(x, u, \cdots, u_{(k)}) = 0 \implies E(\widehat{x}, \widehat{u}, \ldots, \widehat{u}_{(k)}) = 0$

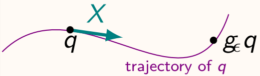


# How to compute them ?

- Restriction to one-parameter symmetries forming a Lie group
  $G = \{g_\epsilon : (x, u) \mapsto (\widehat{x}, \widehat{u}),\ \epsilon \in I\}$ ($g_\epsilon$ depends smoothly on $\epsilon$)

**Infinitesimal variation:** $\mathrm{X} = \xi_x \dfrac{\partial}{\partial x} + \xi_u \dfrac{\partial}{\partial u}$

$$\xi_x(x, u) = \frac{\partial \widehat{x}}{\partial \epsilon}_{|\epsilon=0}, \qquad \eta_u(x, u) = \frac{\partial \widehat{u}}{\partial \epsilon}_{|\epsilon=0}$$



Eg. • $(x, u) \longmapsto (y + \epsilon, u)$, $\mathrm{X} = \dfrac{\partial}{\partial x}$ • $(x, u) \longmapsto (\mathrm{e}^{\alpha\epsilon}\, x, \mathrm{e}^{\beta\epsilon}\, u)$, $\mathrm{X} = \alpha x \dfrac{\partial}{\partial x} + \beta u \dfrac{\partial}{\partial u}$

- X is called infinitesimal generator of $G$, X known $\longrightarrow$ $g_\epsilon$ known

$$\frac{\mathrm{d}\widehat{x}}{\mathrm{d}\epsilon} = \xi_x(\widehat{x}, \widehat{u}), \qquad \frac{\mathrm{d}\widehat{u}}{\mathrm{d}\epsilon} = \eta_u(\widehat{x}, \widehat{u}), \qquad \widehat{x}(\epsilon = 0) = x, \qquad \widehat{u}(\epsilon = 0) = u$$

**Symmetry condition**

$E(x, u, \cdots, u_{(k)}) = 0 \quad \Longrightarrow \quad E(\widehat{x}, \widehat{u}, \ldots, \widehat{u}_{(k)}) = 0$

replaced by

$\mathrm{pr}^{(k)} X \cdot E = 0$ on $\mathcal{S}$

⋆ Under local solvability condition

# Incompressible fluid flow

Navier-Stokes's equation

$$\frac{\partial u}{\partial t} + \operatorname{div}(u \otimes u) + \tfrac{1}{\rho}\nabla p - \nu \Delta u = 0, \qquad \operatorname{div} u = 0$$

### Lie algebra: 5-dimensional and four $\infty$-dimensional

$$\frac{\partial}{\partial t}, \qquad \pi(t)\frac{\partial}{\partial p}$$

$$\alpha^1(t)\frac{\partial}{\partial x^1} + \dot{\alpha}^1(t)\frac{\partial}{\partial x^1} - x^1\ddot{\alpha}(t)\frac{\partial}{\partial p}, \qquad \alpha^2(t)\frac{\partial}{\partial x^2} + \dot{\alpha}^2(t)\frac{\partial}{\partial x^2} - x^2\ddot{\alpha}(t)\frac{\partial}{\partial p}$$

$$\alpha^3(t)\frac{\partial}{\partial x^3} + \dot{\alpha}^3(t)\frac{\partial}{\partial x^3} - x^3\ddot{\alpha}(t)\frac{\partial}{\partial p}$$

$$x^2\frac{\partial}{\partial x^3} - x^3\frac{\partial}{\partial x^2} + u^2\frac{\partial}{\partial u^3} - u^3\frac{\partial}{\partial u^2}, \qquad x^3\frac{\partial}{\partial x^1} - x^1\frac{\partial}{\partial x^3} + u^3\frac{\partial}{\partial u^1} - u^1\frac{\partial}{\partial u^3}$$

$$x^1\frac{\partial}{\partial x^2} - x^2\frac{\partial}{\partial x^1} + u^1\frac{\partial}{\partial u^2} - u^2\frac{\partial}{\partial u^1}$$

$$2t\frac{\partial}{\partial t} + x^1\frac{\partial}{\partial x^1} + x^2\frac{\partial}{\partial x^2} + x^3\frac{\partial}{\partial x^3} - u^1\frac{\partial}{\partial u^1} - u^2\frac{\partial}{\partial u^2} - u^3\frac{\partial}{\partial u^3} - 2p\frac{\partial}{\partial p}$$

$\pi(t)$, and $\alpha^i(t)$ are arbitrary functions of $t$

# Incompressible fluid flow

Navier-Stokes's equation

$$\frac{\partial u}{\partial t} + \operatorname{div}(u \otimes u) + \frac{1}{\rho}\nabla p - \nu \Delta u = 0, \qquad \operatorname{div} u = 0$$

**Lie symmetry group generated by**

- Time translation: $(t, x, u, p) \mapsto (t+\epsilon, x, u, p)$
- Pressure translation: $(t, x, u, p+\zeta(t))$
- Rotation: $(t, Rx, Ru, p)$
- Generalized galilean transformation: $\left(t,\ x+\alpha(t),\ u+\dot{\alpha}(t),\ p-\rho x \cdot \ddot{\alpha}(t)\right)$
- Scale transformation: $(a^2 t,\ ax,\ a^{-1}u,\ a^{-2}p)$

