Rachel Azulay

Christelle Combescure

Prédiction des modes bifurqués périodiques des matériaux architecturés grâce à la théorie des groupes

Predicting the post-bifurcated patterns of architectured materials using group-theoretic tools

R. Azulay, C. Combescure

J. Mech. Phys. Solids 187 (2024) 105631

Introduction

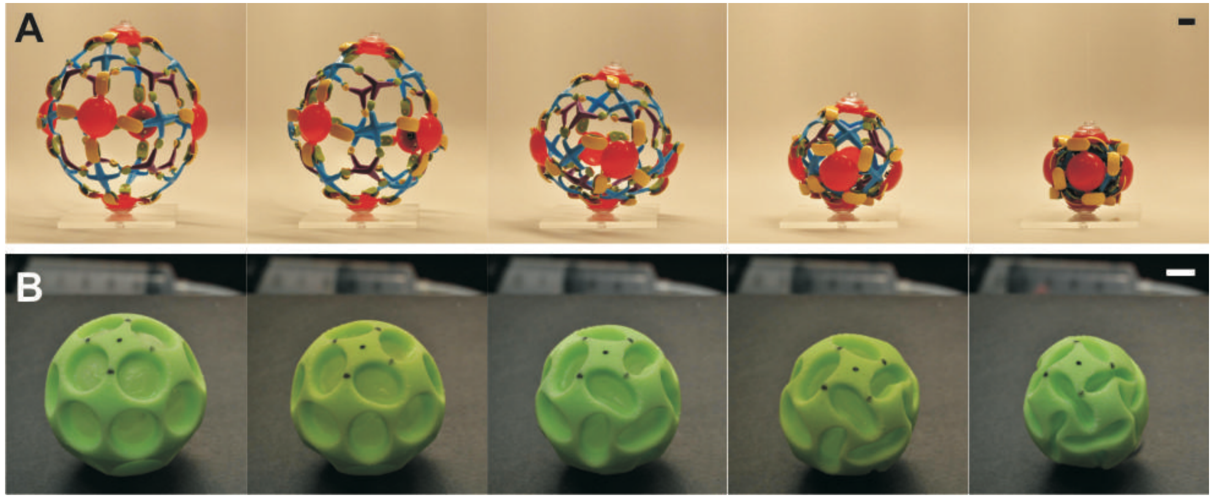
Lattice materials


Lattice cell

- Mesoscopic arrangement

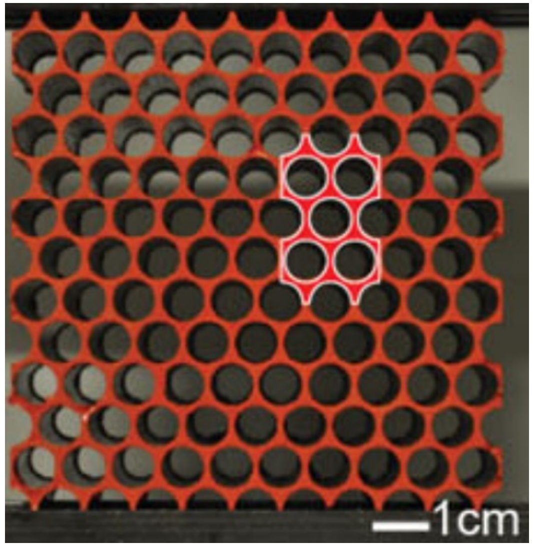
Periodic	Quasiperiodic	Aperiodic
 *Welbeck St Car Park, London*	 *Transbay Transit Center, San Francisco*	 *MUCEM, Marseille*

Pattern Generation in Architectured Materials

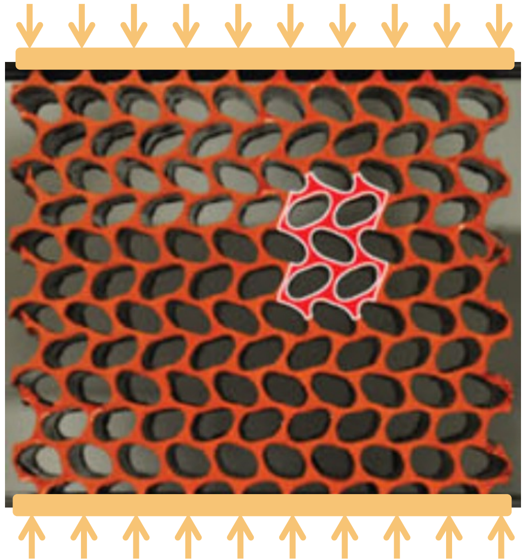


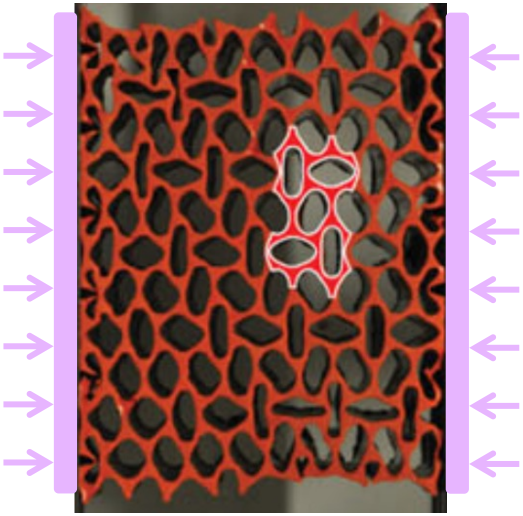
[Shim 2012] Buckling-induced encapsulation of structured elastic shells under pressure.

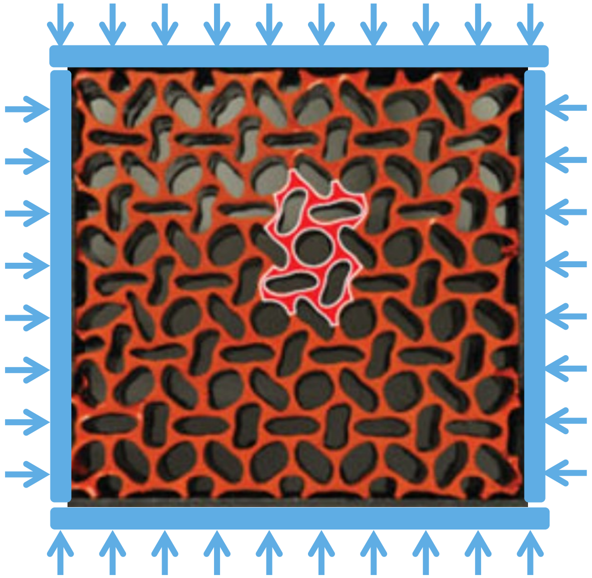
Pattern Generation in Architectured Materials



Undeformed specimen

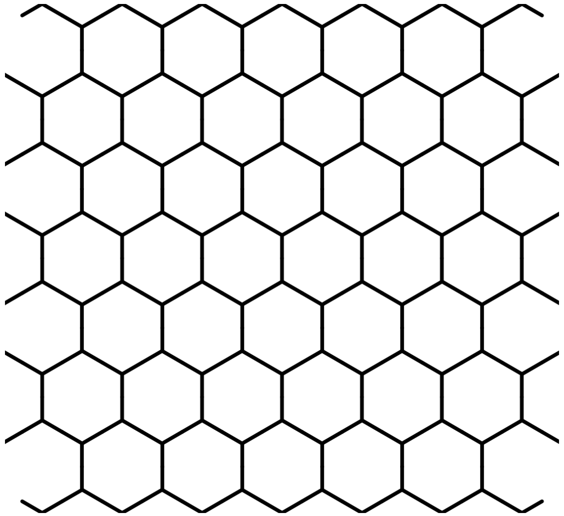






Adapted from [Shan 2014] Harnessing Multiple Folding Mechanisms in Soft Periodic Structures for Tunable Control of Elastic Waves.

Pattern Generation: Bifurcations



Architectured Material

Architectured material's energy:

$$\mathcal{E} : (\mathbb{R}^N \times \mathbb{R}) \longrightarrow \mathbb{R}$$
$$(\mathbf{u}, \lambda) \longmapsto \mathcal{E}(\mathbf{u}, \lambda)$$

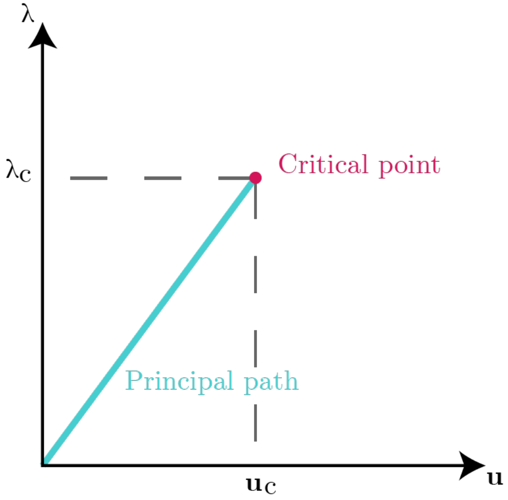
Equilibrium of a finite dof structure:

$$\mathcal{E}_{,\mathbf{u}}\, \delta\mathbf{u} = 0, \quad \forall\, \delta\mathbf{u} \in \mathbb{R}^N$$

Stability operator:

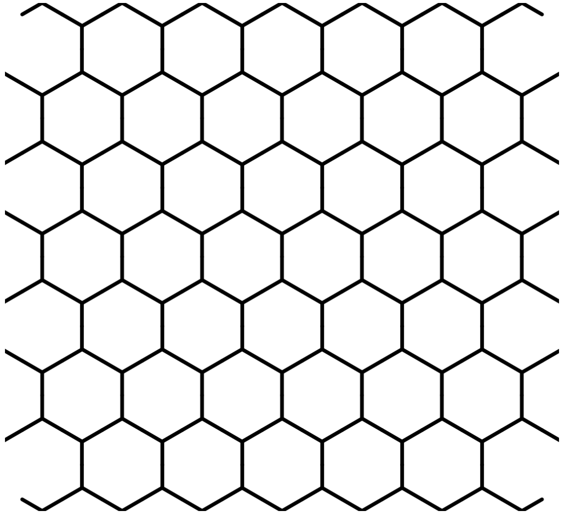
$$\det(\mathcal{E}_{,\mathbf{uu}}(\overset{0}{\mathbf{u}}_c, \lambda_c)) = 0$$ *critical point*

Tangent stiffness matrix



Critical point

Pattern Generation: Bifurcations

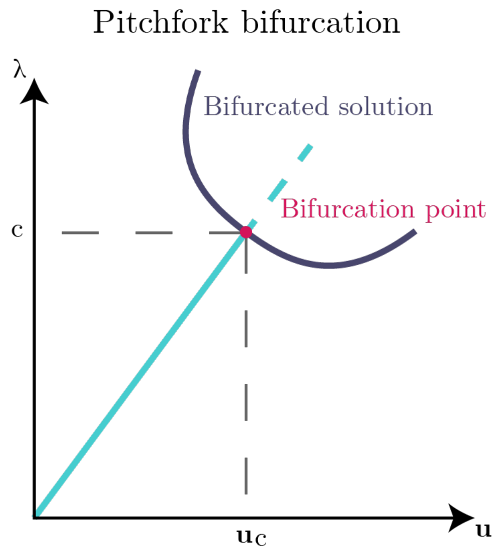


Architectured Material

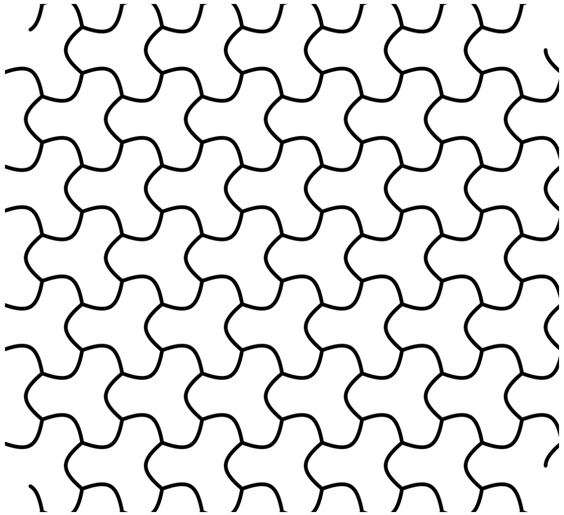
Bifurcation = loss of uniqueness

Equilibrium of a finite dof structure:

$$\mathcal{E}_{,\mathbf{u}}\,\delta\mathbf{u} = 0, \quad \forall\,\delta\mathbf{u} \in \mathbb{R}^N$$



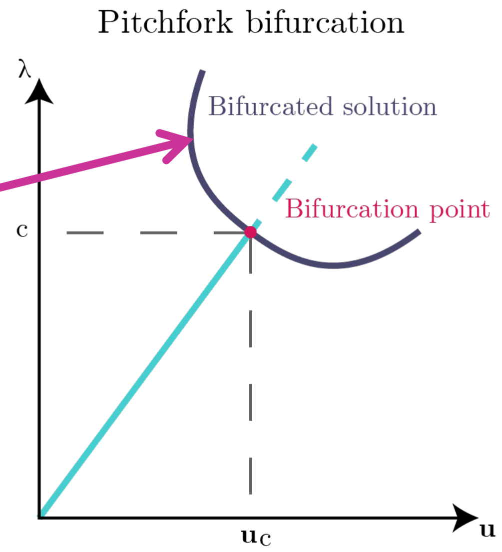
Pattern Generation: Bifurcations



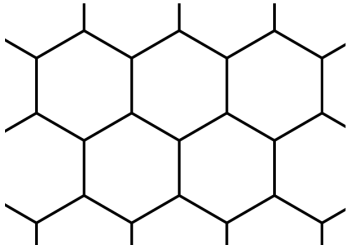
Architectured Material

Bifurcation = loss of uniqueness

PATTERN CHANGE

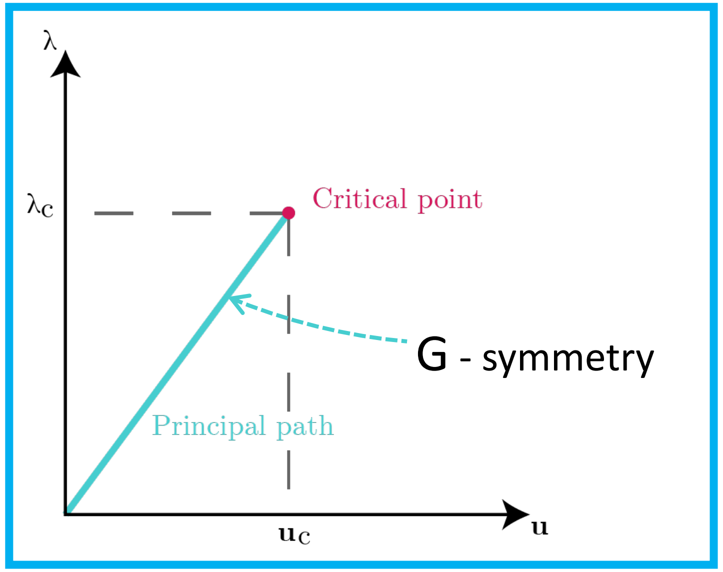


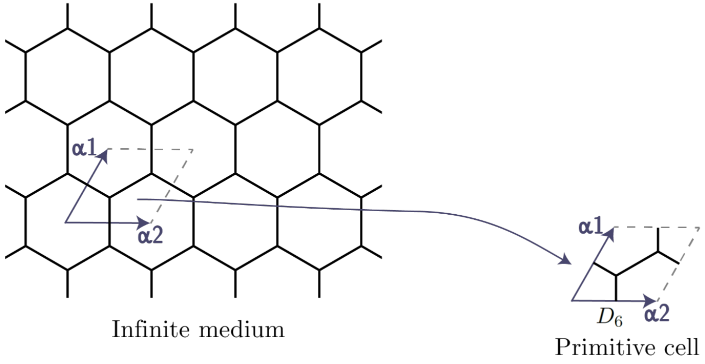
Pattern Generation: Symmetry



(a)

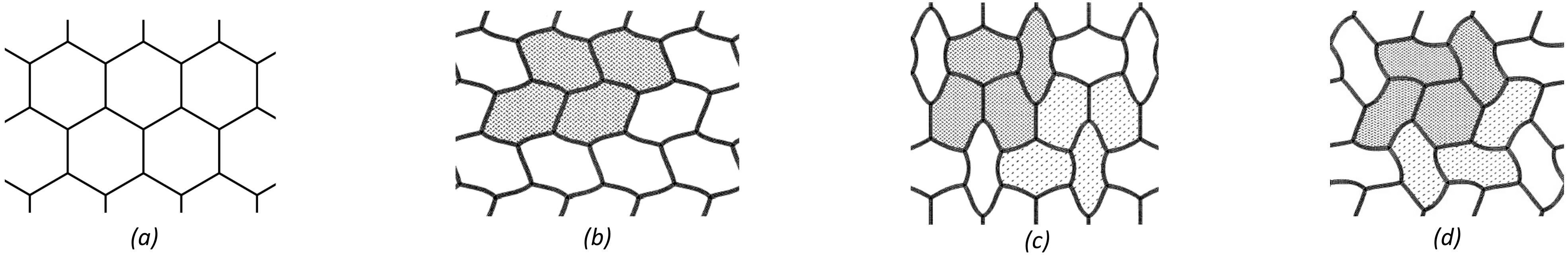
(a) Undeformed hexagonal honeycomb



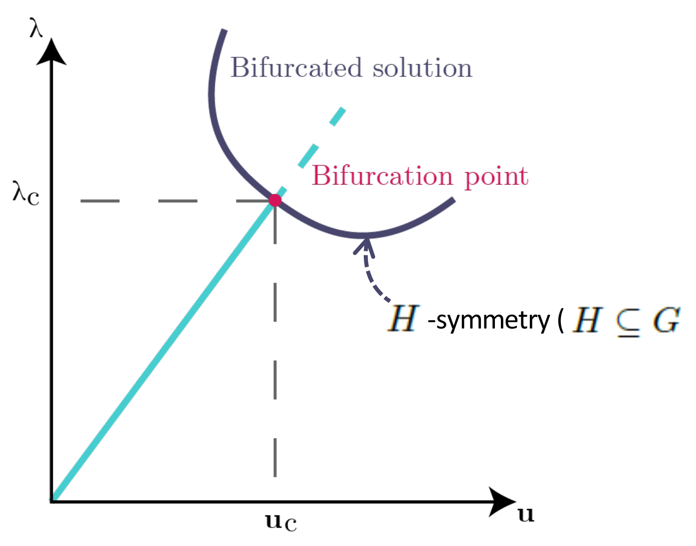


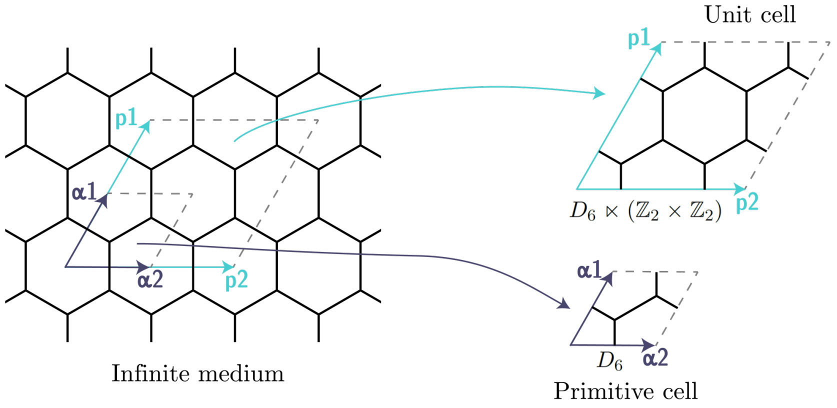
[Okumura 2002] Post-buckling analysis of elastic honeycombs subject to in-plane biaxial compression.
[Ohno 2002] Microscopic symmetric bifurcation condition of cellular solids based on a homogenization theory of finite deformation

Pattern Generation: Symmetry



(a) Undeformed hexagonal honeycomb, (b-d) Adapted from [Ohno 2002] Identified modes for a hexagonal honeycomb under compression. (b) Mode I - uniaxial compression, (c) Mode II - Biaxial compression, (d) Mode III - Equibiaxial compression.



