Time-space formulation of a conservative string subject to finite transformations

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- Study of the behavior of solids for acoustics
- Prediction of non linear effects

Figure – Laser interferometry
https ://www.gurumed.org/2011/05/30/lasers-reveal-exactly-how-guitars-create-music/

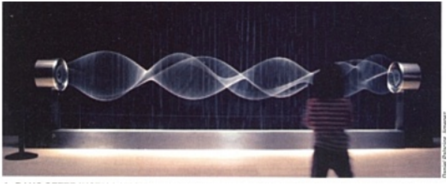

Figure – Vibrating string
https ://www.pedagogie.ac-nantes.fr/physique-chimie/mutualisation/travail-collaboratif/a-la-decouverte-des-ondes-683840.kjsp

String equation

Consider the transverse displacement $w : (0,+\infty) \times (0,L) \to \mathbb{R}$:

$$\mu \ddot{w} = T\partial_x^2 w + f$$
$$\text{with} \quad w(t,0) = w(t,L) = 0 \quad w(0,x) = 0$$

$\mu = \rho A$ [kg.m^{-1}] linear mass density, T [N] tension, f [N/m] external force

Power balance

$$\frac{d}{dt}\left(\int_0^L \left[\frac{\mu \dot{w}^2}{2} + \frac{T(\partial_x w)^2}{2}\right] \mathrm{d}x\right) = \int_0^L f\,\dot{w}\,\mathrm{d}x$$

Port Hamiltonian system for the string

The Hamiltonian $\mathcal{H}(\alpha) := \int_0^L \frac{1}{2}\alpha(x)^T W(x)\alpha(x)\,\mathrm{d}x$ with $W = \mathrm{diag}(1/\mu, T)$
and with

State α	Units	Flow	Units	Effort	Units
$\alpha_1 = \pi = \mu\,\dot{w}$ momentum	[kg/s]	$\dot{\pi} = \mu\ddot{w}$ inertial force	[N/m]	$\pi/\mu = \dot{w}$ material point velocity	[m/s]
$\alpha_2 = \varepsilon = \partial_x w$ infinitesimal strain	adim	$\dot{\varepsilon} = \frac{d}{dt}(\partial_x w)$ strain rate	[s^{-1}]	$T\partial_x w$ elastic force	[N]
Port		$\dot{w}$ contact point velocity	[m/s]	$-f$ force applied to the actuator	[N/m]

PHS

$$\underbrace{\begin{pmatrix}\dot{\alpha} \\ w\end{pmatrix}}_{\text{flow } f} = \mathcal{J}\underbrace{\begin{pmatrix}\delta H(\alpha) \\ -f\end{pmatrix}}_{\text{effort } e} \quad \text{where} \quad \mathcal{J} = \begin{bmatrix} 0 & \partial_x & -1 \\ \partial_x & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

leads to

string equation : $\mu\ddot{w} = T\partial_x(\partial_x w) + f$

kinematic concordance : $\frac{d}{dt}(\partial_x w) = \partial_x(\dot{w})$

idem : $\dot{w} = \dot{w}$

Questions and approach

Here, we focus on

Dynamic of a string undergoing reversible finite transformations

Questions

Can we find a modeling and numerical schemes that

1. verifies the principles of mechanics : principle of linear momentum, principles of thermodynamics ?
2. is adapted to dynamics in finite transformations ?

$\rightarrow$ Review 3D continuum mechanics for the dynamics of solids

$\rightarrow$ Raise issues.

Investigate

Time-space formulation and address 4D covariance

Plan

1. 3D continuum mechanics of the string (under finite transformations)
2. Issues
3. Time-space covariant approach

String kinematics

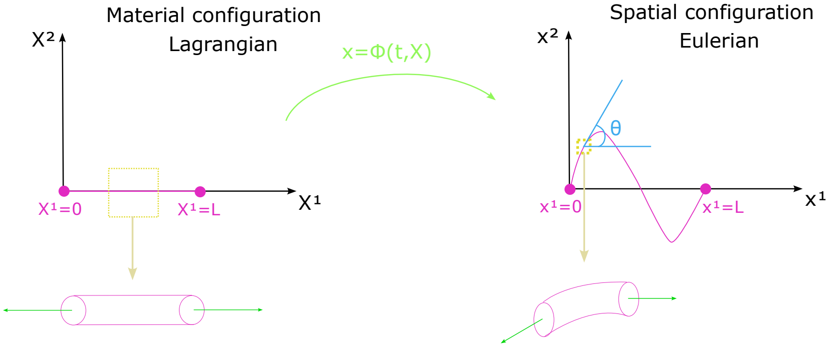

Figure – The string kinematic hypothesis.