Other symmetries:

- Equivalence (scale) transformation: $(t,\ ax,\ au,\ a^2 p,\ a^2 \nu)$
- Reflections $(t,\ \Lambda x,\ \Lambda u,\ p)$

  $\Lambda = diag(\pm 1, \pm 1, \pm 1)$
- 2D material indifference $(t,\ R(t)x,\ R(t)u,\ p-3\omega\psi + \frac{1}{3}\omega^2\|x\|^2)$

  $R(t)$ horizontal rotation with angle $\omega t$, $\psi$ stream function

# Compressible fluid flow

Compressible Navier-Stokes equations

$$\begin{cases} \rho_t + u \cdot \operatorname{grad} \rho + \rho \operatorname{div} u = 0 \\ \rho u_t + (\operatorname{grad} u) u = \operatorname{div} \sigma \\ C_v \left(p_t + u \cdot \operatorname{div} p + p \operatorname{div} u\right) = R \operatorname{tr}(\sigma \mathsf{S}) + \kappa \, \Delta \left(\frac{p}{\rho}\right) \end{cases}$$

$$\mathsf{S} = \frac{\nabla u + {}^{\top}\nabla u}{2}$$

$$\sigma = 2\mu \mathsf{S} - \left(p + \frac{2\mu}{3} \operatorname{div} u\right) \mathsf{I_d}$$

## 12-dimensional Lie algebra $\mathfrak{g} = \operatorname{span}(X_1, X_2, \ldots, X_{12})$

$$X_1 = \frac{\partial}{\partial t}, \qquad X_2 = \frac{\partial}{\partial x^1}, \qquad X_3 = \frac{\partial}{\partial x^2}, \qquad X_4 = \frac{\partial}{\partial x^3}$$

$$X_5 = t\frac{\partial}{\partial x^1} + \frac{\partial}{\partial u^1}, \qquad X_6 = t\frac{\partial}{\partial x^2} + \frac{\partial}{\partial u^2}, \qquad X_7 = t\frac{\partial}{\partial x^3} + \frac{\partial}{\partial u^3}$$

$$X_8 = x^2\frac{\partial}{\partial x^3} - x^3\frac{\partial}{\partial x^2} + u^2\frac{\partial}{\partial u^3} - u^3\frac{\partial}{\partial u^2}, \qquad X_9 = x^3\frac{\partial}{\partial x^1} - x^1\frac{\partial}{\partial x^3} + u^3\frac{\partial}{\partial u^1} - u^1\frac{\partial}{\partial u^3},$$

$$X_{10} = x^1\frac{\partial}{\partial x^2} - x^2\frac{\partial}{\partial x^1} + u^1\frac{\partial}{\partial u^2} - u^2\frac{\partial}{\partial u^1}$$

$$X_{11} = 2t\frac{\partial}{\partial t} + x^1\frac{\partial}{\partial x^1} + x^2\frac{\partial}{\partial x^2} + x^3\frac{\partial}{\partial x^3} - u^1\frac{\partial}{\partial u^1} - u^2\frac{\partial}{\partial u^2} - u^3\frac{\partial}{\partial u^3} - 2p\frac{\partial}{\partial p}$$

$$X_{12} = x^1\frac{\partial}{\partial x^1} + x^2\frac{\partial}{\partial x^2} + x^3\frac{\partial}{\partial x^3} + u^1\frac{\partial}{\partial u^1} + u^2\frac{\partial}{\partial u^2} + u^3\frac{\partial}{\partial u^3} - 2\rho\frac{\partial}{\partial \rho}$$

# Symmetry group and reduction

### 12-dimensional Lie symmetry group $G$ generated by

| | $(\widehat{t}, \widehat{x}, \widehat{u}, \widehat{p}, \widehat{\rho})$ |
|---|---|
| Time translations: | $(t+\varepsilon, x, u, p, \rho)$ |
| Space translations: | $(t, x+\epsilon, u, p, \rho)$ |
| Galilean transformations: | $(t, x+\epsilon t, u+\epsilon, p, \rho)$ |
| Rotations: | $(t, \mathrm{R}x, \mathrm{R}u, p, \rho)$ |
| 1st scale transformations: | $(\mathrm{e}^{2\varepsilon}\, t, \mathrm{e}^{\varepsilon}\, x, \mathrm{e}^{-\varepsilon}\, u, \mathrm{e}^{-2\varepsilon}\, p, \rho)$ |
| 2nd scale transformations: | $(t, \mathrm{e}^{\varepsilon}\, x, \mathrm{e}^{\varepsilon}\, u, p, \mathrm{e}^{-2\varepsilon}\, \rho)$ |