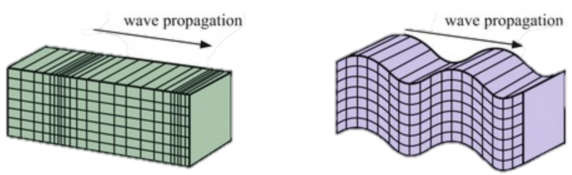


[Okumura 2002] Post-buckling analysis of elastic honeycombs subject to in-plane biaxial compression.

[Ohno 2002] Microscopic symmetric bifurcation condition of cellular solids based on a homogenization theory of finite deformation

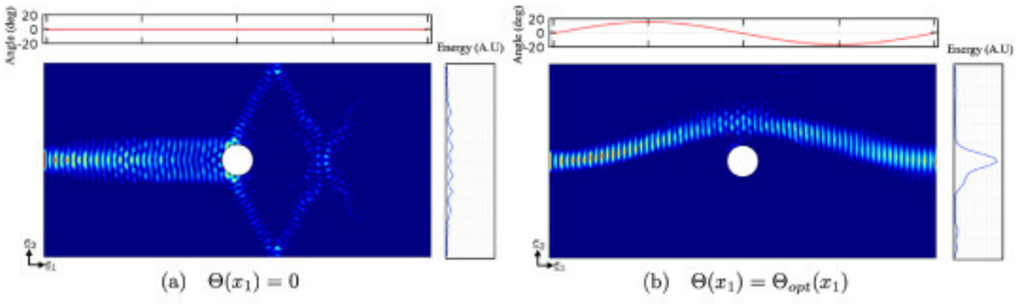
Objective

ANR Project: Max-Oasis

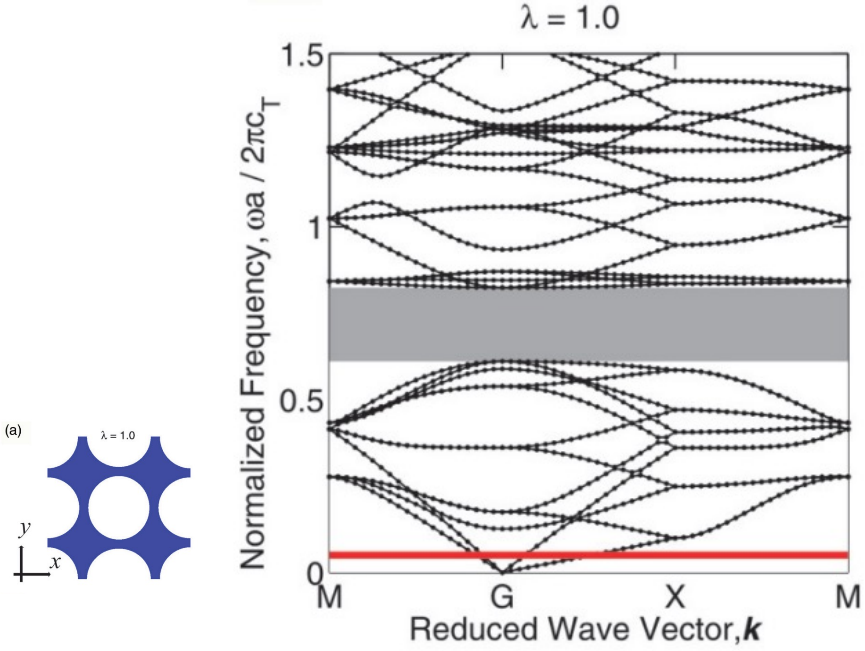


- Interest: Wave propagation properties

Directionality

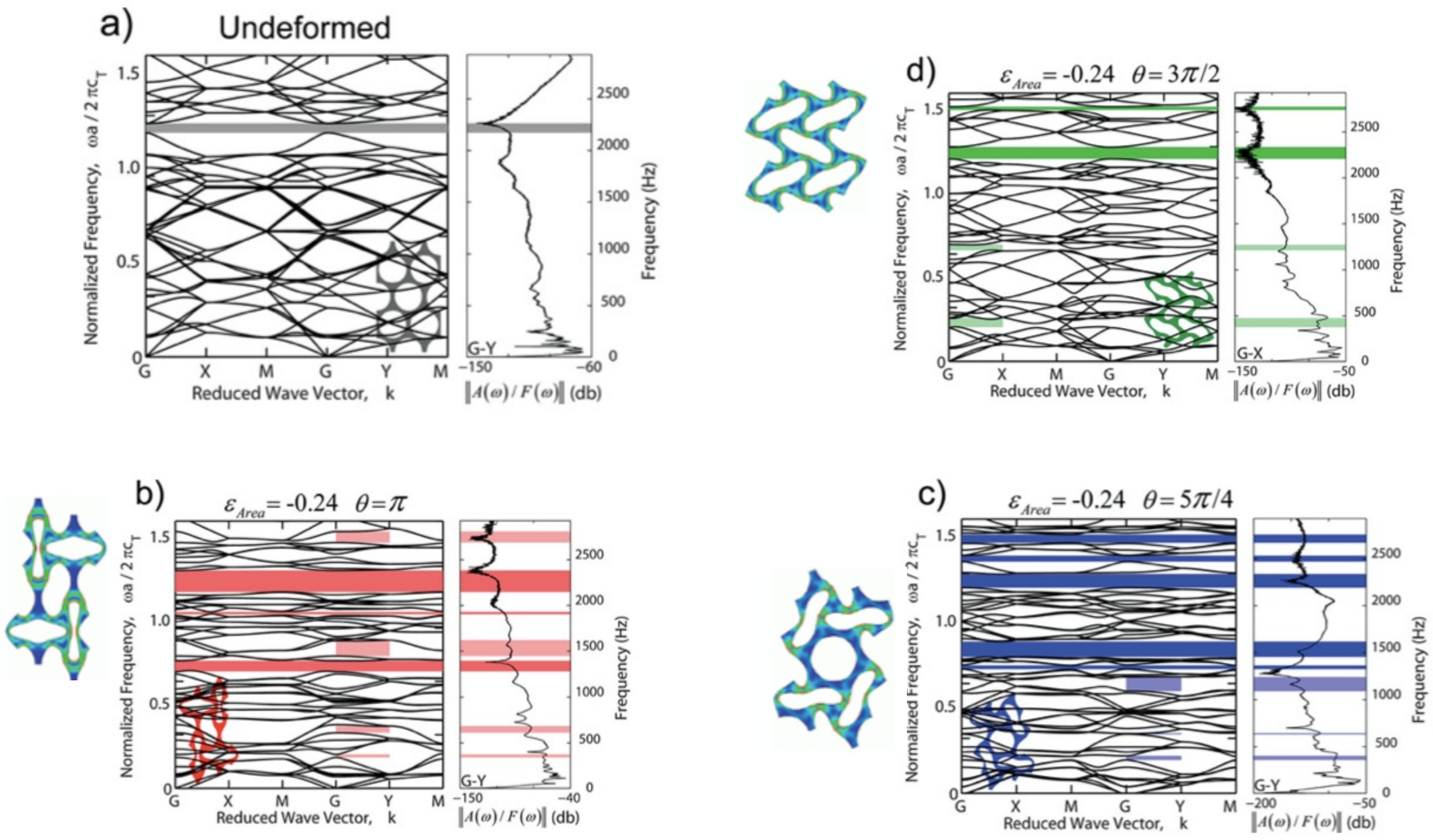


Filtering



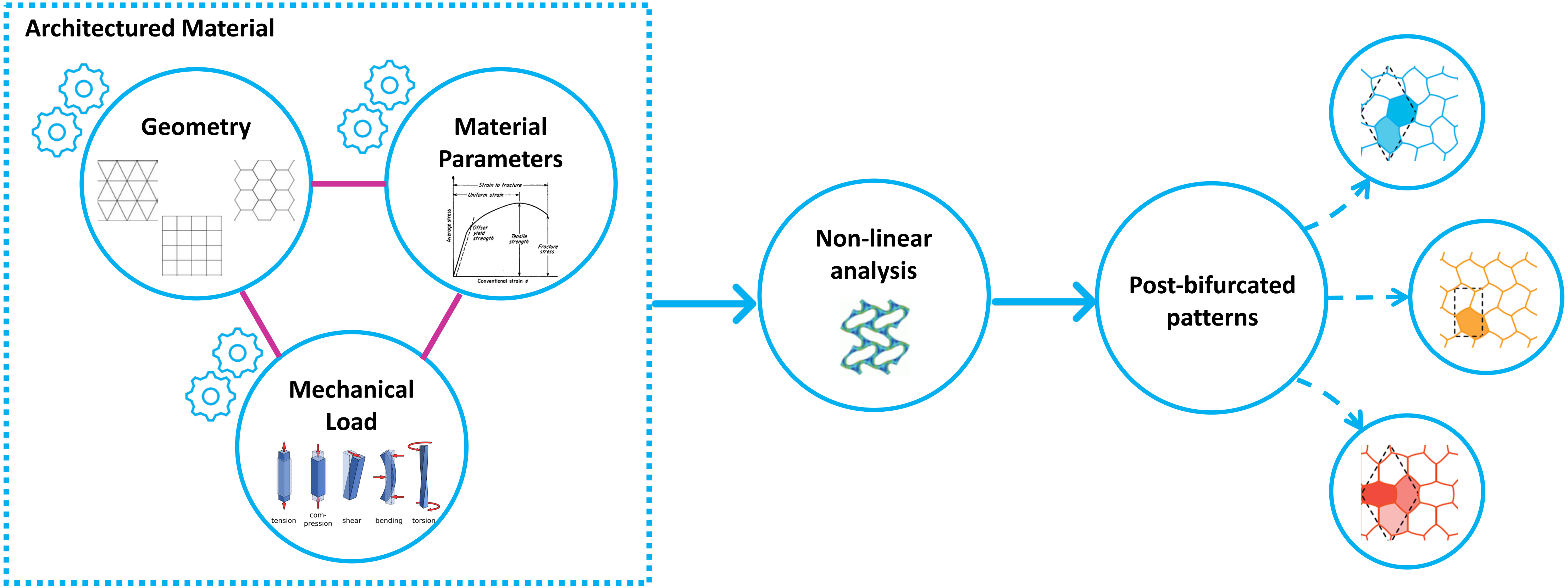
[Rosi 2019] Continuum modelling of frequency-dependent acoustic beam focusing and steering in hexagonal lattices
[Wang 2013] Effects of geometric and material nonlinearities on tunable band gaps and low-frequency directionality of phononic crystals.

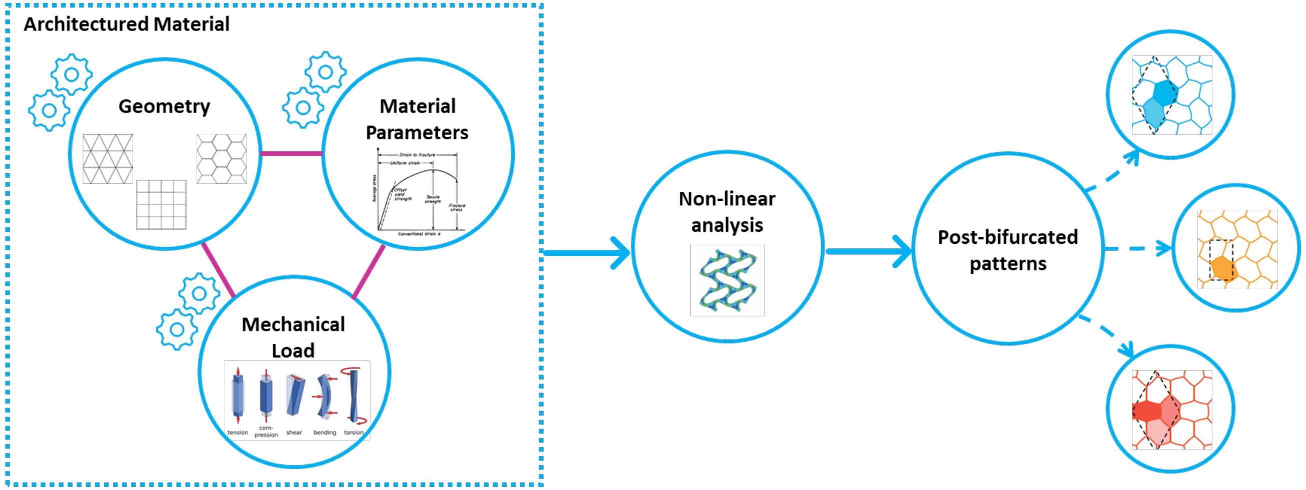
Patterns: new properties



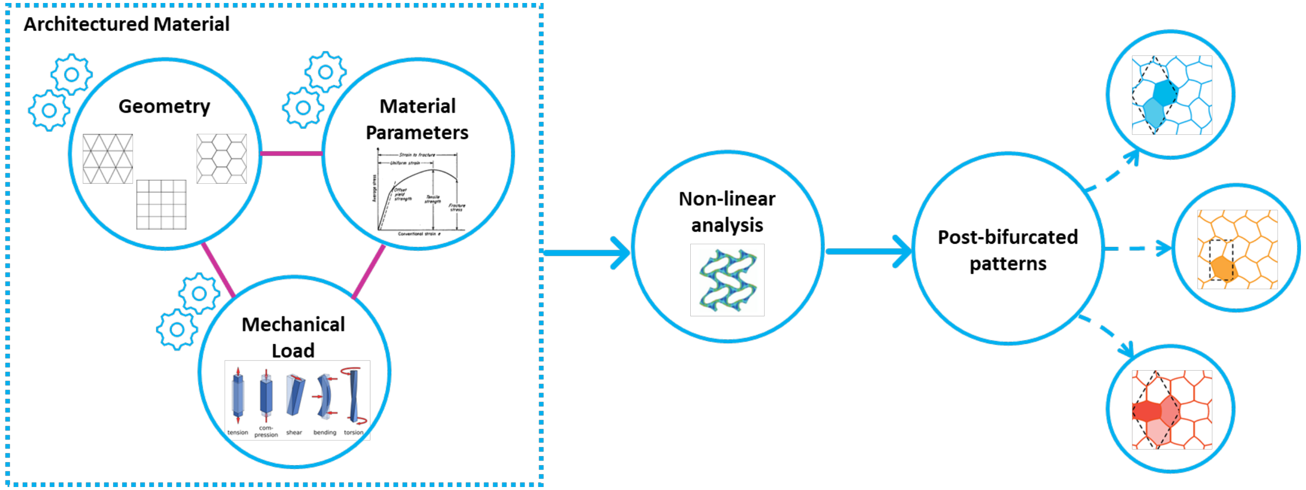
[Shan 2014] Harnessing Multiple Folding Mechanisms in Soft Periodic Structures for Tunable Control of Elastic Waves.

Existing Design Methodology





- Problems due to symmetry
- Issues for period multiplying bifurcations
- Limited to simple geometries
- May involve trial and error
- Lacks robustness



No tool or systematic approach for designing pattern generating architectured materials

Designing Pattern Generating Materials

Mechanical standpoint : analysis

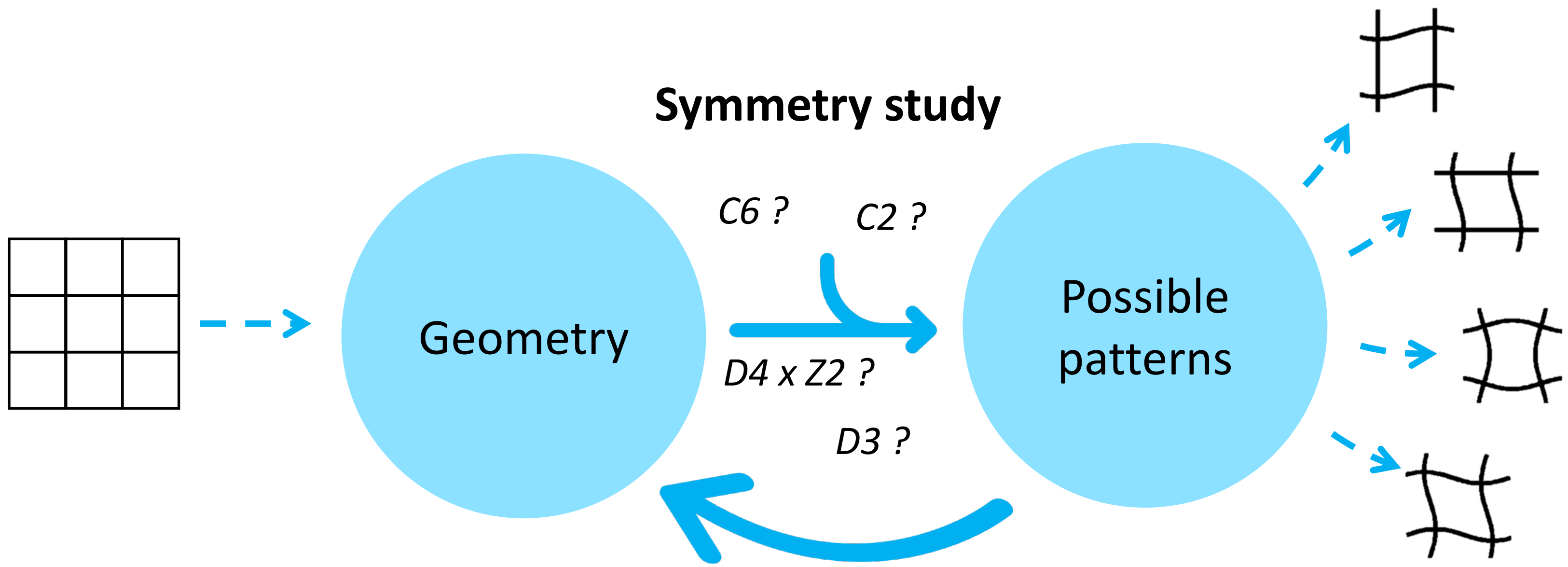
Knowing the system's energy, we find its post-bifurcated behaviour

Mathematical standpoint : design

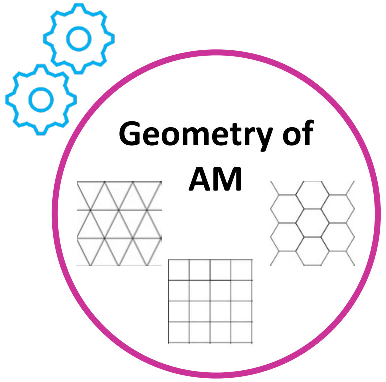
Knowing the system's symmetry, we find all its possible post-bifurcated behaviours