Transformation

$$x = \Phi(t, X_1, X_2, X_3)$$

For the string : Φ composed of longitudinal strain $\boldsymbol{U}(\lambda)$ and rotation $\boldsymbol{R}(\theta)$
where λ is the relative elongation along the string

Material configuration Lagrangian

Lagrange deformation tensor $\boldsymbol{E}$

$$\boldsymbol{E} = \frac{1}{2}\left(\boldsymbol{F}^T\boldsymbol{F} - \boldsymbol{I}\right)$$

$$x = \Phi(t, X_1, X_2, X_3)$$

$$F^i_j = \frac{\partial x^i}{\partial X^j}$$

$$(\boldsymbol{F}^{-1})^T\boldsymbol{E}\boldsymbol{F}^{-1} = \boldsymbol{e}$$

$$\boldsymbol{E} = \boldsymbol{F}^T\boldsymbol{e}\boldsymbol{F}$$

Spatial configuration Eulerian

Euler Almansi deformation tensor $\boldsymbol{e}$

$$\boldsymbol{e} = \frac{1}{2}\left(\boldsymbol{I} - (\boldsymbol{F}^{-1})^T\boldsymbol{F}^{-1}\right)$$

For the string

$$\Phi \begin{cases} x^1 = X^1 + W^1(t, X^1, X^2) \\ x^2 = X^2 + W^2(t, X^1, X^2) \\ x^3 = X^3 \end{cases}$$

with the identification $\boldsymbol{F}(\theta, \lambda) = \boldsymbol{R}(\theta)\boldsymbol{U}(\lambda)$:

$$F^i_j = \begin{pmatrix} 1 + \partial_{X^1} W^1 & \partial_{X^2} W^1 & 0 \\ \partial_{X^1} W^2 & 1 + \partial_{X^2} W^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \lambda\cos\theta & -\sin\theta & 0 \\ \lambda\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and $\boldsymbol{E} = \frac{1}{2}\begin{pmatrix} \lambda^2 - 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Principle of linear momentum (Newton)

Material configuration Lagrangian

Piola-Kirchhoff 2 stress tensor $\boldsymbol{\Sigma}$

$$J^{-1}\boldsymbol{F}\boldsymbol{\Sigma}\boldsymbol{F}^T = \boldsymbol{\sigma}$$
$$\boldsymbol{\Sigma} = J\boldsymbol{F}^{-1}\boldsymbol{\sigma}(\boldsymbol{F}^{-1})^T$$

Spatial configuration Eulerian

Cauchy stress tensor $\boldsymbol{\sigma}$

Principle of linear momentum :
or its weak formulation

$$div(\boldsymbol{\sigma}) + \boldsymbol{f} = \rho\boldsymbol{a}$$

$$\Downarrow$$

Power balance

$$\int_\Omega \left(\rho\boldsymbol{a}.\boldsymbol{v} + \boldsymbol{\sigma} : \boldsymbol{d}\right) \mathrm{d}\Omega = \mathcal{P}^{ext}$$

ρ the mass density
$(\rho\boldsymbol{a}.\boldsymbol{v})$ the power density for inertia
$(\boldsymbol{\sigma} : \boldsymbol{d})$ the mechanical power density with $\boldsymbol{d}$ the rate of deformation
$\mathcal{P}^{ext}$ the external power density

First principle of thermodynamics

Hyper-elasticity : adiabatic or isothermal reversible transformations

Material configuration Lagrangian

$$\rho_0\dot{\mathscr{E}}(\boldsymbol{E}) - \boldsymbol{\Sigma} : \boldsymbol{D} = 0$$