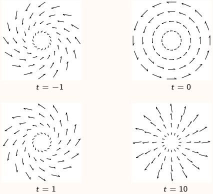
### Reduction with $H \subset G$

$$\begin{tikzcd}
(y, v) \in M \arrow[d, "\text{Orbit}"] & E(y, v, \ldots, v_{(k)}) \arrow[d] & \text{Self-sim. solution } v(y) \\
(s, V) \in M/H & E_{reduced}(\xi, V, \ldots, V_{(k)}) \arrow[r] & \text{Solution } V(\xi) \arrow[u]
\end{tikzcd}$$

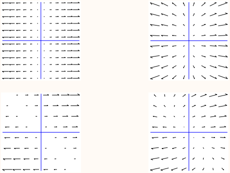
## 2D: 2nd scale transformations

$$u(t,r,\theta) = \frac{at+b}{at^2+2bt+2}\, r\, e_r + \frac{(2a-b^2)^{1/2}}{at^2+2bt+2}\, r\, e_\theta\,, \qquad \rho(t,r,\theta) = \frac{\rho_3}{r^2}\,,$$

$$p(t,r,\theta) = \frac{4\mu R\, \mathrm{e}^{4\kappa t/C_v\rho_3}}{(at^2+2bt+2)^{R/C_v+1}} \int \frac{(at^2+2bt+2)^{R/C_v-1}(at+b)^2}{3C_v\, \mathrm{e}^{4\kappa t/C_v\rho_3}}\, \mathrm{d}t$$


$t=-1$ $t=0$

$t=1$ $t=10$



## 2D: Galilean transf in $x^1$ then scale transf

$$u^1 = \frac{\alpha y}{t(t+\beta)^\delta} + \frac{x}{t}, \qquad u^2 = \frac{y\delta}{t+\beta}, \qquad \rho = \frac{(t+\beta)^\delta \rho_3}{ty^2}, \qquad \delta = 0 \text{ or } 1$$

$$\frac{C_v}{R}p' + \frac{(C_v+R)(2t+\beta)^\delta\rho_3 - 2\kappa t^2}{\rho_3 R t (t+\beta)^\delta}\, p = \mu\, \frac{3\alpha^2 + 4(t^2+\beta t+\beta^2)^\delta}{3t^2(t+\beta)^{2\delta}} \qquad \text{(ODE)}$$

## 3D: Translations spatiales, Galiléennes, Échelles

$$u = \frac{\alpha z + x(t+c)}{(t+a)(t+c)}, \qquad v = \frac{\beta z + y(t+c)}{(t+b)(t+c)}, \qquad w = \frac{z}{t+c}$$

$$\rho = \frac{\rho_3(t+c)}{(t+a)(t+b)z^2}, \qquad p = \frac{\mu R\, h(t)^{-\frac{R}{C_v}-1} f(t)}{3C_v} \int \frac{h(t)^{\frac{R}{C_v}-1} g(t)}{f(t)}\, \mathrm{d}t$$

$$h(t) = (t+a)(t+b)(t+c)$$

$$f(t) = (t+c)^{\frac{2\kappa(a-c)(b-c)}{\rho_3 C_v}} \cdot \exp\left(\frac{\kappa t(t+2a+2b-2c)}{\rho_3 C_v}\right)$$

$$g(t) = \cdots$$

## 3D: Traveling wave, $\xi = t + ax + by + cz$

$$u = u_0\, \mathrm{e}^{\alpha\xi} + u_1\,, \qquad p = p_0\, \mathrm{e}^{\alpha\xi} + p_1\,, \qquad \rho = \left(\rho_0\, \mathrm{e}^{\alpha\xi} + \rho_1\right)^{-1}$$

The constants are linked by algebraic equations

# Other solutions/Other applications

- Other solutions
  - DCDS 2024
  - Non-exhaustive
  - Each symmetry transforms a solution into another one
  - Optimal basis of $\mathfrak{g}$
- Anisothermal incompressible fluid flow
  - Burgers vortex
  - Lundgren vortex
  - Burgers shear flow, $\cdots$
- Invariant numerical schemes
- Turbulence
  - Scaling laws of turbulence (wall laws, . . . )
  - Invariant LES turbulence models
- Conservation laws
  - Nœther: Variational symmetry $\iff$ Conservation law

# Link between symmetries and Integrating factors (IF)

- $\omega^t \,\mathrm{d}t + \omega^u \,\mathrm{d}u$ is exact $\iff$ $\dfrac{\partial \omega^t}{\partial u} = \dfrac{\partial \omega^u}{\partial t}$
- $\dfrac{\mathrm{d}u}{\mathrm{d}t} = \dfrac{tu + u^2}{tu - t^2}$, $\underbrace{(tu + u^2)\,\mathrm{d}x + (t^2 - tu)\,\mathrm{d}u}_{\alpha} = 0$
  - $\mu = \dfrac{1}{2t^2 u}$ is an IF: $\mu\alpha$ is exact
  - Resolution: $\mathrm{d}\left[\ln x + \ln u - \frac{u}{x}\right] = 0$
  - How to find $\mu$ ? ← $X = x\dfrac{\partial}{\partial x} + u\dfrac{\partial}{\partial u}$ is a symmetry
- 1st order Dynamical system: $\dfrac{\mathrm{d}u}{\mathrm{d}t} = F(t, u)$

Lie: $\xi \dfrac{\partial}{\partial t} + \eta \dfrac{\partial}{\partial u}$ is a symmetry $\iff$ $\mu = \frac{1}{\eta - \xi F}$ is an IF

- Higher-order: Exists dynamical systems with IF but without symmetry
- PDE: ?