Objective

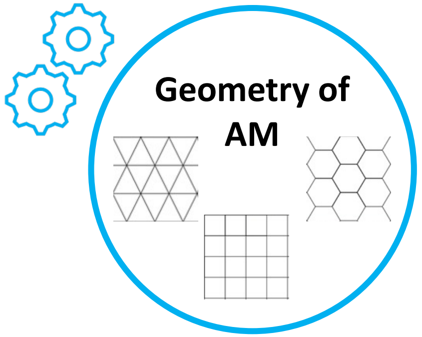
- Design methodology based on group theory for pattern generating architectured materials

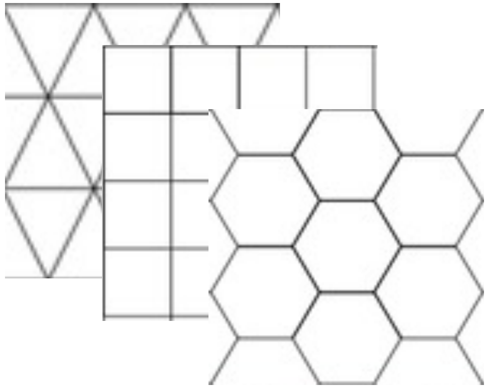


Method

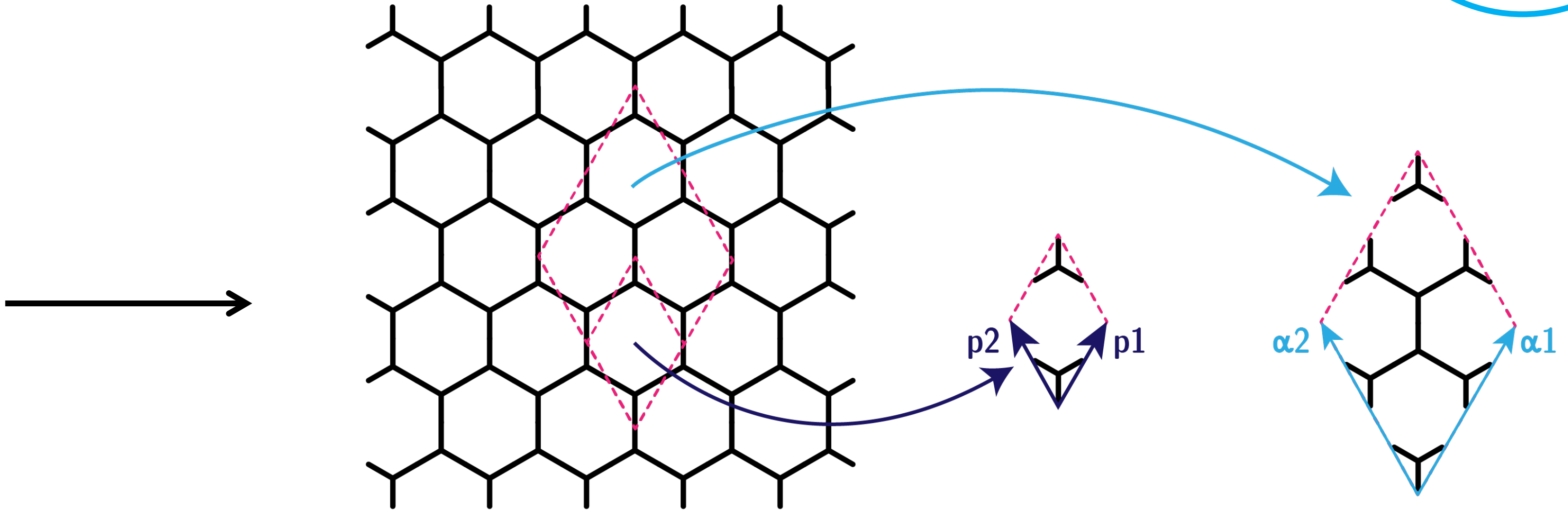

Geometry of
AM

Geometry



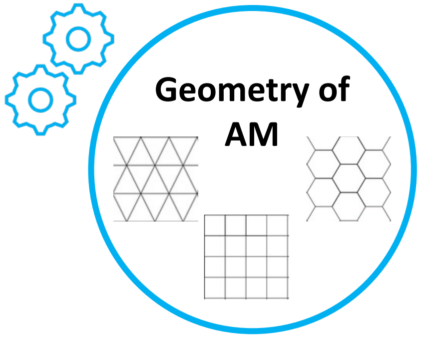


Architectured Material



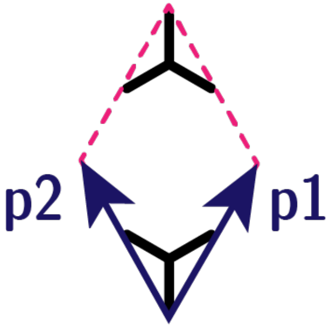
Infinite medium   Primitive cell   Unit cell

Needs a description

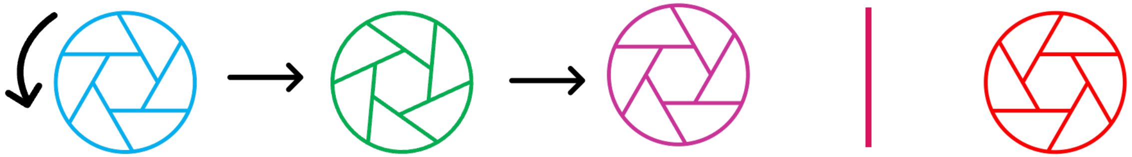


Geometry

- Primitive cell: point group

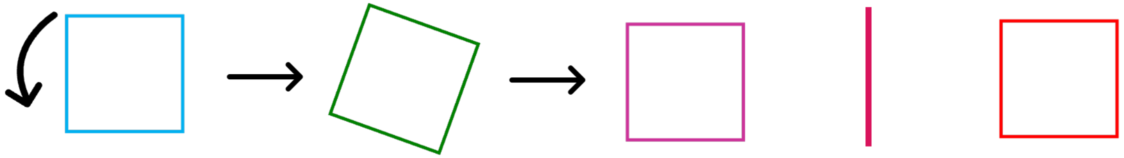


Cyclic group **Cn**: all rotations about a fixed point by 2π/n



Example of **C6** symmetry

Dihedral group **Dn**: rotations of Cn and axial symmetries

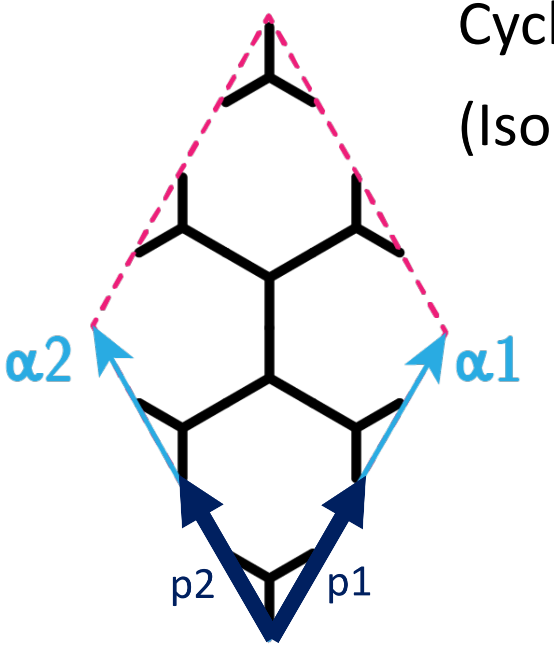


Example of **D4** symmetry

Geometry of AM

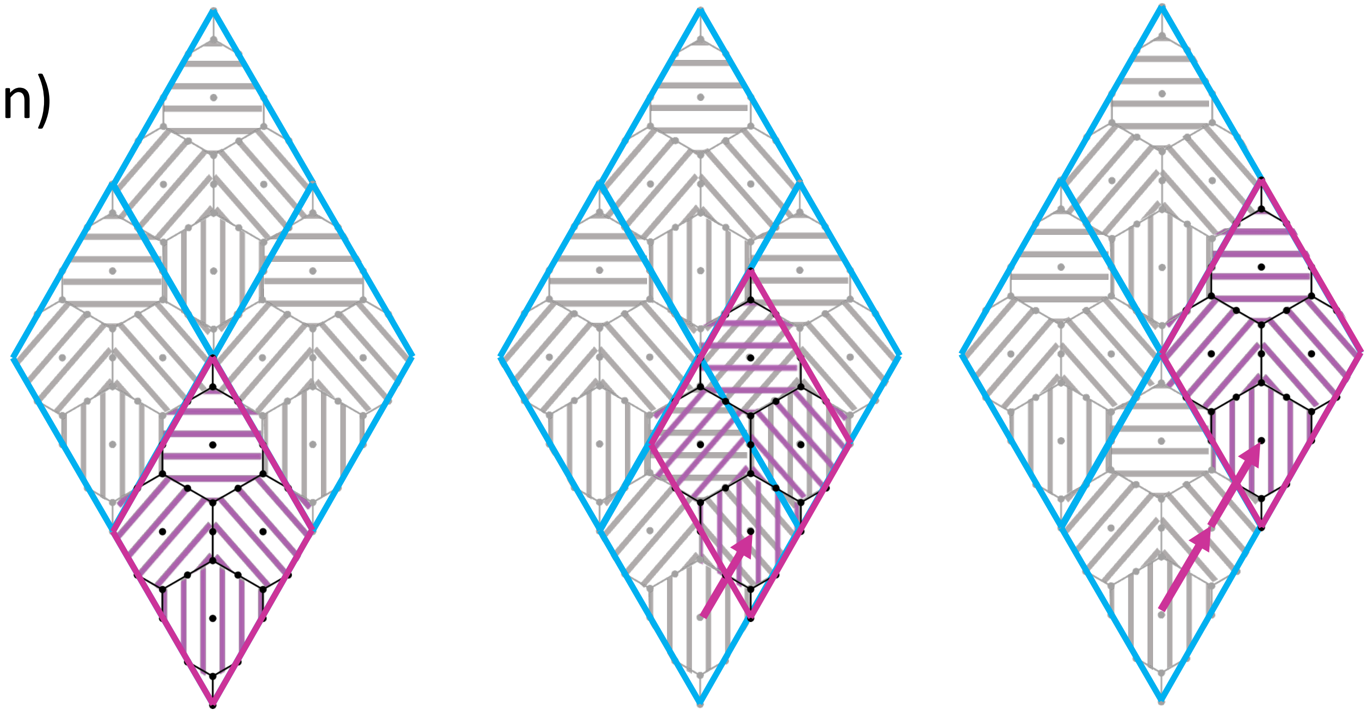
Geometry

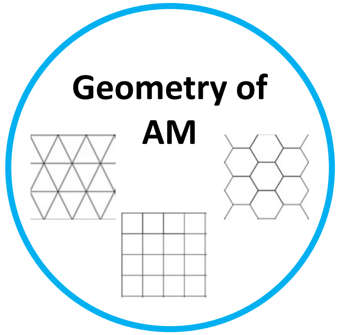
- Unit cell: permutation group (periodicity)



Cyclic group **Zn**:

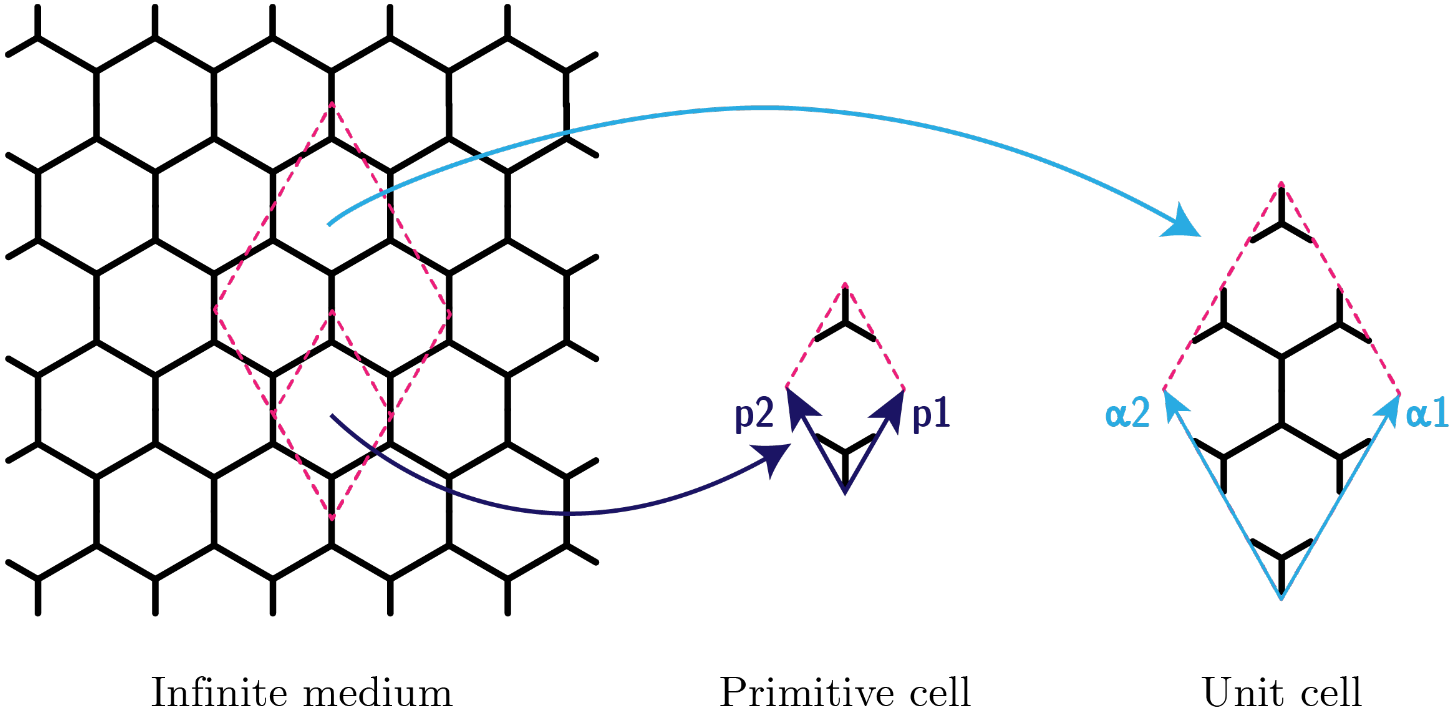
(Isomorphic to Cn)



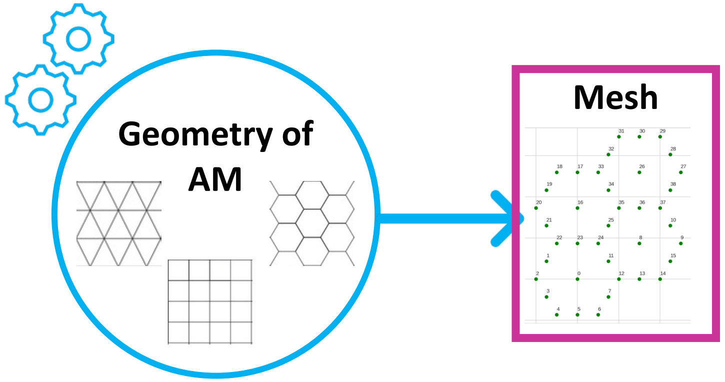


Geometry

- Honeycomb symmetry group: $D_6 \ltimes (Z_2 \times Z_2)$ (Manual Input)



Method



Mesh

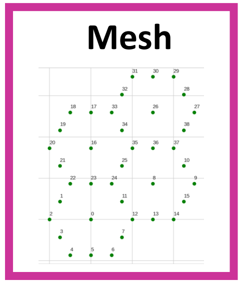
- Vector space of the problem: $\mathbb{V}$

  Configuration space:

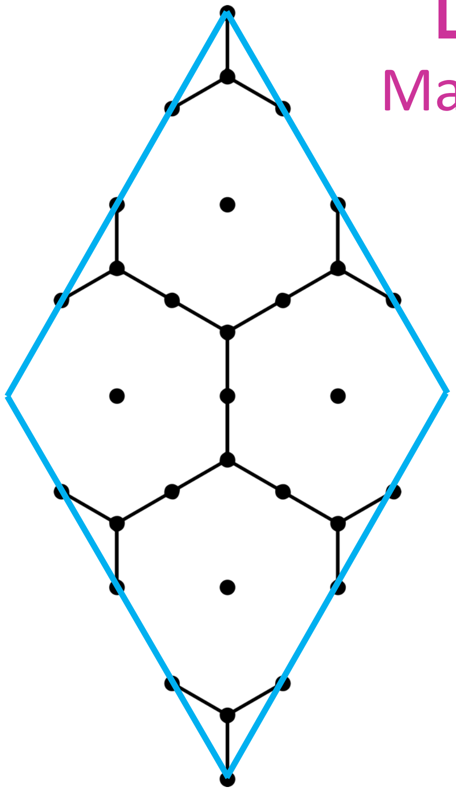
  - Each node is given n degrees of freedom (DOF)

$\mathbb{V}$ is then a N-dimensional vector space such that the dof vector of the problem $\mathbf{u} \in \mathbb{V}$,

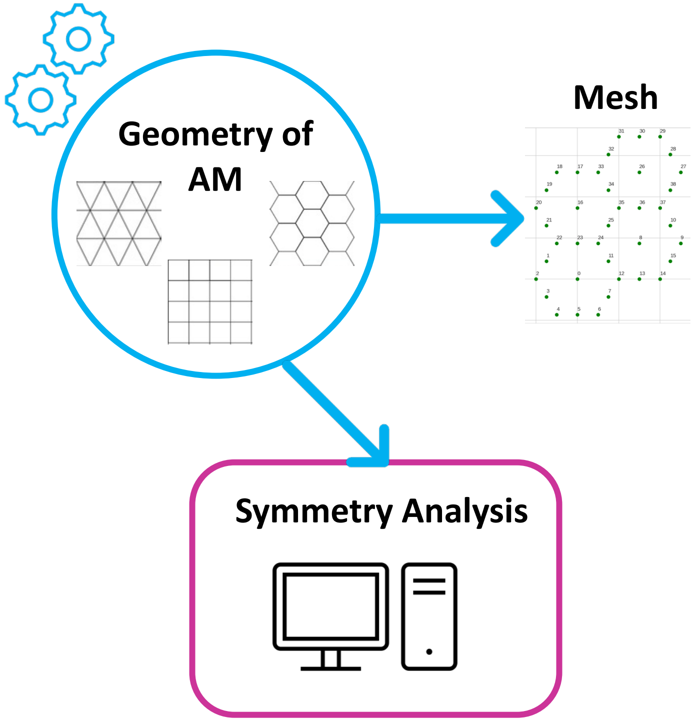
with N the total number of dofs of the problem