$\psi = \rho_0\mathscr{E}$ Energy density

$$\Downarrow$$

$$\dot{\psi}(\boldsymbol{E}) - \boldsymbol{\Sigma} : \boldsymbol{D} = 0$$

$$\dot{\psi}(\boldsymbol{E}) = \frac{\partial\psi}{\partial\boldsymbol{E}} : \boldsymbol{D}$$

$$\dot{\boldsymbol{E}} = \boldsymbol{D}$$

$$\Downarrow$$

$$\boldsymbol{\Sigma} = \frac{\partial\psi}{\partial\boldsymbol{E}}$$

Constitutive model

$\longleftarrow$

$$\boldsymbol{D} = \boldsymbol{F}^T d\boldsymbol{F}$$

$$\boldsymbol{\Sigma} = J\boldsymbol{F}^{-1}\boldsymbol{\sigma}(\boldsymbol{F}^{-1})^T$$

$\longrightarrow$

Spatial configuration Eulerian

$$\rho\dot{\mathscr{E}}(\boldsymbol{e}) - \boldsymbol{\sigma} : \boldsymbol{d} = 0$$

$\mathscr{E}(\boldsymbol{e})$ the *specific* internal energy

$$\boldsymbol{\sigma} = J^{-1}\boldsymbol{F}\boldsymbol{\Sigma}\boldsymbol{F}^T$$

Constitutive model for the string

Constitutive choice : $\psi = \frac{1}{2}\boldsymbol{E} : \boldsymbol{CE}$

$$\text{with } C^{ijkl} : \begin{pmatrix} Y & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} Y \text{ Young modulus} \quad \left(\boldsymbol{E} = \frac{1}{2}\begin{pmatrix} \lambda^2 - 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right)$$

Anisotropic model for the string.

This yields

$$\psi = \frac{Y}{8}\left[\left(\lambda^2 - 1\right)^2\right]$$

The Piola-Kirchhoff 2 stress tensor is then :

$$\Sigma^{ij} = \frac{\partial \psi}{\partial E_{ij}} = \frac{Y}{2}\begin{pmatrix} \lambda^2 - 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Plan

Issue 1 : configurations : Lagrange vs Euler

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Material configuration Lagrangian

Piola-Kirchhoff 2 stress tensor $\boldsymbol{\Sigma}$

No principle of linear momentum on the Lagrangian configuration

$$J^{-1}\boldsymbol{F}\boldsymbol{\Sigma}\boldsymbol{F}^T = \boldsymbol{\sigma}$$

$$\boldsymbol{\Sigma} = J\boldsymbol{F}^{-1}\boldsymbol{\sigma}(\boldsymbol{F}^{-1})^T$$

Spatial configuration Eulerian

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Power balance

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Hyper-elasticity : adiabatic or isothermal reversible transformations

Material configuration
Lagrangian

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$$\dot{\boldsymbol{E}} = \boldsymbol{D}$$

$$\Downarrow$$

$$\boldsymbol{\Sigma} = \frac{\partial\psi}{\partial\boldsymbol{E}}$$

Constitutive model

$\longleftarrow$

$$\boldsymbol{D} = \boldsymbol{F}^T \boldsymbol{dF}$$

$$\boldsymbol{\Sigma} = J\boldsymbol{F}^{-1}\boldsymbol{\sigma}(\boldsymbol{F}^{-1})^T$$

$\longrightarrow$

Spatial configuration
Eulerian

$$\rho\dot{\mathscr{E}}(\boldsymbol{e}) - \boldsymbol{\sigma} : \boldsymbol{d} = 0 \qquad (1)$$

$\mathscr{E}(\boldsymbol{e})$ the specific internal energy

No constitutive model derived directly from (1)

Issue 2 : strain rate in power balance

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Material configuration Lagrangian

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Issue 2 : strain rate in power balance

Mechanical power

The mechanical power density $(\boldsymbol{\sigma} : \boldsymbol{d})$ in the power balance
is not the differentiation with respect to time of the mechanical energy *density* because

$$\dot{\boldsymbol{e}} \neq \boldsymbol{d} \quad \text{infinitesimal transformations } \dot{\varepsilon} \approx \boldsymbol{d}$$

$$\dot{\boldsymbol{\rho}} \neq 0 \quad \text{infinitesimal transformations } \dot{\rho} \approx 0$$

These issues prevent from writing a proper PHS formulation for finite transformations involving the time derivative of a state whether in the Lagrangian or the Eulerian point of view.
Every other nonlinear model considered in the literature should encounter the same difficulties.