# Use of conservation laws

- Better understanding of the dynamics of the system
- Theoretical analysis:
  - Resolution
  - Intregrability
  - Existence of solution
  - Stablility
  - ...
- Design of robust numerical schemes
  - Energy-preserving schemes
  - (Multi-)symplectic integrators

# Conservation laws

- ODE: $E(t, u, \dot{u}, \dots) = 0$ Solution manifold $\mathcal{S} \subset \mathbb{R} \times \mathbb{R}^{n_u}$

### First integral

Scalar function $F(t, u, \dot{u}, \dots)$ such that $\dfrac{\mathrm{d}F}{\mathrm{d}t} = 0$ on $\mathcal{S}$

- PDE: $E(y, u, u_{(1)}, \dots) = 0,$ Solution manifold $\mathcal{S} \subset \mathbb{R}^{n_y} \times \mathbb{R}^{n_u}$

### Conservation law:

$$Div\ F = 0 \qquad \text{on} \quad \mathcal{S}$$

- Flux: $F = F(y, u, u_{(1)}, u_{(2)}, \dots)$ is a $\mathbb{R}^{n_y}$-valued function
- Total divergence: $\mathrm{Div}\ F = D_1 F^1 + D_2 F^2 + \cdots + D_{n_y} F^{n_y}$

Total derivative $D_i = \dfrac{\partial}{\partial y^i} + u^a_i \dfrac{\partial}{\partial u^a} + u^a_{ij} \dfrac{\partial}{\partial u^a_j} + \cdots$

- If $y = (t, x)$ and $F = (F^t, -F^x)$ then $D_t F^t = Div_x F^x$

(total density variation) $\dfrac{\mathrm{d}}{\mathrm{d}t} \displaystyle\int_\Omega F^t \,\mathrm{d}x = \int_{\partial\Omega} F^x \,\mathrm{d}x$ (flux through $\partial\Omega$)

# Variational problem and infinitesimal transformation

## (Dirichlet) Variational problem and Euler-Lagrange equations

- $\mathcal{L} = \int_\Omega L(y, u, u_{(1)}, \dots)\ \mathrm{d}y$
- $\delta\mathcal{L} = 0 \qquad \Longrightarrow \qquad \mathrm{EL}(y, u, u_{(1)}, \dots) = 0$

$EL_a := \frac{\delta L}{\delta u^a} = \frac{\partial L}{\partial u^a} - D_i\left(\frac{\partial L}{\partial u^a_i}\right) + \cdots$

Lie group $\quad G = \{g_\epsilon : (y, u) \longmapsto (\widehat{y}\ , \widehat{u}\ )\} \xrightarrow[\text{transf.}]{\text{Infinit.}} \mathrm{X} = \xi(y, u)\frac{\partial}{\partial y} + \eta(y, u)\frac{\partial}{\partial u}$

## Generalized infinitesimal transformation:

- $\mathrm{X} = \xi(y, u, u_{(1)}, \dots)\frac{\partial}{\partial y} + \eta(y, u, u_{(1)}, \dots)\frac{\partial}{\partial u}$
- Prolongation: $\quad \mathrm{pr}\,\mathrm{X} = \mathrm{X} + \eta^a_J \frac{\partial}{\partial u^a_J} \quad$ with $\quad \eta^a_J = D_J Q^a + \xi^j u^a_{J,j}$
- Characteristic: $(Q^1, \dots, Q^{n_u})$ where $\quad Q^a = \eta^a - \xi^j u^a_j$

## Nœther's identity

$$(\mathrm{pr}\,\mathrm{X}) \cdot L + L\,\mathrm{Div}\,\xi = Q^a\,\mathrm{EL}_a + \mathrm{Div}\,P$$

$P^i = \xi^i L + Q^a \frac{\partial L}{\partial u^a_i} + \cdots$

Obtained from integration by parts

# (First) Nœther's theorem

**Variational symmetry group: $G = \{g_\epsilon : (y, u) \longmapsto (\,\widehat{y}, \widehat{u})\}$ s.t**

$$\int_{\widehat{\Omega}} L(\widehat{y}, \widehat{u}, \dots, \widehat{u}_{(p)})\ \mathrm{d}\widehat{y} = \int_{\Omega} L(y, u, \dots, u_{(p)})\ \mathrm{d}y$$

$G$ variational symmetry $\Longrightarrow (\mathrm{pr}\,X)\cdot L + L\,\mathrm{Div}\,\xi = 0$

**Generalized (divergence) variational symmetry: Generalized infinitesimal $X$**

$(\mathrm{pr}\,X)\cdot L + L\,\mathrm{Div}\,\xi = \mathrm{Div}\,B$ for some $B(y, u, u_{(1)}, \dots)$

**Nœther's theorem**

Variational symmetry group $\mathcal{L} \Longleftrightarrow$ Conservation law of $\mathrm{EL} = 0$