Manual Input



Method



Symmetry Analysis

- Representations

A representation of a group G on $\mathbb{V}$ is a homomorphism

$$\tilde{T} : G \longrightarrow GL(\mathbb{V})$$

which satisfies:

$$\tilde{T}(gh) = \tilde{T}(g)\tilde{T}(h) \quad g, h \in G$$

Constructed in Python

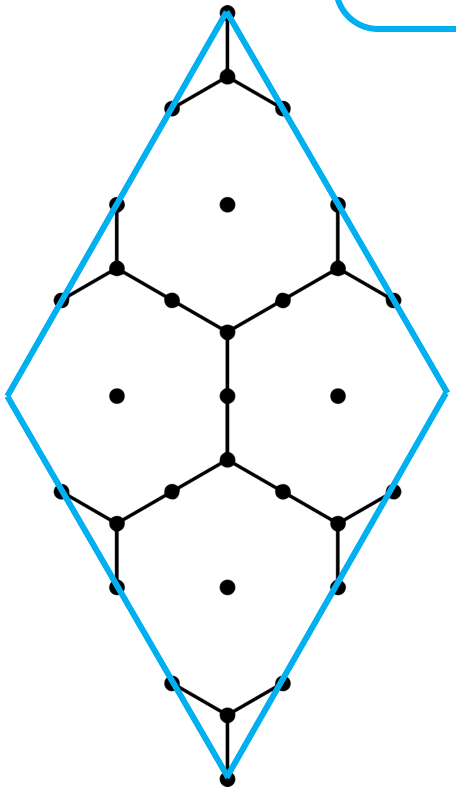
Matrix Representations

$\underline{\mathbf{T}}$

GAP Output

Irreducible Representations

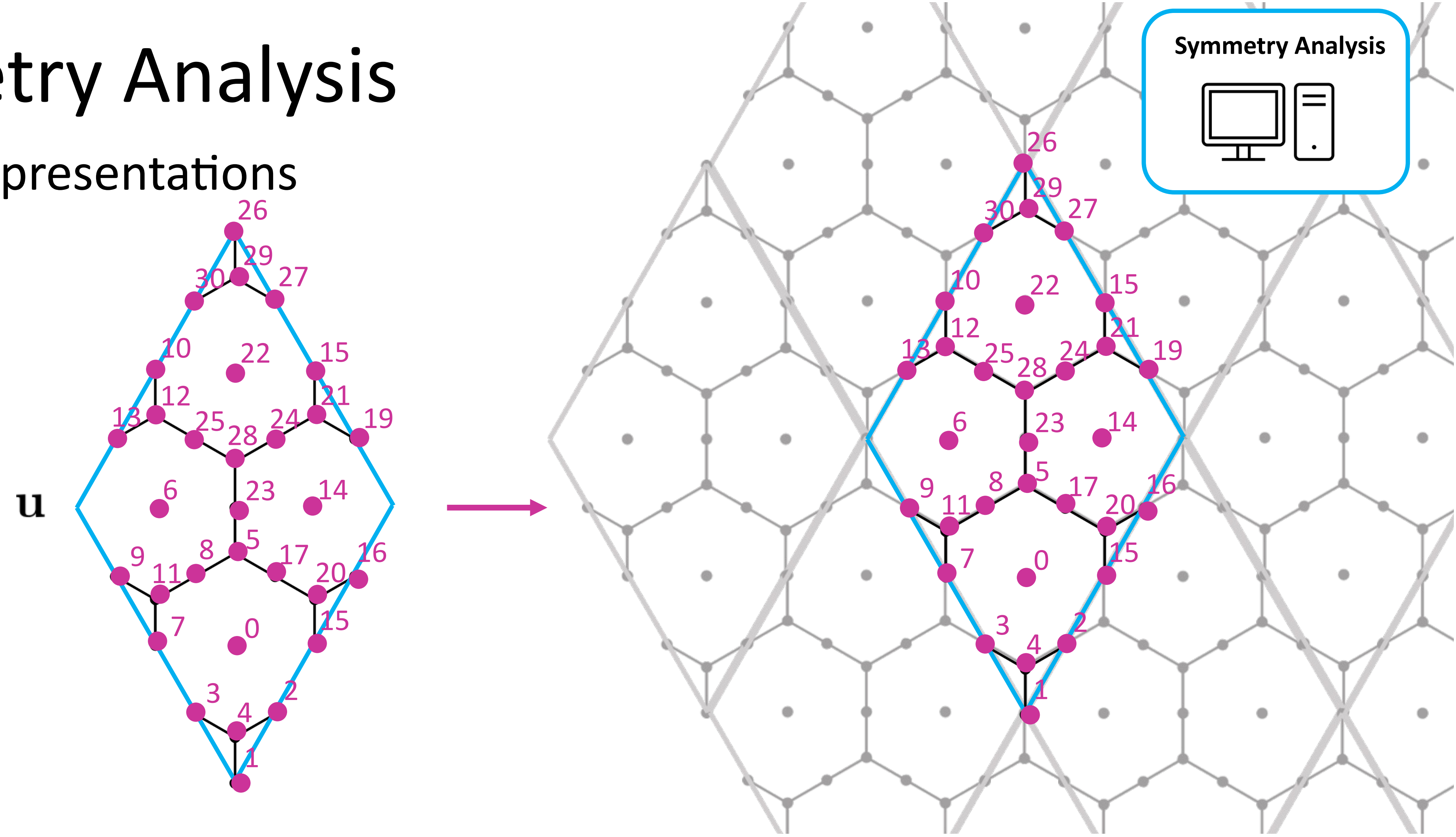
$\underline{\mathbf{T}}^{\mu}$



Symmetry Analysis

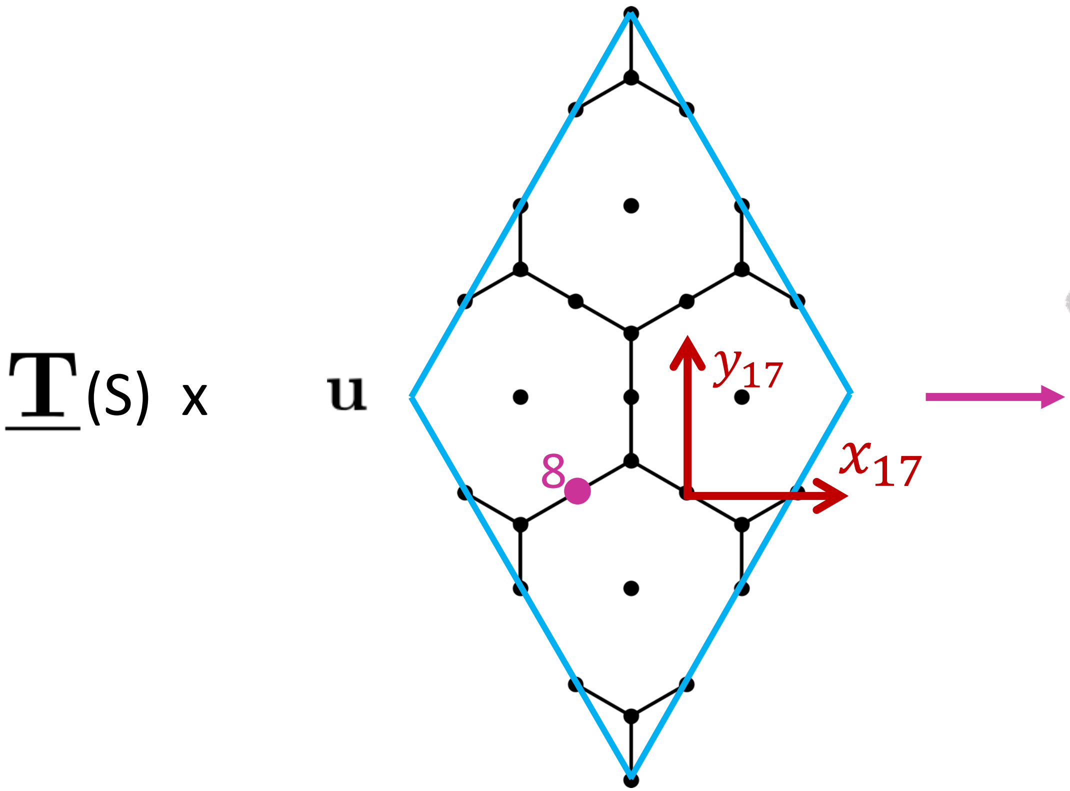
Symmetry Analysis

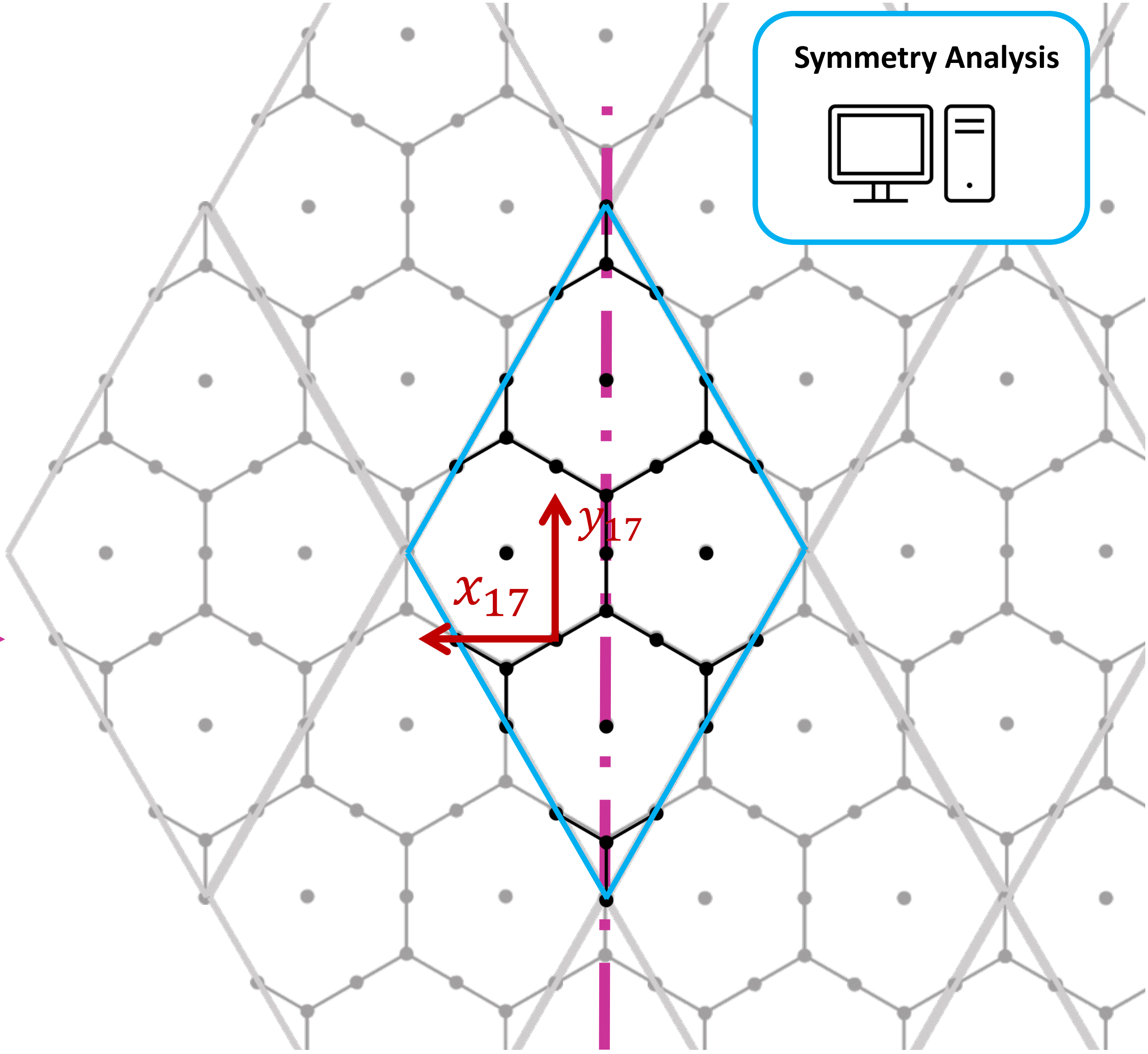
- Matrix Representations



Symmetry Analysis

- Matrix Representations

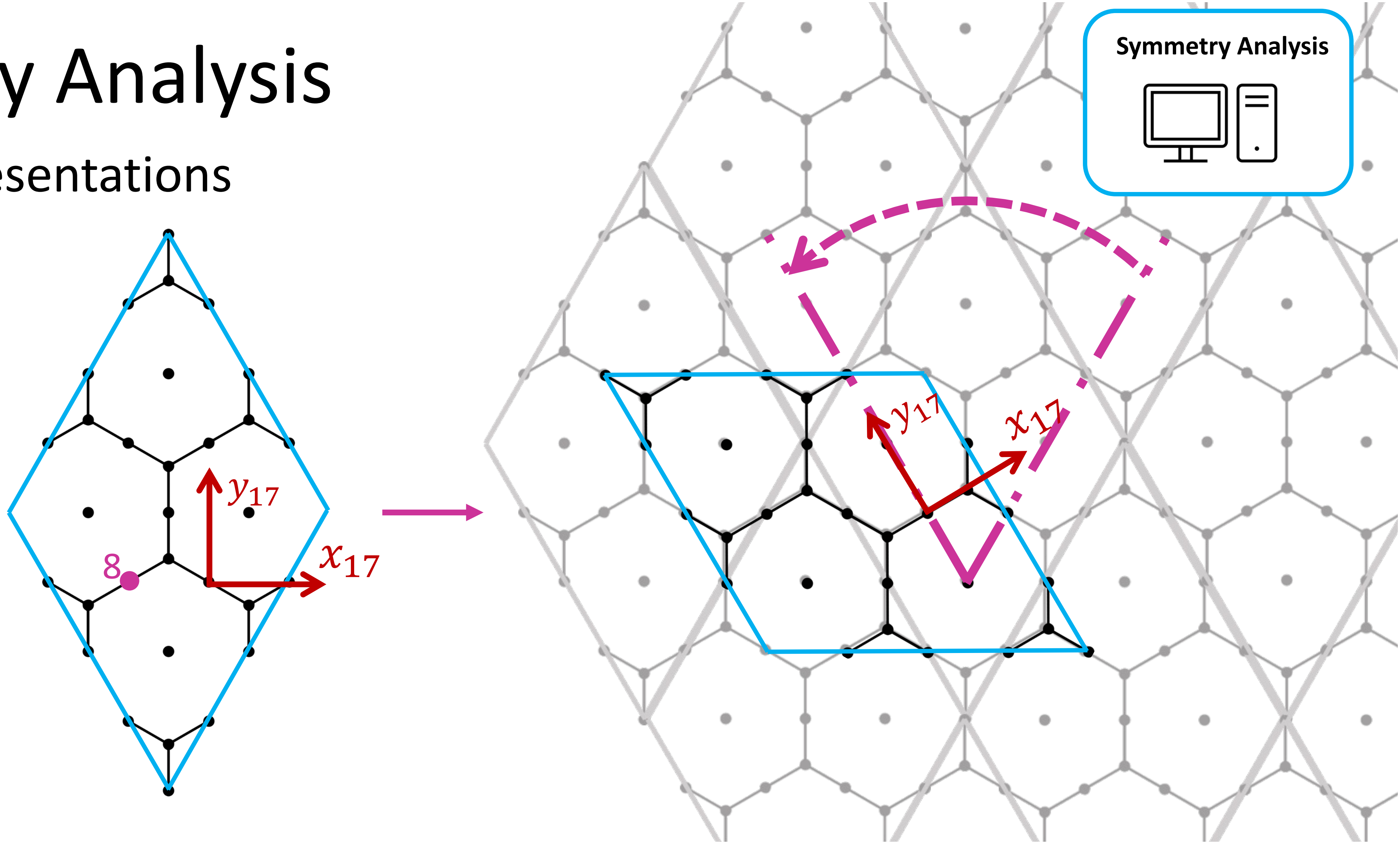




Symmetry Analysis

- Matrix Representations

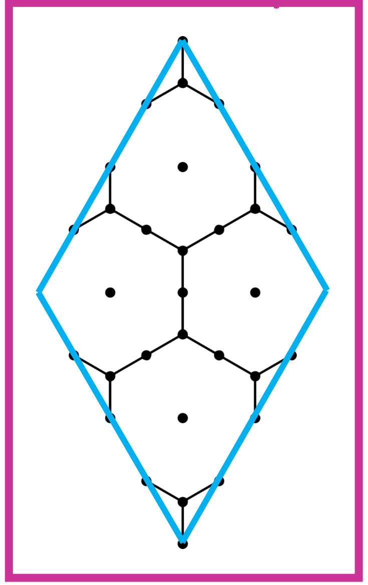
$\underline{\mathbf{T}}$(c6) x $\mathbf{u}$



Symmetry Analysis

- Inputs

Manual Input



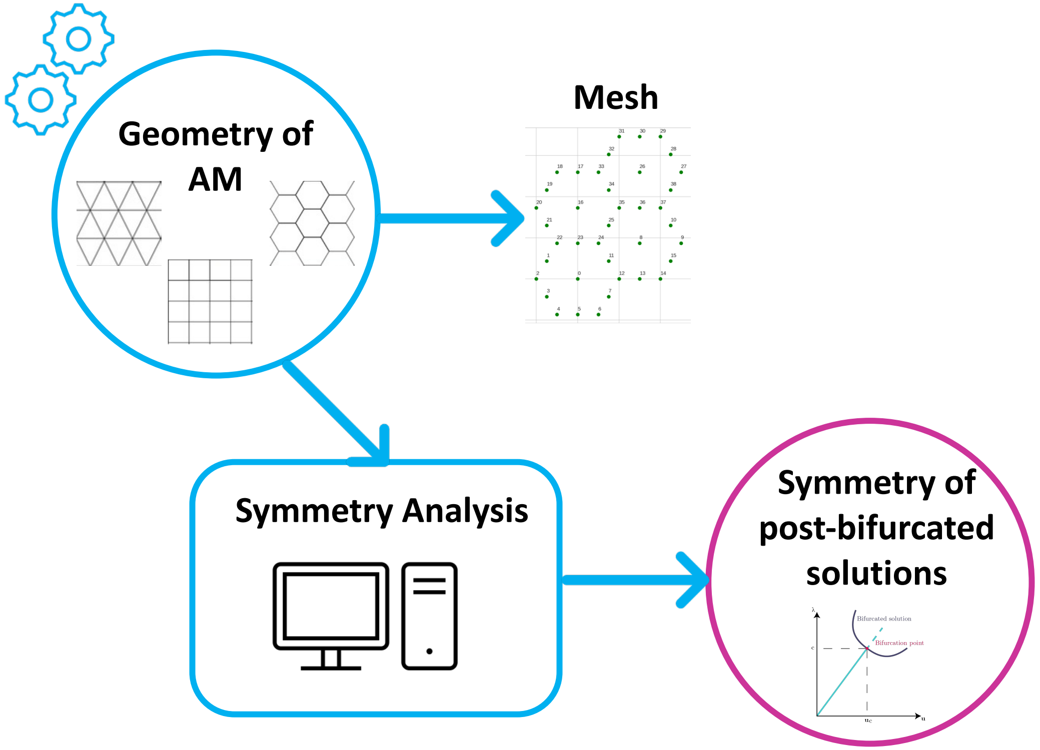
GAP Manual Input

$$D_6 \ltimes (Z_2 \times Z_2)$$

Subgroups, Elements, Generators, Irreducible representations

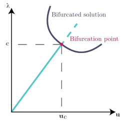
Difficulty: Getting GAP to work with Python

Method


Geometry of AM
Mesh
Symmetry Analysis
Symmetry of post-bifurcated solutions
Bifurcated solution
Bifurcation point

Equivariant Bifurcation Theory

Symmetry of post-bifurcated solutions



- Isotypic decomposition

$\mathbb{V}$ can be decomposed as a direct sum of G-irreducible subspaces

$$\mathbb{V} = \bigoplus_{\mu=1}^{m} \mathbb{V}_\mu$$

Some $\mathbb{V}_\mu$ may not appear in the decomposition.