Plan

World-tube

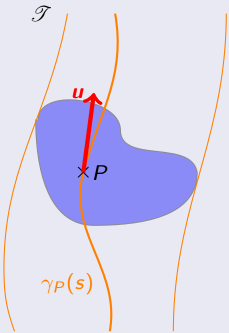

Figure – Representation of a world-tube $\mathscr{T}$ with world-lines.

4D manifold

Universe : connected orientable $4D$ Riemannian manifold $\mathcal{M}$

- Continuum (continuously distributed matter : rest mass density $\widetilde{\rho}_c$...)
- Metric tensor field $\boldsymbol{g}$
- Event $P \in \mathcal{M}$ identified by a set x of four coordinates $(x^\mu)_{\mu\in\{0,1,2,3\}}$
- Worldline $\gamma_P(s)$ curve in $\mathcal{M}$,

$$\gamma_P : \mathbb{R} \to \mathcal{M}, s \mapsto \gamma_P(s) = \{\phi_s(P)\}.$$

time-space counterpart of trajectories

- 4-velocity noted $\boldsymbol{u}$:

$$u^\mu = \frac{dx^\mu}{ds} \quad \text{with} \quad ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu.$$

time-like future oriented unit vector

World-tube

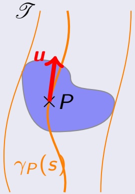

Figure – Event P of coordinates x or $\widetilde{x}$ or $\widehat{x}$.

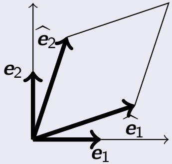

Figure – Example for current basis vectors $\boldsymbol{e}_\mu$ and proper basis vectors $\widehat{\boldsymbol{e}}_\mu$ for sliding.

Covariance

Principle of covariance : the formulation of the laws of physics should be invariant to changes of observers.

- Observer : a 4D coordinate system
- Transformations of coordinates :

$$\varphi : \widetilde{x}^\nu \in S_\alpha \mapsto x^\mu = \varphi^\mu(\widetilde{x}^\nu) \in S_\beta .$$

basis vectors $\boldsymbol{e}_\mu$ and $\widetilde{\boldsymbol{e}}_\mu$;
Jacobian matrix of φ : $\left(\frac{\partial x^\kappa}{\partial \widetilde{x}^\mu}\right) \in GL(4, \mathbb{R})$

- A vector V is invariant to changes of observers : $\boldsymbol{V} = V^\mu\ \boldsymbol{e}_\mu = \widetilde{V}^\mu\ \widetilde{\boldsymbol{e}}_\mu = \widehat{V}^\mu\ \widehat{\boldsymbol{e}}_\mu$
- *Proper* observer $\widehat{x}$, a coordinate system with basis vectors $\widehat{\boldsymbol{e}}_\mu$ such that :

$$\widehat{u}^\mu = (1, 0, 0, 0) \quad \text{for all events.}$$

time-space counterpart of the Lagrangian description
→ Only one manifold (no configuration) Issue 1

Time-space kinematics

World-tube

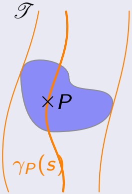

Figure – Event P of coordinates x or $\widetilde{x}$ or $\widehat{x}$.

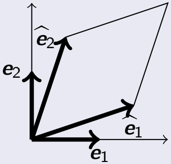

Figure – Example for current basis vectors $\boldsymbol{e}_\mu$ and proper basis vectors $\widehat{\boldsymbol{e}}_\mu$ for sliding.