**Proof**

X is a generalized variational symmetry of $L$

$\Longleftrightarrow \quad Q^a\,\mathrm{EL}_a + \mathrm{Div}(P - B) = 0$ $\qquad (\mathrm{pr}\,X)\cdot L + L\,\mathrm{Div}\,\xi = Q^a\,\mathrm{EL}_a + \mathrm{Div}\,P$

$\Longleftrightarrow \quad \mathrm{Div}(P - B) = 0 \quad$ on $\quad \mathcal{S}_{EL}$

# Kepler's problem

- Position of the planet: $u = u(t)$, masse $= 1$

$$L = \frac{1}{2}\|\dot{u}\|^2 + \frac{\mu}{\|u\|}, \qquad \mathrm{EL} \equiv \ddot{u} + \frac{\mu u}{\|u\|^3} = 0$$

  - Time translation $\mathrm{X} = \dfrac{\partial}{\partial t}$ $\longleftrightarrow$ Energy: $\dfrac{1}{2}\|\dot{u}\|^2 - \dfrac{\mu}{\|u\|}$
  - Rotation $\mathrm{X} = u^i \dfrac{\partial}{\partial u^j} - u^j \dfrac{\partial}{\partial u^i}$ $\longleftrightarrow$ Angular momentum: $u \times \dot{u}$
  - ??? $\longleftrightarrow$ Runge-Lenz vector: $\dot{u} \times (u \times \dot{u}) - \mu \dfrac{u}{\|u\|}$

- Nœther's identity $(\mathrm{pr}\,\mathrm{X}) \cdot L + L\,\mathrm{Div}\,\xi = Q^a\,\mathrm{EL}_a + \mathrm{Div}\,P$

$$\mathrm{X} = \Big(\dot{u} \otimes u - 2u \otimes \dot{u} + (u \cdot \dot{u})\,\mathrm{Id}\Big)\frac{\partial}{\partial u}$$

# For a non-variational problem: Multipliers

$E[u] \equiv E(y, u, \dots, u_{(p)}) = 0,$ Solution manifold $\mathcal{S}_E$

## Multipliers of CoLas and their determining equation

$\mathrm{Div}\, P = 0$ on $\mathcal{S}_E$

$\Longleftrightarrow$ $\mathrm{Div}\, P = \Lambda^k E_k$ for some $\Lambda(y, u, u_{(1)}, \dots)$

(unique for Cauchy-Kovalevskaya PDE's)

$$\Longleftrightarrow \qquad \frac{\delta(\Lambda \cdot E)}{\delta u} = 0$$

$L = \mathrm{Div}\, \ell$ for some $\ell$ $\Longleftrightarrow$ $\dfrac{\delta L}{\delta u} \equiv 0$

i.e. Euler-Lagrange: $0 = 0$

Solve $\dfrac{\delta(\Lambda \cdot E)}{\delta u} = 0$ ⟶ “All” local CoLas (up to a prescribed order)

# From a "Bilagrangian" (Ibragimov)

$E[u] \equiv E(y, u, \dots, u_{(p)}) = 0,$ Solution manifold $\mathcal{S}_E$

- Adjoint variable: $v = (v^1, \dots, v^{n_u})$
- "Bilagrangian": $L[y, u, \dots, u_{(p)}, v] = v \cdot E$ Up to a total divergence
- Euler-Lagrange eq.: $\begin{cases} \dfrac{\delta(v \cdot E)}{\delta v} \equiv E[u] = 0 \\ \dfrac{\delta(v \cdot E)}{\delta u} \equiv E^*[u, v] = 0 \end{cases}$
- Nœther's theorem ⟶ Local and non-local CoLas in $(u, v)$

- Condition: $\dim E = \dim u = n_u$
- $\Lambda$ verifies $E^*[u, \Lambda] = 0$ while $v$ verifies $E^*[u, v]_{|\mathcal{S}_E} = 0$
- How to find variational symmetries ?

X variational symmetry $\Longrightarrow$ X (dynamical) symmetry of EL $= 0$

Search among (combinations of generalized) dynamical symmetries

# Outline of the bilagrangian approach

$$\begin{tikzcd}
E[y,u]=0 \arrow[r] & \begin{cases} E[y,u]=0 \\ E^{*}[y,u,v]=0 \end{cases} \arrow[r] & \begin{array}{c}\text{Dynamical symmetries}\\ (\operatorname{pr} X)\cdot\begin{bmatrix} E \\ E^{*} \end{bmatrix}=0\end{array} \arrow[d] \\
\cancel{L[y,u]} \arrow[u] & L[y,u,v] \arrow[u] \arrow[r] & \begin{array}{c}\text{Variational symmetries}\\ (\operatorname{pr} X)\cdot L + L\operatorname{Div}\xi = \operatorname{Div} B\end{array} \arrow[d, "\text{Nœther}"] \\
 & & \text{Conservation laws}
\end{tikzcd}$$