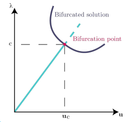
Equivariant Bifurcation Theory

- Irreducible representations: Block diagonalisation

$$\mathbb{V} = \bigoplus_{\mu=1}^{m} \mathbb{V}_\mu \longrightarrow \mathcal{E}_{,\mathbf{uu}}(\mathbf{u}, \lambda) = (\underline{\mathbf{T}}^\mu)^{-1}(g)\, \mathcal{E}_{,\mathbf{uu}}(\mathbf{u}, \lambda)\, \underline{\mathbf{T}}^\mu(g)$$

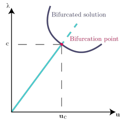
$$\begin{bmatrix} \boxed{B_1} & 0 & 0 & 0 \\ 0 & \boxed{B_2} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \boxed{B_k} \end{bmatrix}$$

Symmetry of post-bifurcated solutions



Equivariant Bifurcation Theory

Symmetry of post-bifurcated solutions



- Critical point occurs when $\det(\mathcal{E}_{,\mathbf{uu}}(\overset{0}{\mathbf{u}}_c, \lambda_c)) = 0$

Generically, only blocks corresponding to one irreducible representation vanish

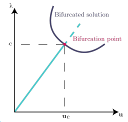
$$\begin{bmatrix} B_1 & 0 & 0 & 0 \\ 0 & B_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & B_k \end{bmatrix}$$

$$\mathbb{V} = \bigoplus_{\mu=1}^{m} \mathbb{V}_\mu$$

Bifurcation takes place in one of the G-irreducible subspaces $\mathbb{V}_\mu$

Equivariant Branching Lemma [Vanderbauwhede, 1980]

Symmetry of post-bifurcated solutions



Apply Equivariant Branching Lemma for each irreducible subspace $\mathbb{V}^{\mu}$

For each symmetry subgroup H of $\mathbf{u} \in \mathbb{V}^{\mu}$:
if $\mathrm{dimFix}_{\mathbb{V}^{\mu}}(H) = 1$ a bifurcated solution with symmetry group H exists

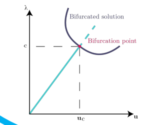
Symmetry Group: $G_{\mathbf{u}} = \{g \in G \mid \mathbf{T}(h)\mathbf{u} = \mathbf{u}\}$

Fixed point subspace: $\mathrm{Fix}_{\mathbb{V}^{\mu}}(H) = \{\mathbf{u} \in \mathbb{V}^{\mu} | \mathbf{T}^{\mu}(h)\mathbf{u} = \mathbf{u}, \forall h \in H)$

And its dimension: $\mathrm{dimFix}_{\mathbb{V}^{\mu}}(H) = \frac{1}{|\mathrm{H}|} \sum_{h \in H} tr\big(\mathbf{T}(h)\big)$

Equivariant Bifurcation Theory

Symmetry of post-bifurcated solutions



- Symmetry of the solutions

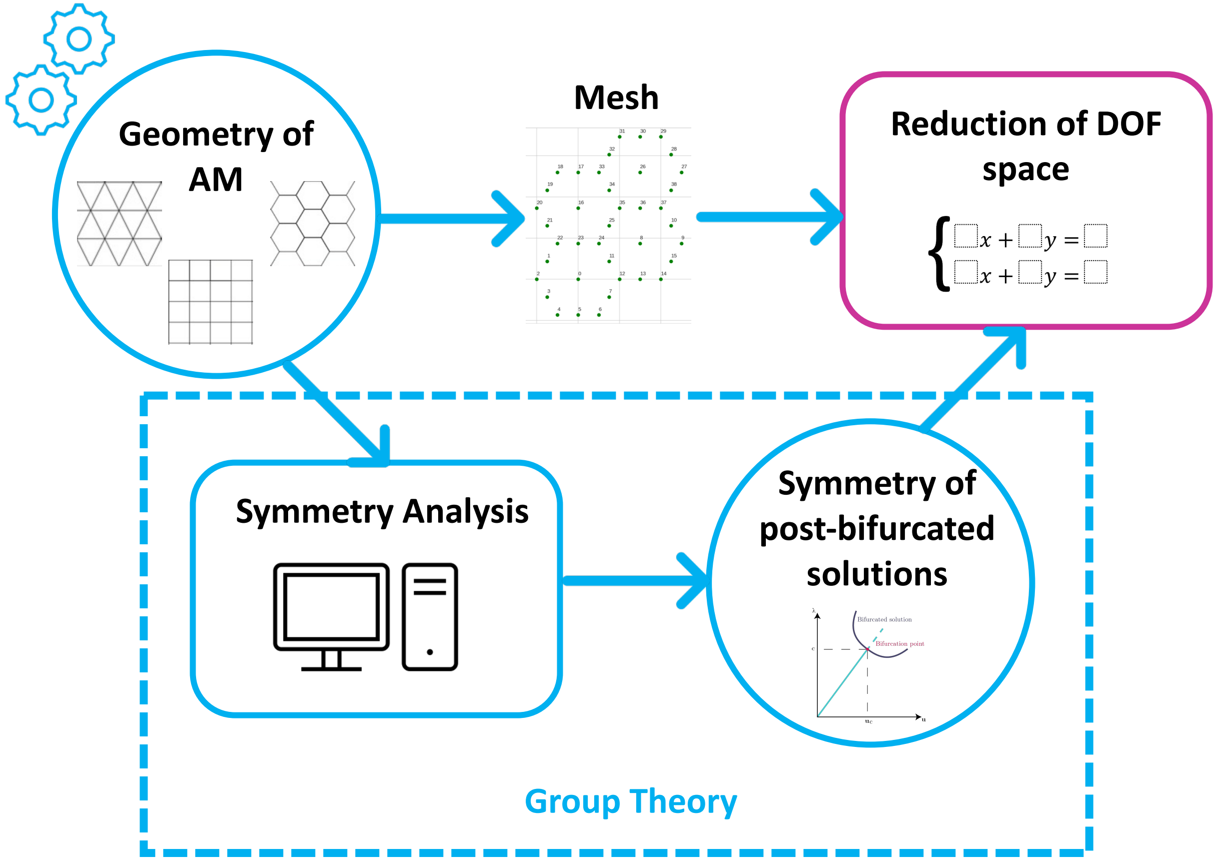
Isotropy Subgroup = Symmetry Group

$$G_{\mathbf{u}} = \{g \in G,\ \underline{\mathbf{T}}(g)\mathbf{u} = \mathbf{u}\}$$

- Symmetry of the critical displacement eigenvector

$$G_{\ker(\mathcal{E}_{,\mathbf{uu}}(\overset{0}{\mathbf{u}}_c,\lambda_c))} = \{g \in G \mid \underline{\mathbf{T}}(g)\mathbf{u} = \mathbf{u},\ \forall\, \mathbf{u} \in \ker(\mathcal{E}_{,\mathbf{uu}}(\overset{0}{\mathbf{u}}_c, \lambda_c))\}$$

Method


Geometry of AM
Mesh
Reduction of DOF space
Symmetry Analysis
Symmetry of post-bifurcated solutions
Bifurcated solution
Bifurcation point
Group Theory

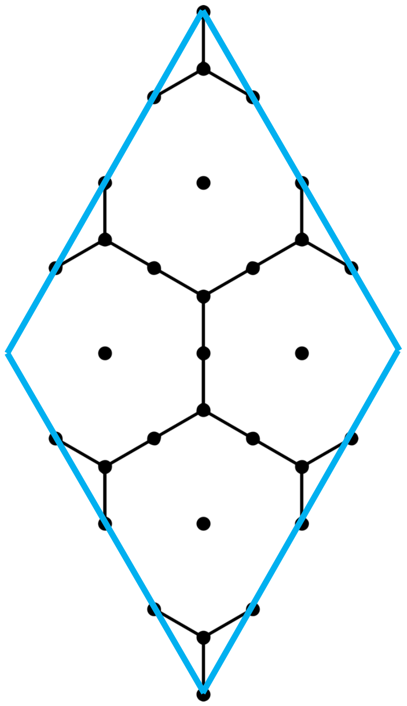
Reduction of DOFs

Reduction of DOF space

$$\begin{cases} \square x + \square y = \square \\ \square x + \square y = \square \end{cases}$$

- Generalised displacement vector

$$\mathbf{u} = (\mathbf{u}_1, \ldots, \mathbf{u}_n)^T, \quad \mathbf{u} \in \mathbb{R}^N$$



- For each post-bifurcated symmetry group

$$\{\underline{\mathbf{T}}(g)\overset{1}{\mathbf{u}} = \overset{1}{\mathbf{u}}\}$$

Symmetry adapted decomposition of the generalised displacement vector

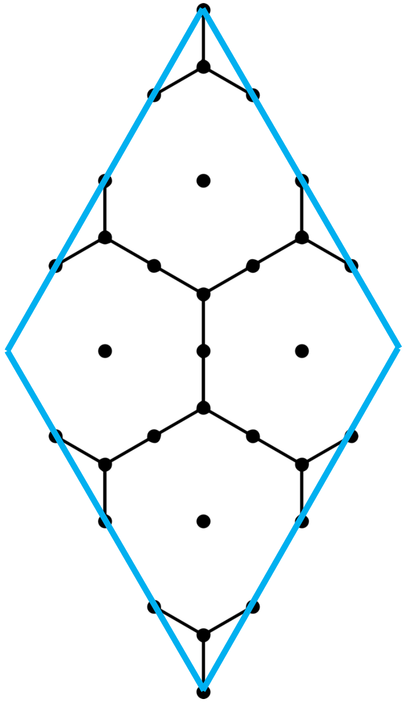
Reduction of DOFs

Reduction of DOF space

$$\begin{cases} \square x + \square y = \square \\ \square x + \square y = \square \end{cases}$$

- Example: $C_6 \ltimes (Z_2 \times Z_2)$

Initial mesh and displacements

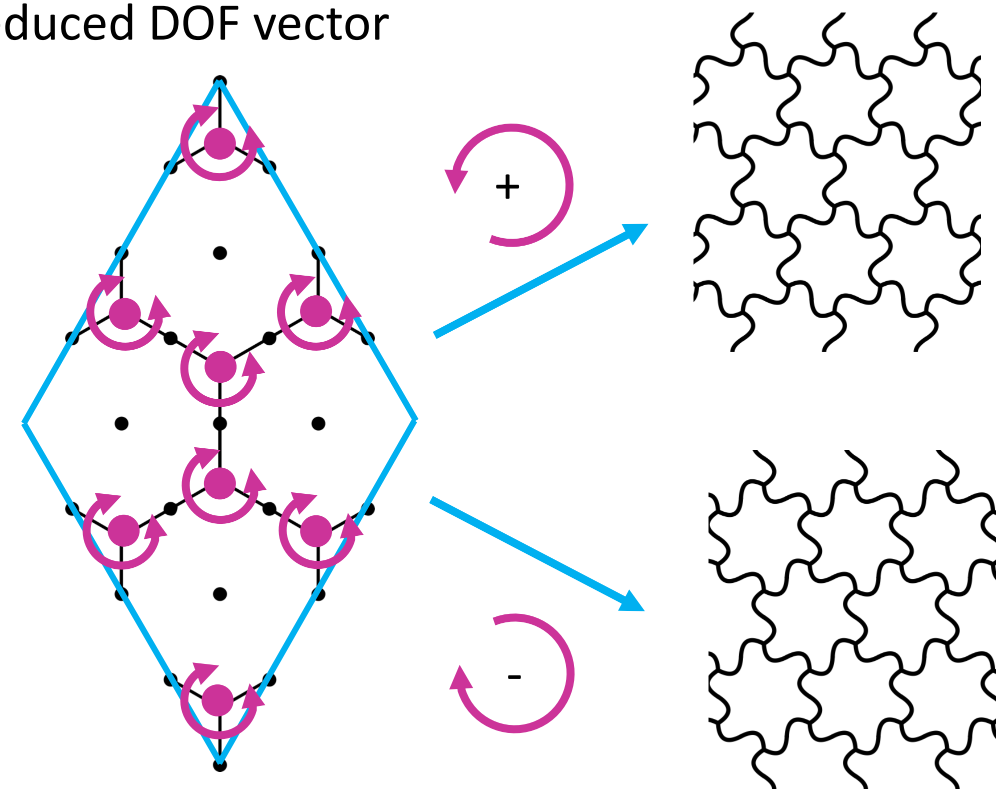


Solve

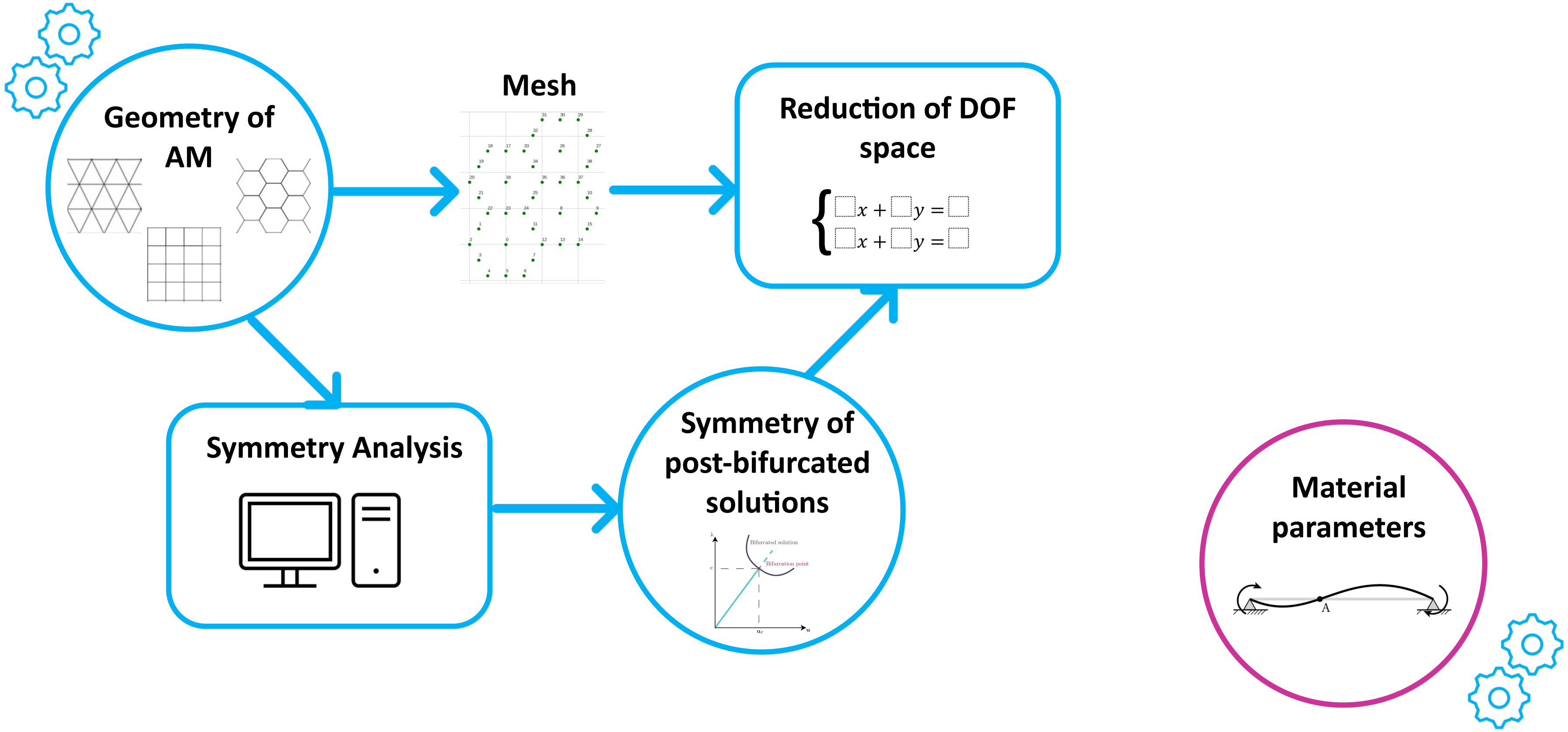
$$\{\underline{\mathbf{T}}(g)\overset{1}{\mathbf{u}} = \overset{1}{\mathbf{u}}\}$$

For all elements of $C_6 \ltimes (Z_2 \times Z_2)$

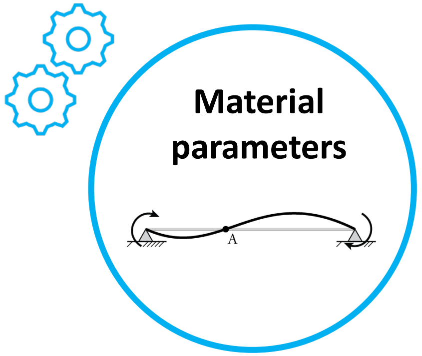
Reduced DOF vector



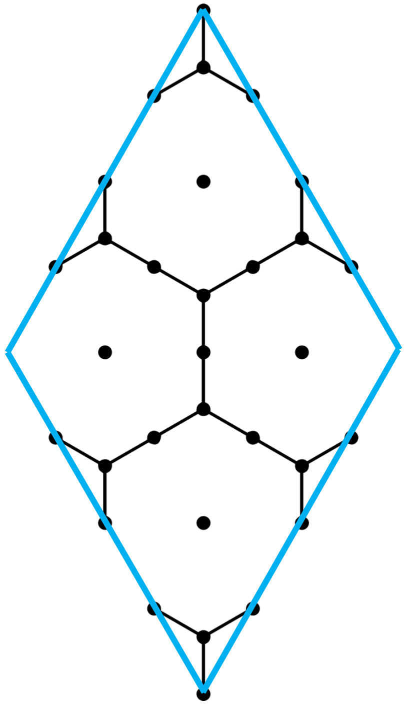
Method


Geometry of AM
Mesh
Reduction of DOF space
Symmetry Analysis
Symmetry of post-bifurcated solutions
Material parameters
A

Material Parameters



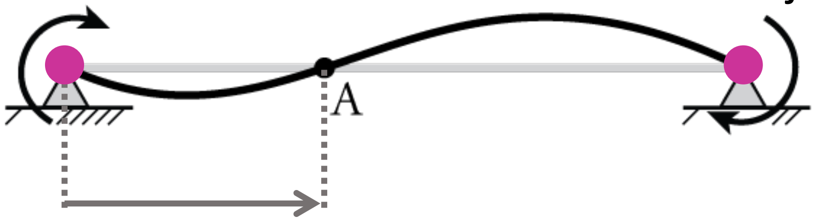
- Euler-Bernoulli beams



Elementary displacement vector:

$$\mathbf{u}_e^T = \begin{bmatrix} u_i & v_i & \theta_i & u_j & v_j & \theta_j \end{bmatrix}$$