Time-space deformation $\boldsymbol{b}$

Second order covariant tensor

$$\boldsymbol{b} = \eta_{\mu\nu} \widehat{\boldsymbol{e}}^{\boldsymbol{\mu}} \otimes \widehat{\boldsymbol{e}}^{\boldsymbol{\nu}}$$

with $\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

Time-space strain $\boldsymbol{e}$

Second order covariant tensor

$$\boldsymbol{e} = \frac{1}{2}(\boldsymbol{g} - \boldsymbol{b})$$

Time-space counterpart of Euler-Almansi tensor

Energy-momentum tensor $\boldsymbol{T}$

$$T^{\mu\nu} = 2\frac{\partial \mathcal{L}_\mathrm{M}}{\partial g_{\mu\nu}}$$

where $\mathcal{L}_\mathrm{M}$ is the material contribution to the energy density Issue 2

Time-space counterpart of the stress tensor

Definition verified for any observer

The conservation of energy and momentum

$$\nabla_\nu T^{\mu\nu} = 0$$

where ∇ denotes the covariant derivative.

Time-space counterpart of BOTH :

- power balance
- principle of linear momentum

Conservation verified for any observer

Projectors

To identify the time-like or space-like contributions, define

- A *time projector* : projection on the 4-velocity $\boldsymbol{u}$.
- A *spatial projector*

Time and space decomposition of $\boldsymbol{T}$

$$T^{\mu\nu} = \mathcal{U} u^\mu u^\nu + \underline{T}^\mu u^\nu + \underline{T}^\nu u^\mu + \underline{\underline{T}}^{\mu\nu}$$

A scalar quantity $\mathcal{U}$, the projection of $\boldsymbol{T}$ twice on time, material energy density :

$$\mathcal{U} = T_{\kappa\lambda} u^\kappa u^\lambda = \mathcal{L}_\mathrm{M}$$

A vector noted $\underline{\boldsymbol{T}}$, its projection on time and space, a heat flux,

$$\underline{T}^\mu = T_{\kappa\lambda} \underline{\underline{g}}^{\mu\kappa} u^\lambda = 0$$

A second order contravariant tensor noted $\underline{\underline{\boldsymbol{T}}}$, its projection twice on space, a stress tensor.

String model with a time-space approach

Hypotheses

Phenomena limited to :

- mechanical effects
- small speed (compared to the speed of light in vaccum)
- no gravitation

Constitutive model

$$\mathcal{L}_{\mathrm{M}}(\boldsymbol{g}, \widetilde{\rho}_c, \boldsymbol{\mathcal{C}}, \boldsymbol{b}) = \widetilde{\rho}_c c^2 + W(\boldsymbol{g}, \boldsymbol{\mathcal{C}}, \boldsymbol{b})$$

$$W = \frac{1}{2}\mathcal{C}^{\alpha\beta\kappa\lambda}\underline{e}_{\alpha\beta}\underline{e}_{\kappa\lambda}$$

W is the time-space counterpart of ψ

The components of the fourth order tensor $\boldsymbol{\mathcal{C}}$ are measured in the proper coordinate system, in Voigt notation, to give :

$$\widehat{\mathcal{C}}^{\alpha\beta\kappa\lambda} : \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & Y & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

where Y is a constant scalar.
The energy-momentum tensor :

$$\begin{aligned} T^{\mu\nu} &= 2\frac{\partial \mathcal{L}_{\mathrm{M}}}{\partial g_{\mu\nu}} \\ &= 2\frac{\partial W}{\partial g_{\mu\nu}} + (\widetilde{\rho}_c c^2 + W) g^{\mu\nu} \end{aligned}$$

Conclusion and perspectives

	HPP	Finite transformations		
		Advantages	Drawbacks	TBD
PDE	OK (linear)		Linear momentum : Euler config. Material law : Lagrange config. Covariance ?	
PHS	OK (linear)	Passive formulation	Power balance : Which flow for the strain ? Covariance ?	
TSC	OK (nonlinear) and useless	Passive covariant formulation		Write the numerical scheme for dynamic case
PHSTSC				Find a state space representation