# Examples in fluid mechanics

- Heat equation (Kolsrud, Ibragimov, Brandao)
  - $E[u] \quad\equiv u_t - \kappa u_{xx} = 0$
  - $E^*[u,v] \equiv v_t + \kappa v_{xx} = 0$
  - $L[u,v] \; = \frac{1}{2}\left(u_t v - u v_t\right) + \kappa u_x v_x$
- Burgers'equation (Kolsrud, Ibragimov, Brandao)
  - $E[u] \quad\equiv u_t + u u_x - \kappa u_{xx} = 0$
  - $E^*[u,v] \equiv v_t + u v_x + \kappa v_{xx} = 0$
  - $L[u,v] \; = \frac{1}{2}\left(v u_t - u v_t\right) + \kappa u_x v_x + \frac{1}{3}(u v_x - v u_x)$
- Navier-Stokes equation (Hamdouni, Razafindralandy)
  - $E[u,p] : \qquad \dfrac{\partial u}{\partial t} + (\nabla u)u + \dfrac{1}{\rho}\nabla p - \nu \Delta u = 0, \qquad \operatorname{div} u = 0$
  - $E^*[u,p,v,q] : \dfrac{\partial v}{\partial t} + (\nabla v)u - ({}^\top\nabla u)v + \dfrac{1}{\rho}\nabla q + \nu\Delta v = 0, \qquad \operatorname{div} v = 0$
  - $L = \dfrac{1}{2}\left(\dfrac{\mathrm{d}u}{\mathrm{d}t}\cdot v - u\cdot\dfrac{\mathrm{d}v}{\mathrm{d}t}\right) + \left(\dfrac{q}{\rho} - \dfrac{u\cdot v}{2}\right)\operatorname{div} u - \dfrac{p}{\rho}\operatorname{div} v + \nu\ \operatorname{tr}({}^\top\nabla u\,\nabla v)$

# Navier-Stokes + adjoint equations

$$\begin{cases} \dfrac{\partial u}{\partial t} + (\nabla u)u + \dfrac{1}{\rho}\nabla p - \nu\Delta u = 0, & \operatorname{div} u = 0 \\ \dfrac{\partial v}{\partial t} + (\nabla v)u - ({}^{\top}\nabla u)v + \dfrac{1}{\rho}\nabla q + \nu\Delta v = 0, & \operatorname{div} v = 0 \end{cases}$$

## Infinitesimal dynamical symmetries

$\pi(t)$, $\varphi(t)$, $w^i(t)$ and $z^i(t)$ are arbitrary functions of $t$

- $\dfrac{\partial}{\partial t}, \qquad \pi(t)\dfrac{\partial}{\partial p}, \qquad \varphi(t)\dfrac{\partial}{\partial q}$
- $w^i\dfrac{\partial}{\partial v^i} + \rho\left(w^i u^i - w^i_t x^i\right)\dfrac{\partial}{\partial q},$ $\quad i = 1, 2, 3$
- $x^j\dfrac{\partial}{\partial v^i} - x^i\dfrac{\partial}{\partial v^j} + \rho(x^j u^i - x^i u^j)\dfrac{\partial}{\partial q}$ $\quad (i, j) \in \{(1, 2), (2, 3), (3, 1)\}$
- $x^j\dfrac{\partial}{\partial x^i} - x^i\dfrac{\partial}{\partial x^j} + u^j\dfrac{\partial}{\partial u^i} - u^i\dfrac{\partial}{\partial u^j} + v^j\dfrac{\partial}{\partial v^i} - v^i\dfrac{\partial}{\partial v^j},$ $\quad (i, j) \in \{(1, 2), (2, 3), (3, 1)\}$
- $z^i\dfrac{\partial}{\partial x^i} + z^i_t\dfrac{\partial}{\partial u^i} - \rho x^i z^i_{tt}\dfrac{\partial}{\partial p},$ $\quad i = 1, 2, 3$
- $2t\dfrac{\partial}{\partial t} + x^k\dfrac{\partial}{\partial x^k} - u^k\dfrac{\partial}{\partial u^k} - 2p\dfrac{\partial}{\partial p} - q\dfrac{\partial}{\partial q}$ $\quad$ (sum over $k$)
- $v^k\dfrac{\partial}{\partial v^k} + q\dfrac{\partial}{\partial q}.$ $\quad$ (sum over $k$)

# Navier-Stokes + adjoint equations

$$\begin{cases} \dfrac{\partial u}{\partial t} + (\nabla u)u + \dfrac{1}{\rho}\nabla p - \nu\Delta u = 0, & \operatorname{div} u = 0 \\ \dfrac{\partial v}{\partial t} + (\nabla v)u - ({}^{\top}\nabla u)v + \dfrac{1}{\rho}\nabla q + \nu\Delta v = 0, & \operatorname{div} v = 0 \end{cases}$$