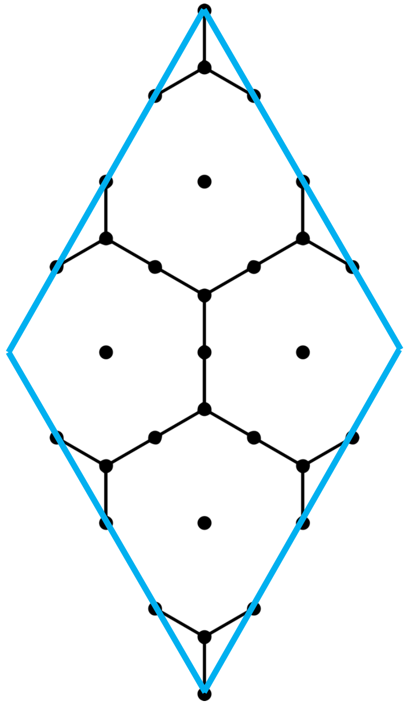
Standard Hermitian cubic interpolation:



Displacements of beam: functions of node displacements

Material Parameters

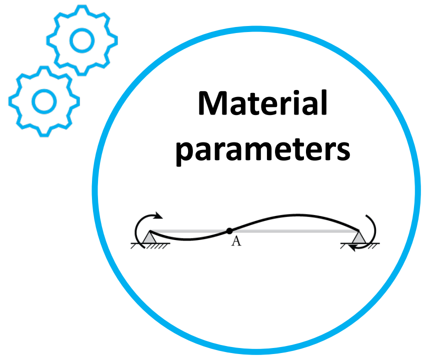
- Euler-Bernoulli beams



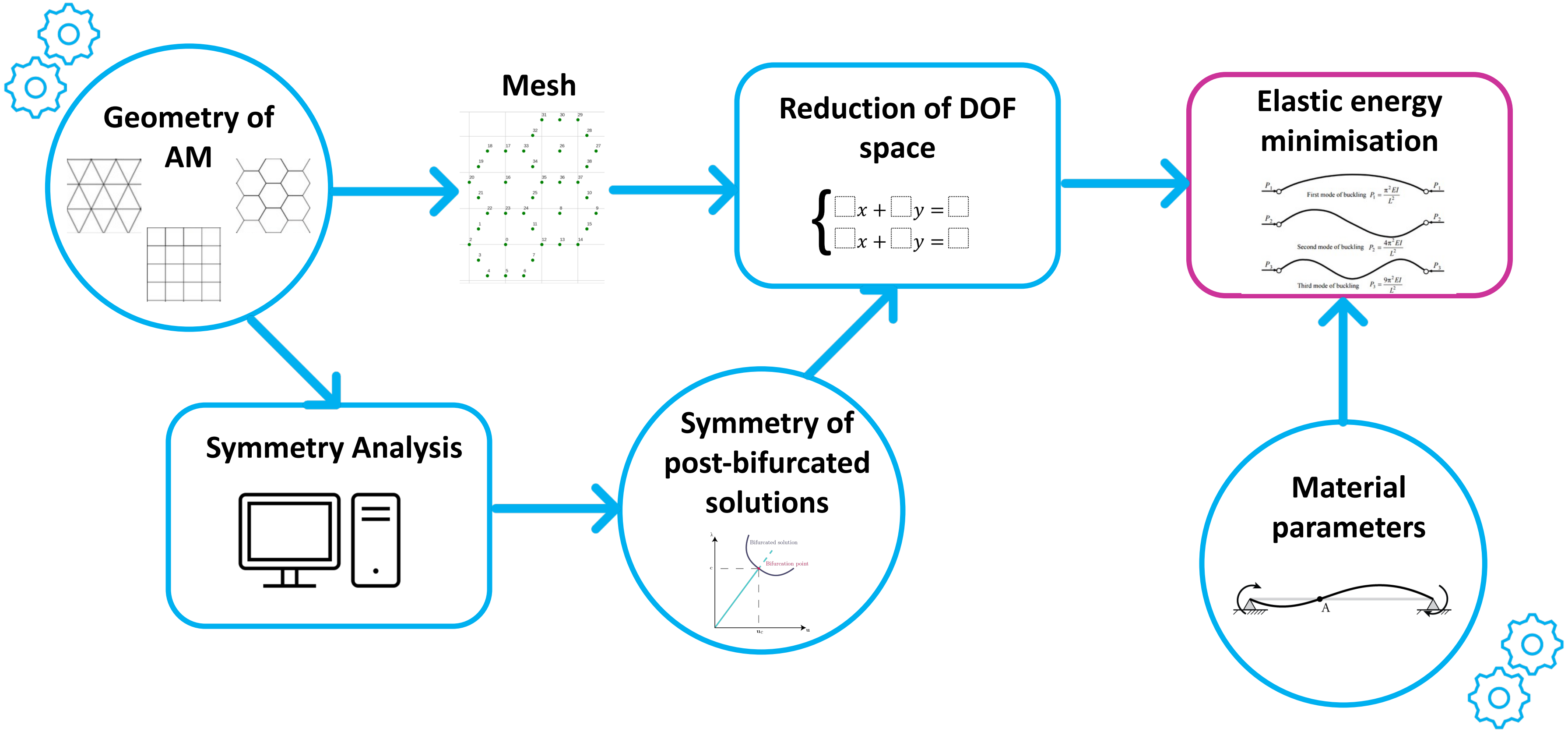
Beam parameters:

L	Length
S	Surface of cross-section
I	Quadratic moment
E	Young’s Modulus

Manual Input

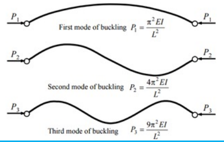


Method

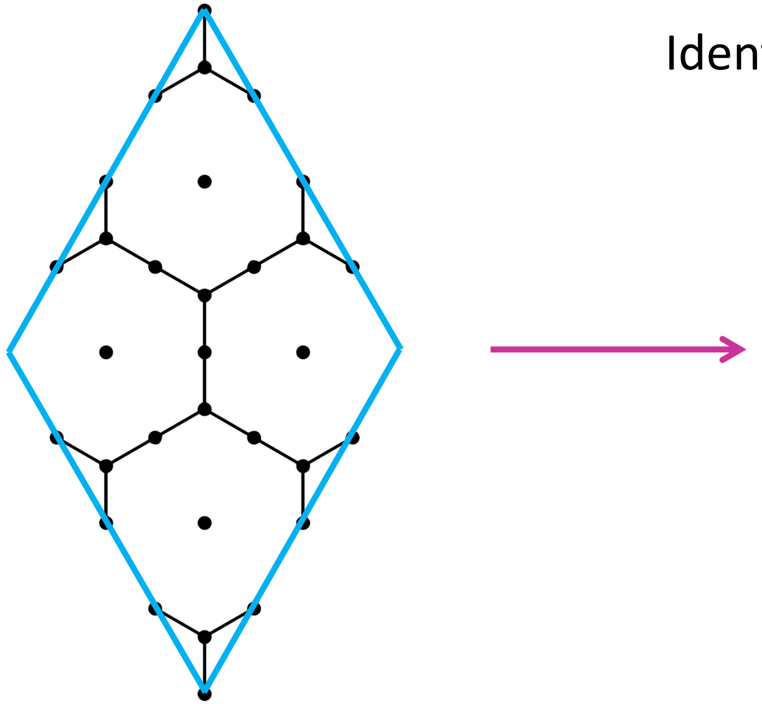

Geometry of AM
Mesh
Reduction of DOF space
Elastic energy minimisation
First mode of buckling
Second mode of buckling
Third mode of buckling
Symmetry Analysis
Symmetry of post-bifurcated solutions
Bifurcated solution
Bifurcation point
Material parameters
A

Elastic Energy Minimisation

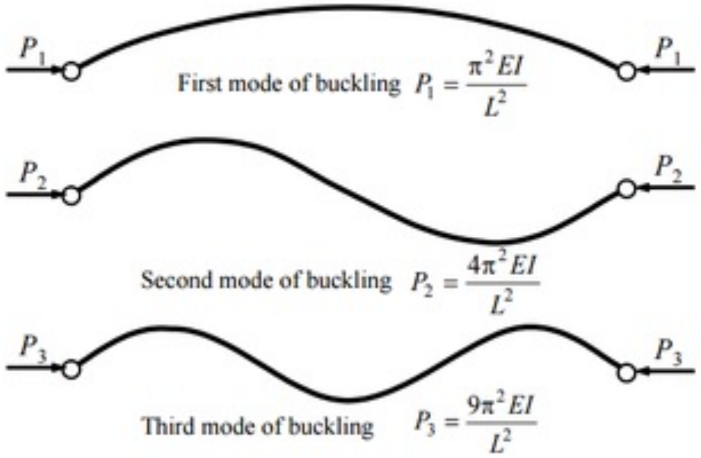
Elastic energy minimisation



- Aim: Obtain the post bifurcated patterns

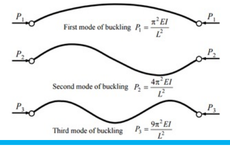


Identify how each of the beams are going to buckle

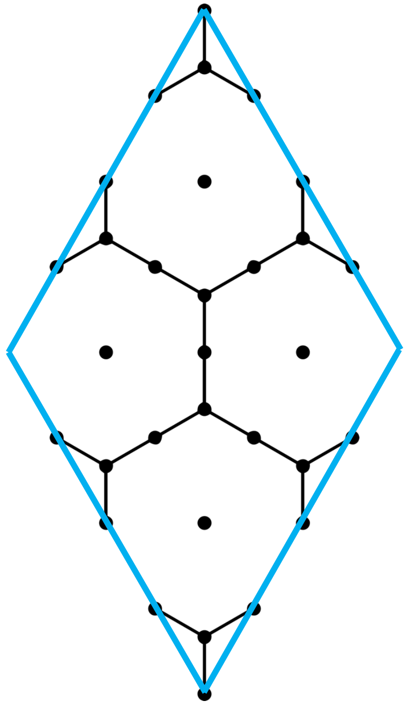


Elastic Energy Minimisation

Elastic energy minimisation



- Compute the energy of the unit cell



Energy:

Periodic boundary conditions

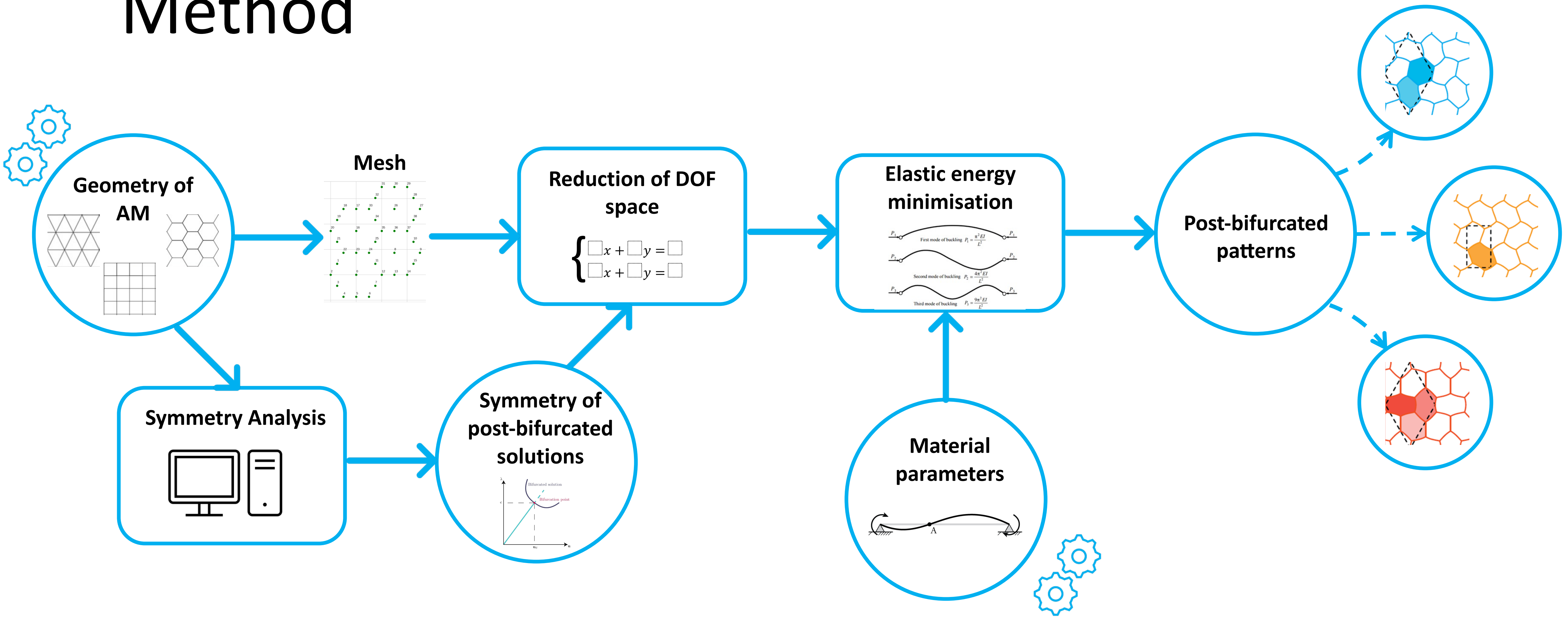
$$\mathcal{E} = \frac{1}{2}\mathbf{u}^T \underline{\mathbf{K}}_e \mathbf{u} - \left[\int_{\mathcal{D}} \mathbf{f} \cdot \mathbf{u} \, d\Omega + \oint_{\partial\mathcal{D}} \mathbf{t} \cdot \mathbf{u} \, d\Gamma \right]$$

No body forces

We want to minimise:

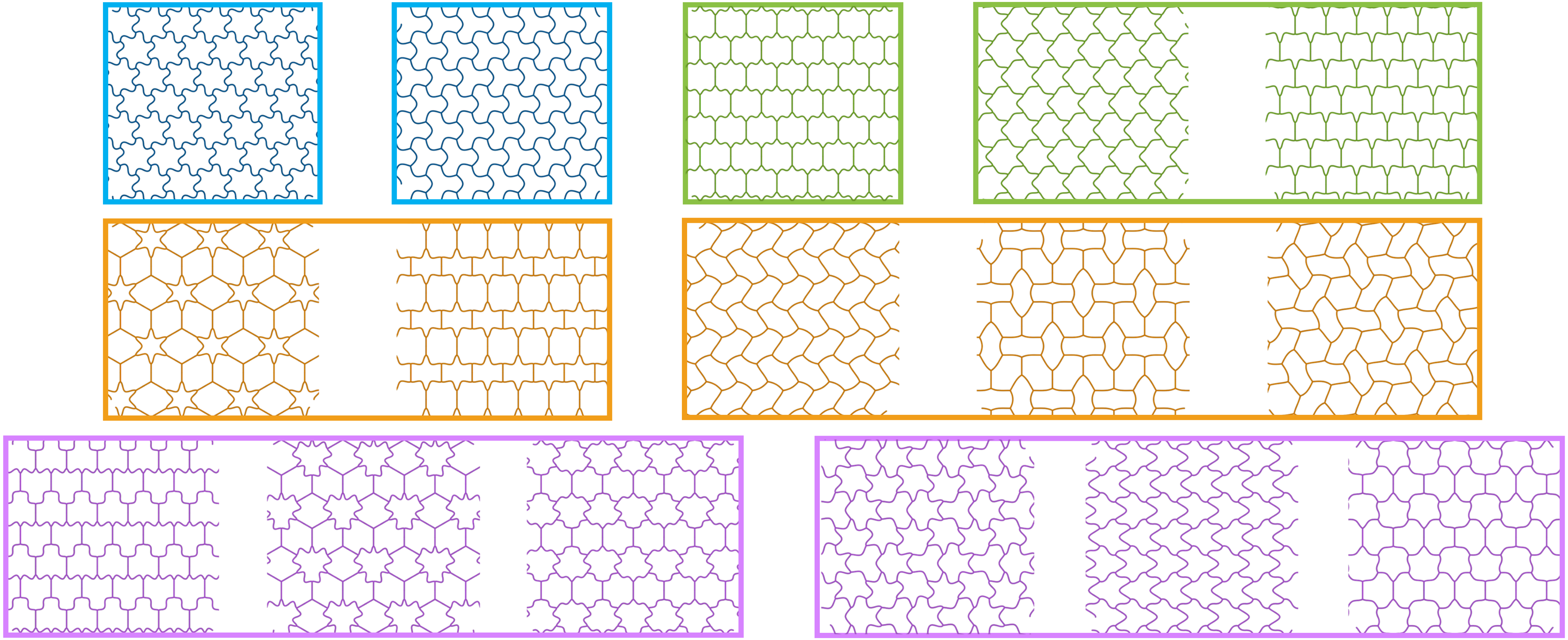
$$\mathcal{E} = \frac{1}{2}\mathbf{u}_c{}^T \underline{\mathbf{K}}_e \mathbf{u}_c \quad \text{subject to} \quad \mathbf{u}_c^T \mathbf{u}_c = 1$$

Method

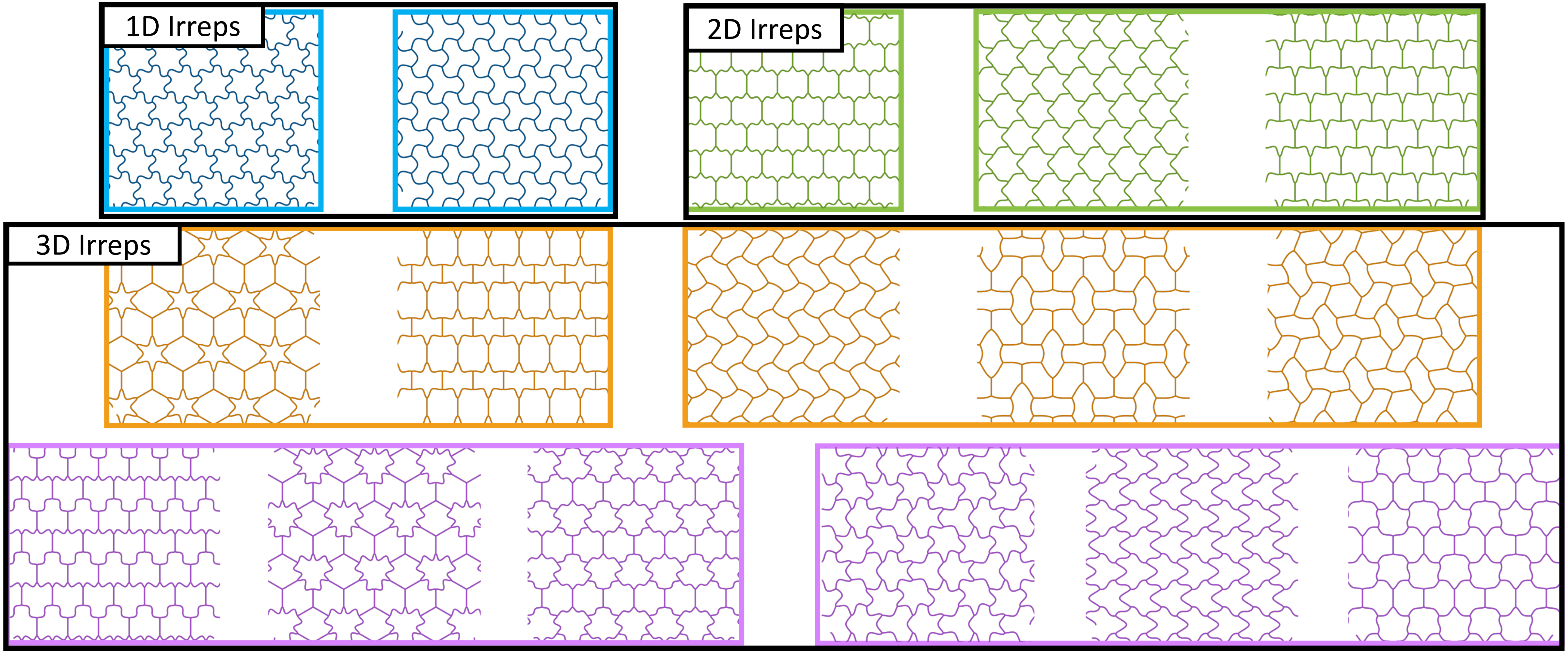

Geometry of AM
Mesh
Reduction of DOF space
Elastic energy minimisation
Post-bifurcated patterns
Symmetry Analysis
Symmetry of post-bifurcated solutions
Material parameters