## Local dynamical symmetry group

$\pi(t)$, $\varphi(t)$, $w^i(t)$ and $z^i(t)$ are arbitrary functions of $t$

- Time translation: $(t, x, u, p, v, q) \longmapsto (t+\epsilon, x, u, p, v, q)$
- Pressure translation: $(t, x, u, p+\pi(t), v, q)$
- Adjoint-pressure translation: $(t, x, u, p, v, q+\varphi(t))$
- 1st $(v, q)$ translation : $(t, x, u, p, v+w, q+\rho w \cdot u - \rho w_t \cdot x)$
- 2nd $(v, q)$ translation ($\omega$ is a constant vector): $(t, x, u, p, v+\omega \times x, q+\rho x \cdot \omega \times u)$
- Constant rotation matrix $R$: $(t, Rx, Ru, p, Rv, q)$
- Generalized Galilean transformation: $(t, x+z, u+z_t, p+\rho z_{tt} \cdot x, v, q)$
- 1st scale transformation: $(e^{2\epsilon}t, e^{\epsilon}x, e^{-\epsilon}u, e^{-2\epsilon}p, v, e^{-\epsilon}q)$
- 2nd scale transformation: $(t, x, u, p, e^{\epsilon}v, e^{\epsilon}q)$

# Some conservation laws

- Generator: $\mathrm{V}_{ij} = x^j \dfrac{\partial}{\partial v^i} - x^i \dfrac{\partial}{\partial v^j} + \rho(x^j u^i - x^i u^j) \dfrac{\partial}{\partial q}$ $\quad (i,j) \in \{(1,2),(2,3),(3,1)\}$

- Translation : $(t, x, u, p, v, q) \longmapsto (t, x, u, p, v + \omega \times x, q + \rho x \cdot \omega \times u)$

▶ $(i,j) = (1,2)$

- Divergence symmetry: $\mathrm{pr}\,\mathrm{V}_{12} \cdot L + L\,\mathrm{Div}\,\xi = \mathrm{Div}\,B$ with

$$B = \tfrac{1}{2}(x^2 u^1 - x^1 u^2) \begin{pmatrix} 1 \\ u^1 \\ u^2 \\ u^3 \end{pmatrix} - \nu \begin{pmatrix} 0 \\ u^2 \\ -u^1 \\ 0 \end{pmatrix}$$

- Flux: $P = \dfrac{1}{2}(x^1 u^2 - x^2 u^1) \begin{pmatrix} 1 \\ u^1 \\ u^2 \\ u^3 \end{pmatrix} - \dfrac{p}{\rho} \begin{pmatrix} 0 \\ x^2 \\ -x^1 \\ 0 \end{pmatrix} + \nu \begin{pmatrix} 0 \\ x^2 u^1_1 - x^1 u^2_1 \\ x^2 u^1_2 - x^1 u^2_2 \\ x^2 u^1_3 - x^1 u^2_3 \end{pmatrix}$

- Local conservation law: $\mathrm{Div}(P - B) = 0$

▶ $(i,j) = (2,3)$ et $(i,j) = (3,1)$: ...

- Generator $\mathrm{R}_{ij} = x^j \frac{\partial}{\partial x^i} - x^i \frac{\partial}{\partial x^j} + u^j \frac{\partial}{\partial u^i} - u^i \frac{\partial}{\partial u^j} + v^j \frac{\partial}{\partial v^i} - v^i \frac{\partial}{\partial v^j}$, $(i,j) \in \{(1,2),(2,3),(3,1)\}$
- Rotation $(t,x,u,p,v,q) \longmapsto (t, Rx, Ru, p, Rv, q)$

▶ $(i,j) = (1,2)$

- Variational symmetry: $\mathrm{pr}\,\mathrm{R}_{12} \cdot L + L\,\mathrm{Div}\,\xi = 0$
- Non-local conservation law with

$$P = \begin{pmatrix} u^2 v^1 - u^1 v^2 + \frac{1}{2}\left(v \cdot \mathrm{R}^{(0)}_{12} u - u \cdot \mathrm{R}^{(0)}_{12} v\right) \\ x^2 L + u^2 \frac{\partial L}{\partial u^1_1} - u^1 \frac{\partial L}{\partial u^2_1} + \mathrm{R}^{(0)}_{12} u^k \frac{\partial L}{\partial u^k_1} + v^2 \frac{\partial L}{\partial v^1_1} - v^1 \frac{\partial L}{\partial v^2_1} + \mathrm{R}^{(0)}_{12} v^k \frac{\partial L}{\partial v^k_1} \\ -x^1 L + u^2 \frac{\partial L}{\partial u^1_2} - u^1 \frac{\partial L}{\partial u^2_2} + \mathrm{R}^{(0)}_{12} u^k \frac{\partial L}{\partial u^k_2} + v^2 \frac{\partial L}{\partial v^1_2} - v^1 \frac{\partial L}{\partial v^2_2} + \mathrm{R}^{(0)}_{12} v^k \frac{\partial L}{\partial v^k_2} \\ u^2 \frac{\partial L}{\partial u^1_3} - u^1 \frac{\partial L}{\partial u^2_3} + \mathrm{R}^{(0)}_{12} u^k \frac{\partial L}{\partial u^k_3} + v^2 \frac{\partial L}{\partial v^1_3} - v^1 \frac{\partial L}{\partial v^2_3} + \mathrm{R}^{(0)}_{12} v^k \frac{\partial L}{\partial v^k_3} \end{pmatrix}$$

where $\mathrm{R}^{(0)}_{ij} = x^j \frac{\partial}{\partial x^i} - x^i \frac{\partial}{\partial x^j}$ is the base part of $\mathrm{R}_{ij}$

▶ $(i,j) = (2,3)$ et $(i,j) = (3,1)$: …

- Generator $S_1 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - u\frac{\partial}{\partial u} - 2p\frac{\partial}{\partial p} - q\frac{\partial}{\partial q}$
- 1st scale transformation: $(e^{2\epsilon}t, e^{\epsilon}x, e^{-\epsilon}u, e^{-2\epsilon}p, v, e^{-\epsilon}q)$

$\mathrm{pr}\, S_1 \cdot L + L\, \mathrm{Div}\, \xi = 2L \longrightarrow$ not a generalized variationnal symmetry

- Générateur $S_2 = v\frac{\partial}{\partial v} + q\frac{\partial}{\partial q}$
- 2nd scale transformation: $(t, x, u, p, e^{\epsilon}v, e^{\epsilon}q)$

$\mathrm{pr}\, S_2 \cdot L + L\, \mathrm{Div}\, \xi = L \longrightarrow$ not a generalized variationnal symmetry

But: $S = S_1 - 2S_2$
$\mathrm{pr}\, S \cdot L + L\, \mathrm{Div}\, \xi = 0 \longrightarrow$ variational symmetry

- Generator $$\mathrm{S} = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - u\frac{\partial}{\partial u} - 2p\frac{\partial}{\partial p} - q\frac{\partial}{\partial q} - 2v\frac{\partial}{\partial v} - 2q\frac{\partial}{\partial q}$$

- Scale transformation: $(\mathrm{e}^{2b}t, \mathrm{e}^{b}x, \mathrm{e}^{-b}u, \mathrm{e}^{-2b}p, \mathrm{e}^{-2b}v, \mathrm{e}^{-3b}q)$

- Variational symmetry: $\mathrm{pr}\,\mathrm{S}\cdot L + L\,\mathrm{Div}\,\xi = 0$

- Non-local conservation law with flux $P = (P^0, P^1, P^2, P^3)$

$$P^0 = \tfrac{1}{2}(u\cdot v) + t(u\cdot v_t - u_t\cdot v) + x\cdot\left(\nabla\tfrac{u\cdot v}{2} - (\nabla u)v\right)$$

$$P^i = x^i L - U\cdot\left[\overline{q}\,\mathrm{e}_i + \tfrac{u^i v}{2} + \nu v_i\right] - V\cdot\left[-\tfrac{p}{\rho}\,\mathrm{e}_i - \tfrac{u^i u}{2} + \nu u_i\right], \quad i = 1, 2, 3,$$

where $U = u + 2tu_t + (\nabla u)x, \qquad V = 2v + 2tv_t + (\nabla v)x$

- Generator $\mathrm{Z}_i = z^i \dfrac{\partial}{\partial x^i} + z_t^i \dfrac{\partial}{\partial u^i} - \rho x^i\, z_{tt}^i \dfrac{\partial}{\partial p}, \qquad i = 1, 2, 3$

- Generalized Galilean transformation: $(t, x+z, u+z_t, p+\rho z_{tt}\cdot x, v, q)$

▶ $i = 1$

- Divergence symmetry: $\mathrm{pr}\,\mathrm{Z}_1 \cdot L + L\,\mathrm{Div}\,\xi = \mathrm{Div}\,B$ where

$$B = -\tfrac{1}{2} z_t^1 v^1 \begin{pmatrix} 1 \\ u^1 \\ u^2 \\ u^3 \end{pmatrix}$$

- Case $z^1$ constant

- Flux

$$P = \begin{pmatrix} \dfrac{1}{2}\,(v_1\cdot u - u_1\cdot v)\,, \\ L + \dfrac{1}{2} u^1\,(v_1\cdot u - u_1\cdot v) - 2\nu u_1\cdot v_1 - u_1^1\overline{q} + v_1^1\overline{p}, \\ \dfrac{1}{2} u^2\,(v_1\cdot u - u_1\cdot v) - \nu\,(u_1\cdot v_2 + u_2\cdot v_1) - u_1^2\overline{q} + v_1^2\overline{p}, \\ \dfrac{1}{2} u^3\,(v_1\cdot u - u_1\cdot v) - \nu\,(u_1\cdot v_3 + u_3\cdot v_1) - u_1^3\overline{q} + v_1^3\overline{p}. \end{pmatrix}$$

$$\overline{p} = p + \rho\tfrac{u\cdot u}{2}$$

$$\overline{q} = q - \rho\tfrac{u\cdot v}{2}$$

▶ $i = 2, 3$: ...

▶ $z$ non constant ...

# Conclusion

- Interpretation of these conservation laws
  Integral form

- Non exhaustive
  - Other combinations of dynamical symmetries
  - Higher-order conservation laws: Bäcklund

- Inviscid flow $\nu = 0$: $$\frac{\partial u}{\partial t} + (\nabla u)u + \frac{1}{\rho}\nabla p = 0, \qquad \operatorname{div} u = 0$$
  - Derive from the Lagrangian $L = \frac{1}{2}\|v\|^2$ in Euler-Poincaré sense
  - Nœther's theorem in Euler-Poincaré sense
  - Compare with the previous conservation laws with $\nu = 0$