Results

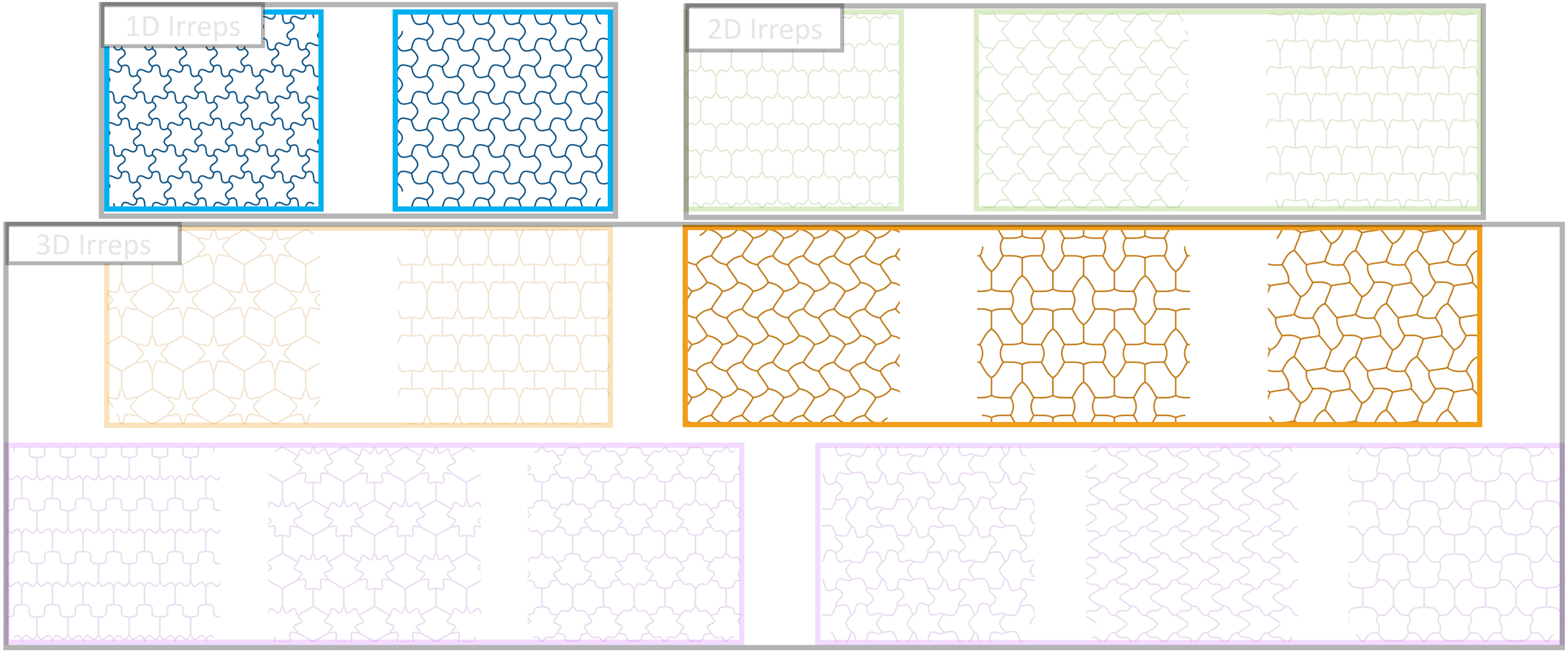
Honeycomb Patterns



Honeycomb Patterns

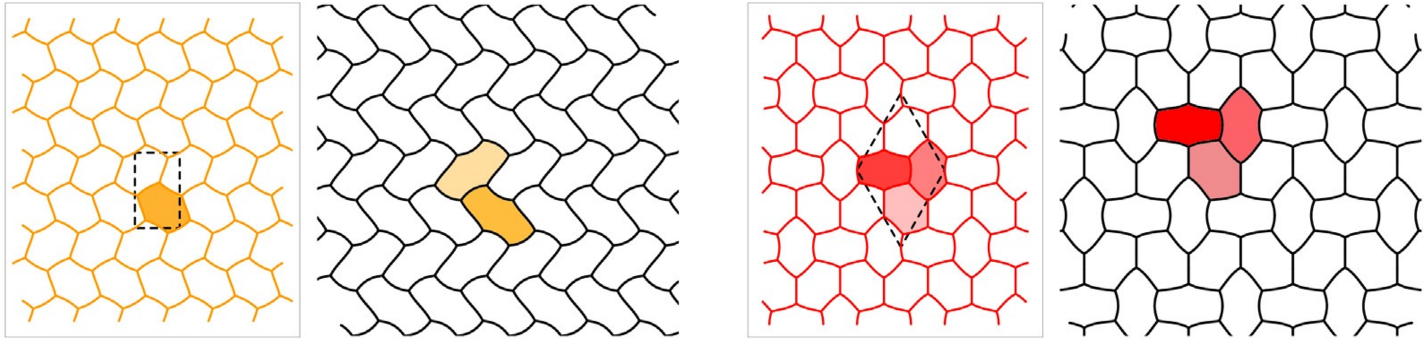


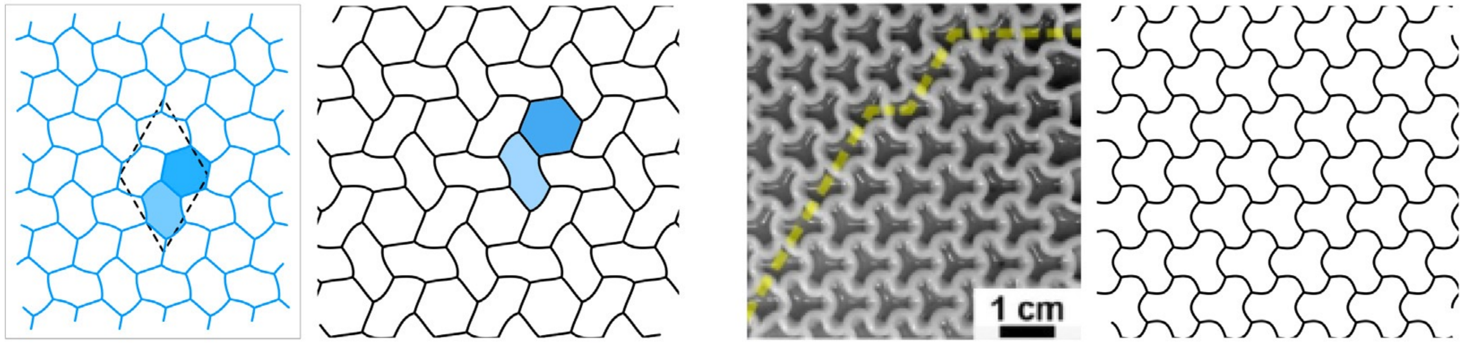
Honeycomb Patterns



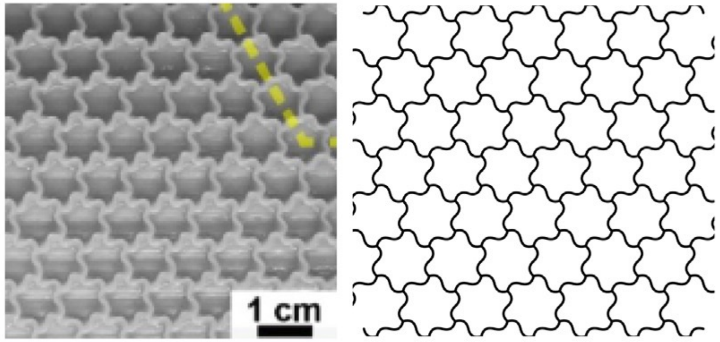
Previously Observed Patterns

From [Combescure 2016] *This study* *From [Combescure 2016]* *This study*



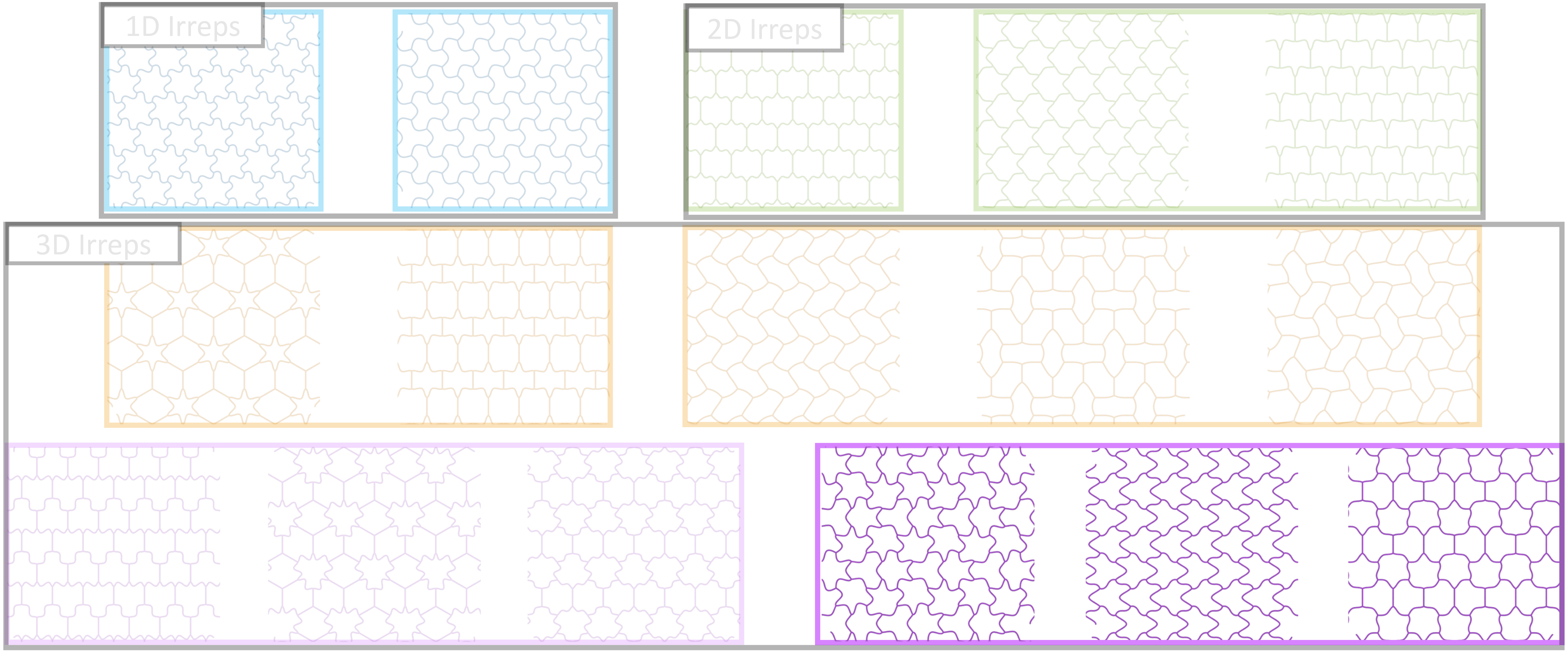


From [Kang 2013]

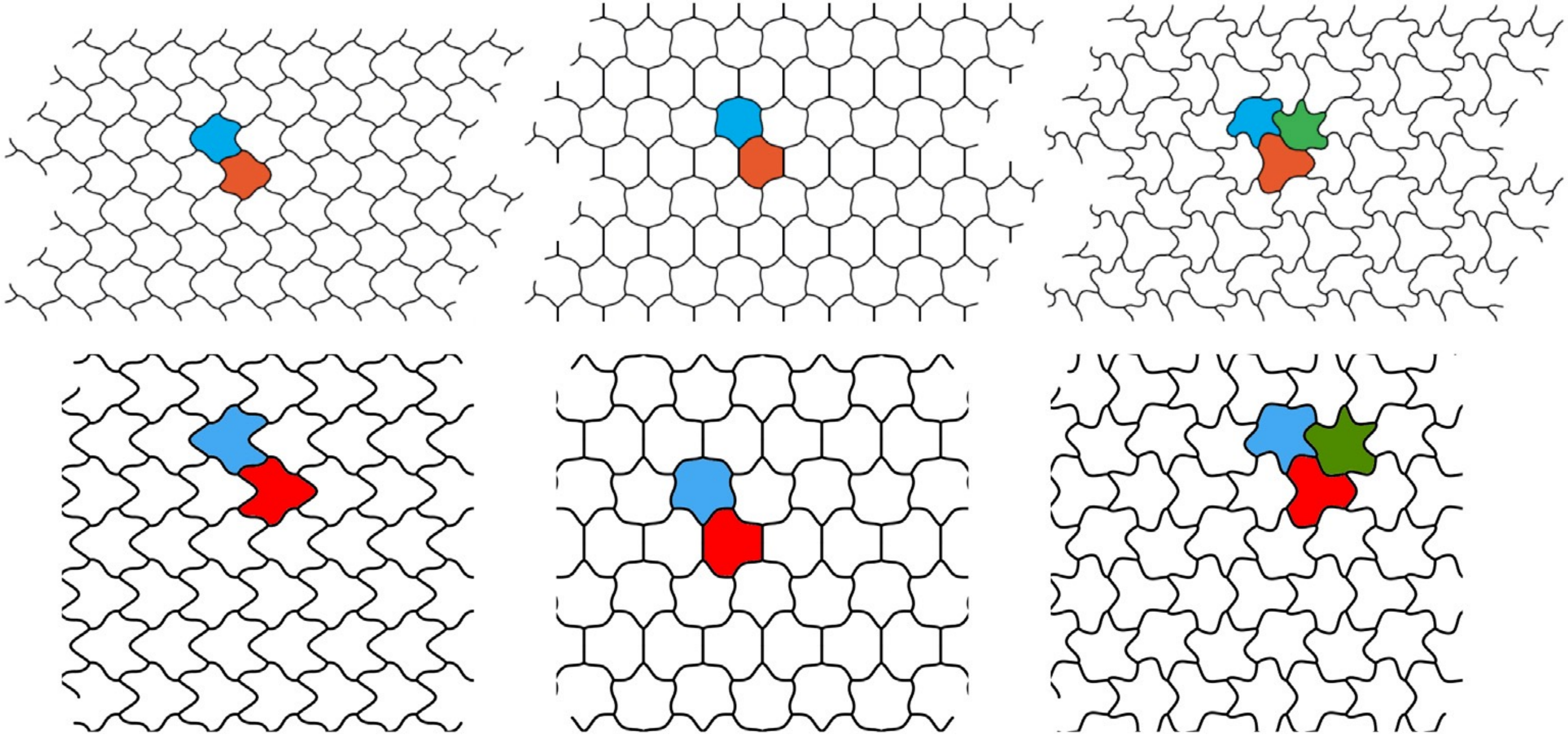


This study

Honeycomb Patterns



Previously Predicted Patterns

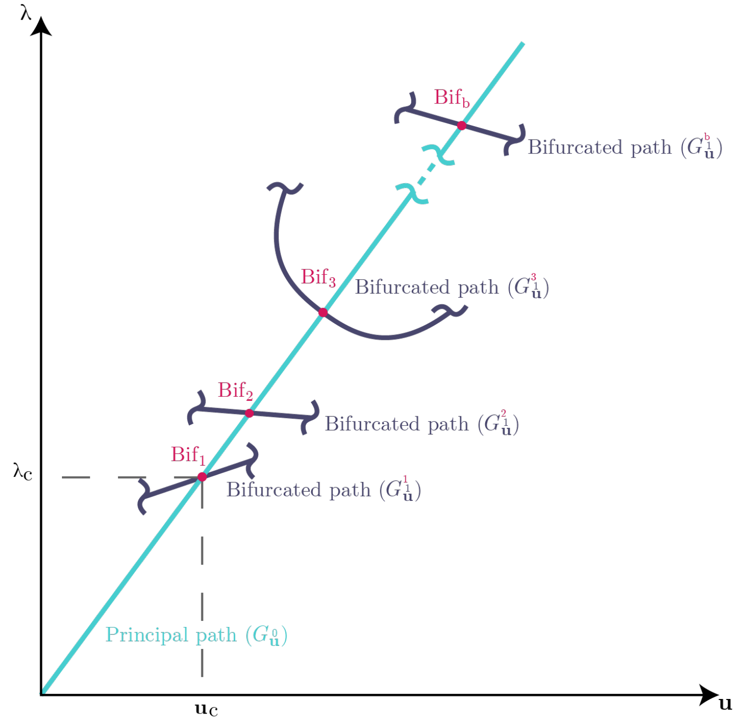


From [Combescure 2016]

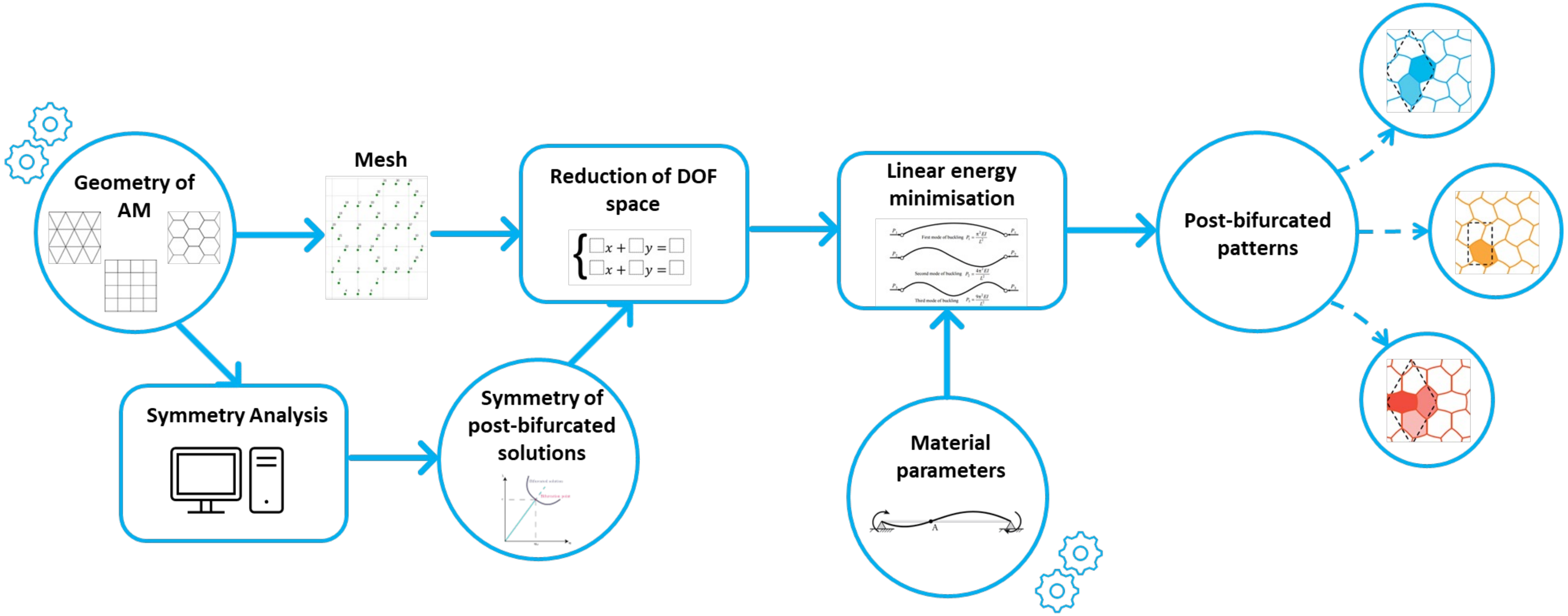
This study

Summary

- Results aligned with the literature
- Validation on other architectures
  - Results
  - Automation process
- Other works:
  - First bifurcation point computation



Conclusion



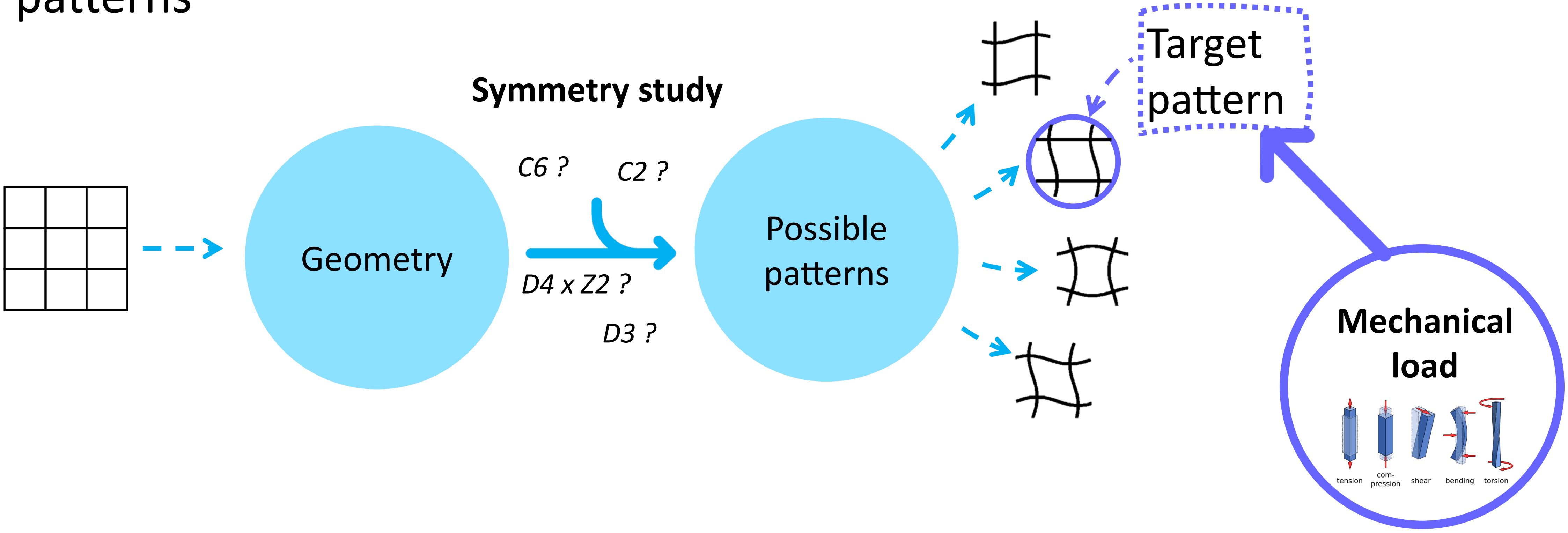
- Group Theoretic approach
- Set size of unit cell
- Any group, elements can be implemented
- Decorrelated trial and error
- Improved robustness: design based

Further work

- Equivariant Branching Lemma is not exhaustive
  - We only obtain generic bifurcation points
- Secondary bifurcation points could be computed
  - Iteratively using the existing method
  - By digging deeper into the underlying concepts in group theory
- Stability analyses for each pattern

Further work

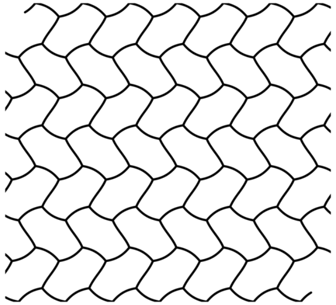
- Finding the appropriate mechanical load to obtain the desired patterns



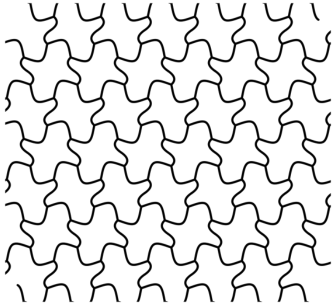
Additional Slides

Comparison

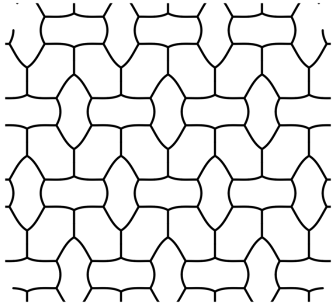
Mode I



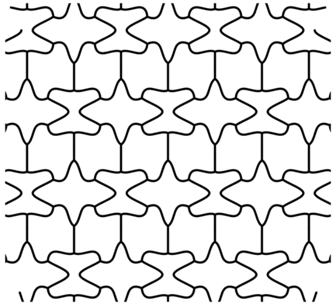
Mode I, higher order



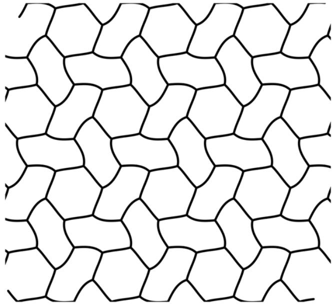
Mode II



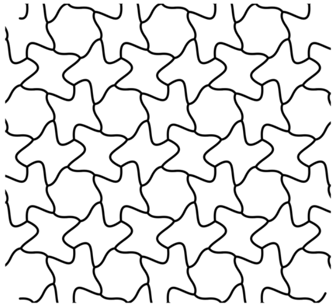
Mode II, higher order

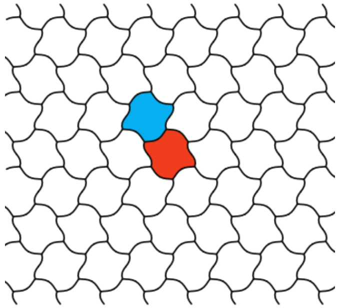


Mode III

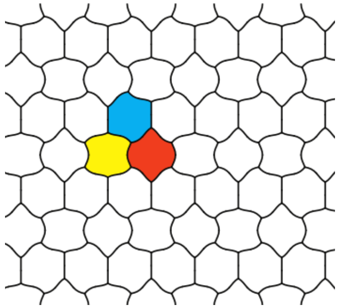


Mode III, higher order

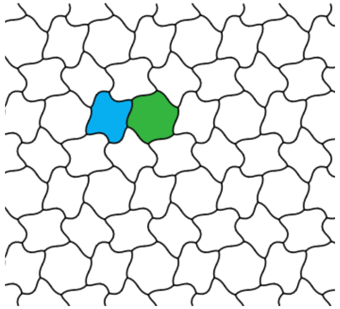




Mode I, higher order
From Combescure 2016



Mode II, higher order
From Combescure 2016



Mode III, higher order
From Combescure 2016

Semi-direct product

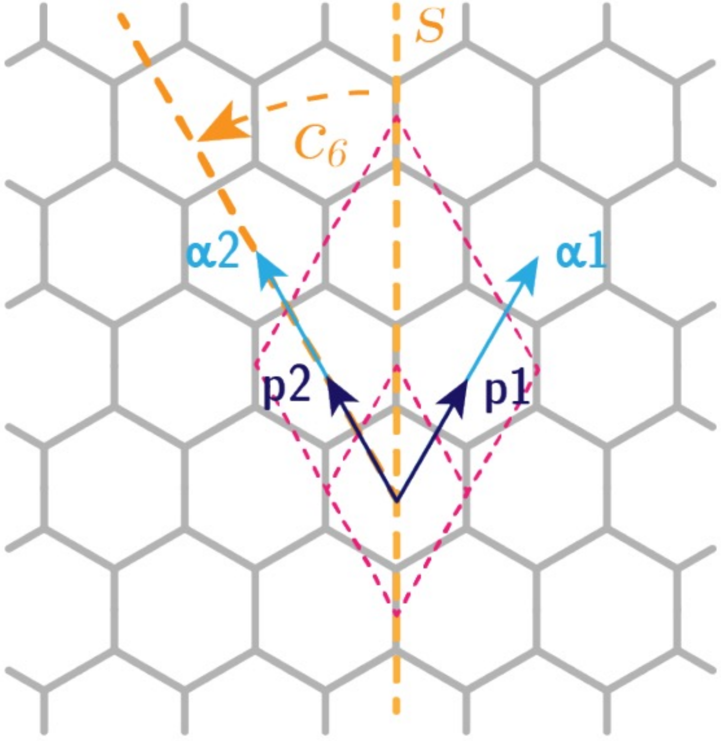
Definition 21. If $\alpha : G \longrightarrow Aut(\Gamma)$ is a group homomorphism, then the group operation of the semidirect product of G and Γ with respect to α is:

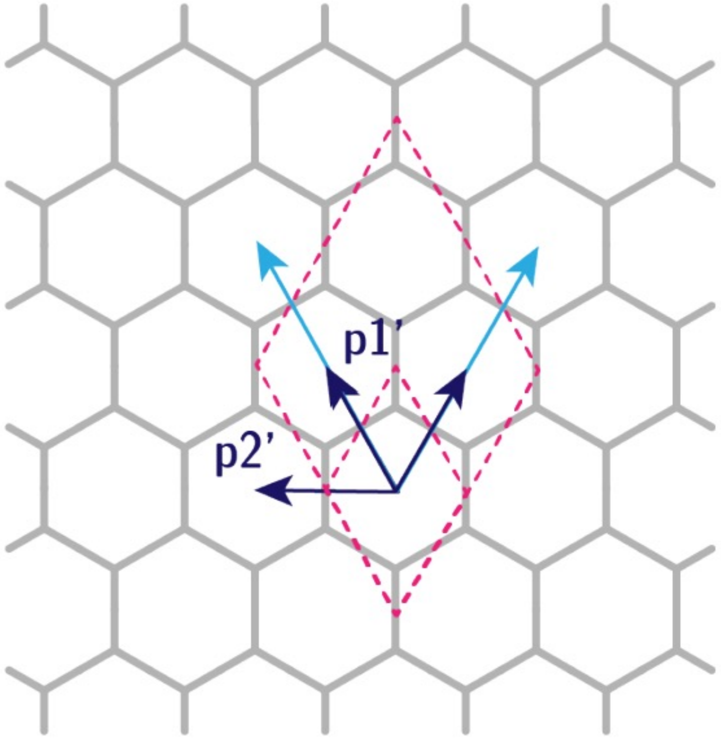
$$(g_1, \gamma_1)(g_2, \gamma_2) = (g_1 g_2, \gamma_1 \alpha_{g_1}(\gamma_2)), \ \forall g_1, g_2 \in G, \forall \gamma_1, \gamma_2 \in \Gamma \quad (3.6)$$

The product is not unique as it depends on the choice of homomorphism

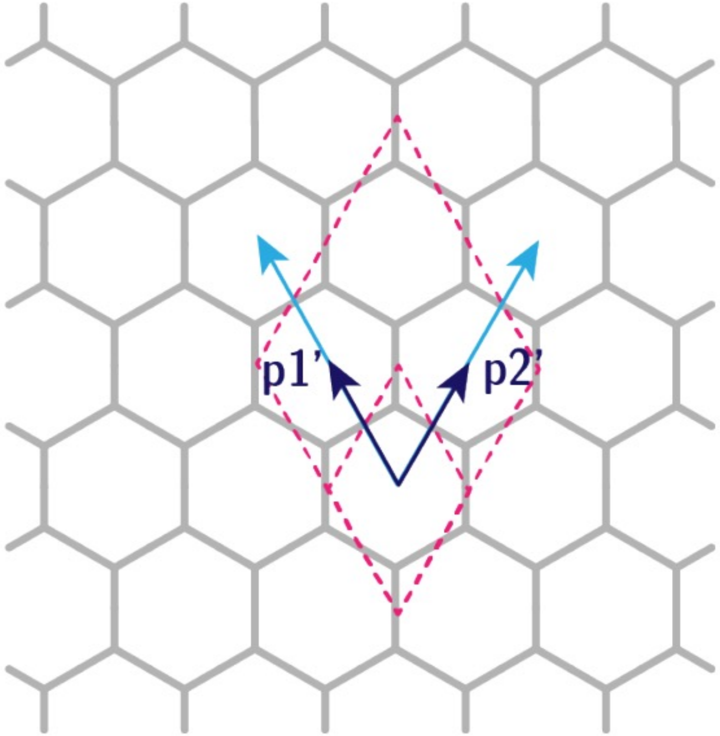
Semi-direct product

$$(g_1, \gamma_1)(g_2, \gamma_2) = (g_1 g_2, \gamma_1 \alpha_{g_1}(\gamma_2)), \ \forall g_1, g_2 \in G, \forall \gamma_1, \gamma_2 \in \Gamma$$



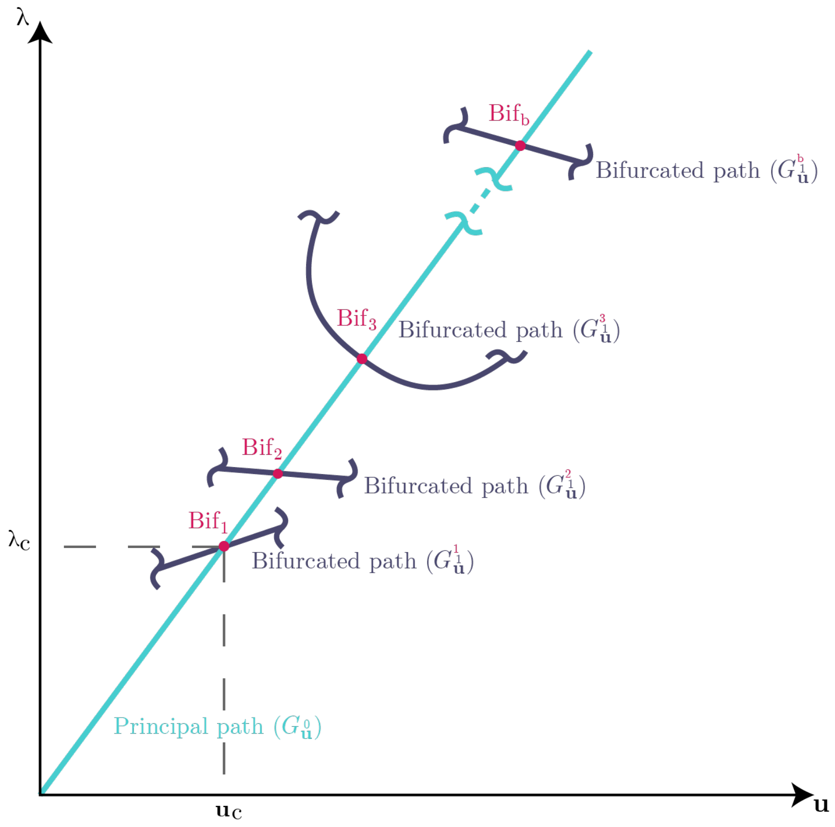


Action of c6 on p1, p2



Action of s on p1, p2

Linear Buckling Analysis



Stability operator: $\det(\mathcal{E}_{,\mathbf{uu}}(\overset{0}{\mathbf{u}}_c, \lambda_c)) = 0$

In mechanics: $\det(\underline{\mathcal{K}}_T) = 0$

The tangent stiffness matrix can be separated into two parts:

$$\underline{\mathcal{K}}_T = \underline{\mathcal{K}}_e + \lambda \underline{\mathcal{K}}_g, \quad \lambda \in \mathbb{R}$$

Stability: $\det(\underline{\mathcal{K}}_e + \lambda \underline{\mathcal{K}}_g) = 0$

Equivalent to the generalised eigenvalue problem:

$$(\underline{\mathcal{K}}_e + \lambda \underline{\mathcal{K}}_g)\mathbf{u} = 0$$

Linear Buckling Analysis

- Elementary stiffness matrices in Euler-Bernoulli beams:

$$\underline{\mathbf{K}}_e = \begin{bmatrix} \frac{ES}{L} & 0 & 0 & \frac{-ES}{L} & 0 & 0 \\ 0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & \frac{-12EI}{L^3} & \frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{4EI}{L} & 0 & \frac{-6EI}{L^2} & \frac{2EI}{L} \\ \frac{-ES}{L} & 0 & 0 & \frac{ES}{L} & 0 & 0 \\ 0 & \frac{-12EI}{L^3} & \frac{-6EI}{L^2} & 0 & \frac{12EI}{L^3} & \frac{-6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{2EI}{L} & 0 & \frac{-6EI}{L^2} & \frac{4EI}{L} \end{bmatrix} \qquad \underline{\mathbf{K}}_g = \frac{\mathcal{N}}{60L} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 36 & 3L & 0 & -36 & 3L \\ 0 & 3L & 4L^2 & 0 & -3L & -L^2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -36 & -3L & 0 & 36 & -3L \\ 0 & 3L & L^2 & 0 & -3L & L^2 \end{bmatrix}$$

$$\mathcal{E}_{nl} = \frac{1}{2}\int_0^L \left(ES\frac{du}{dx}^2 + EI_z\frac{d^2v}{dx^2}^2 + EI_y\frac{d^2w}{dx^2}^2 + GJ_0\frac{d\theta_x}{dx}^2 + N_0\frac{dv}{dx}^2 + N_0\frac{dw}{dx}^2 \right) dx$$

$$\mathcal{E}_{nl} = \frac{1}{2}\int_0^L \left(ES\frac{du}{dx}^2 + EI_z\frac{d^2v}{dx^2}^2 + EI_y\frac{d^2w}{dx^2}^2 + GJ_0\frac{d\theta_x}{dx}^2 \right) dx + \frac{1}{2}\int_0^L \left(N_0\frac{dv}{dx}^2 + N_0\frac{dw}{dx}^2 \right) dx$$

[Yang 1986] Stiffness matrix for geometric nonlinear analysis.