De la Thermodynamique des Processus Irréversibles aux structures de Dirac

L’exemple de la thermo-visco-élasticité en grandes transformations

Rencontre du GDR-GDM 2024

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Sous la direction des Pr. Anthony GRAVOUIL et David DUREISSEIX

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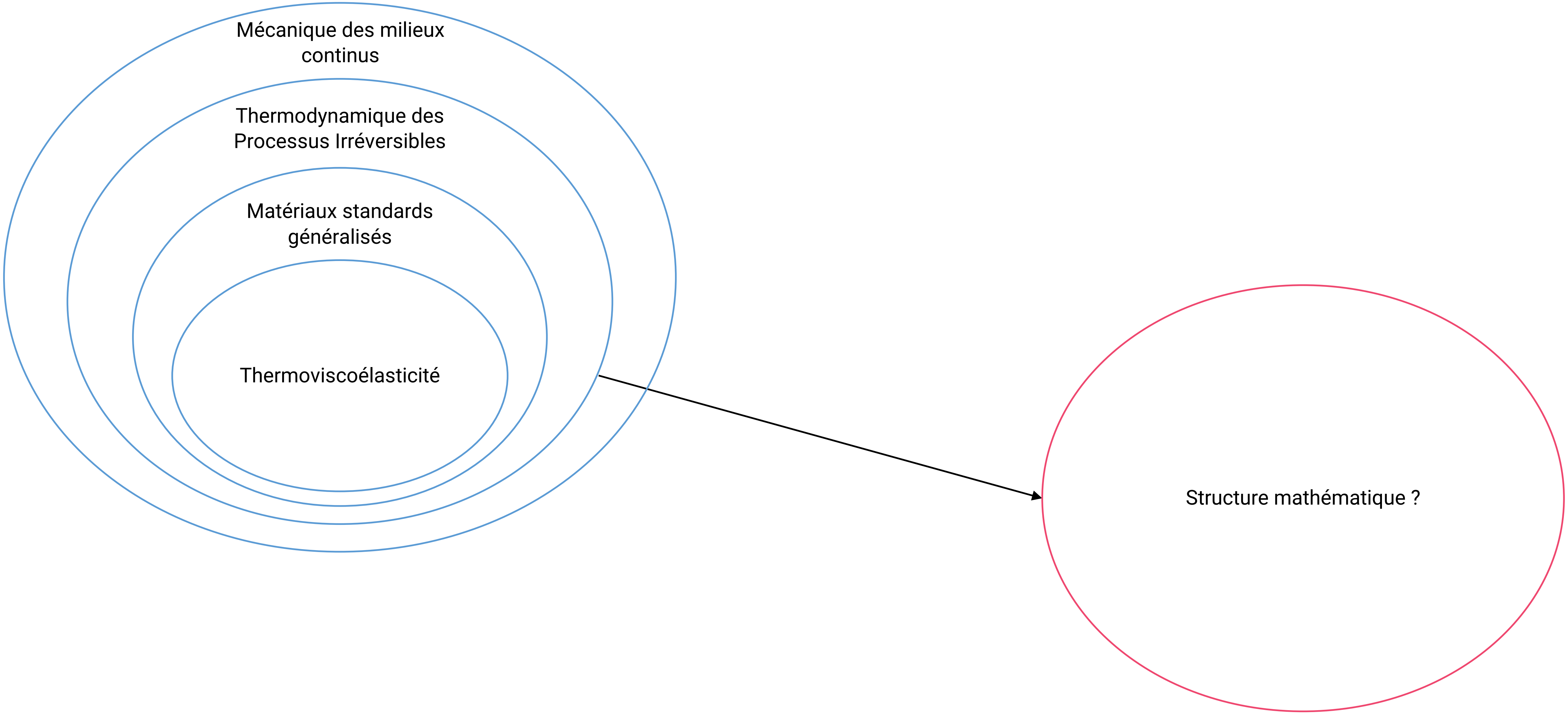
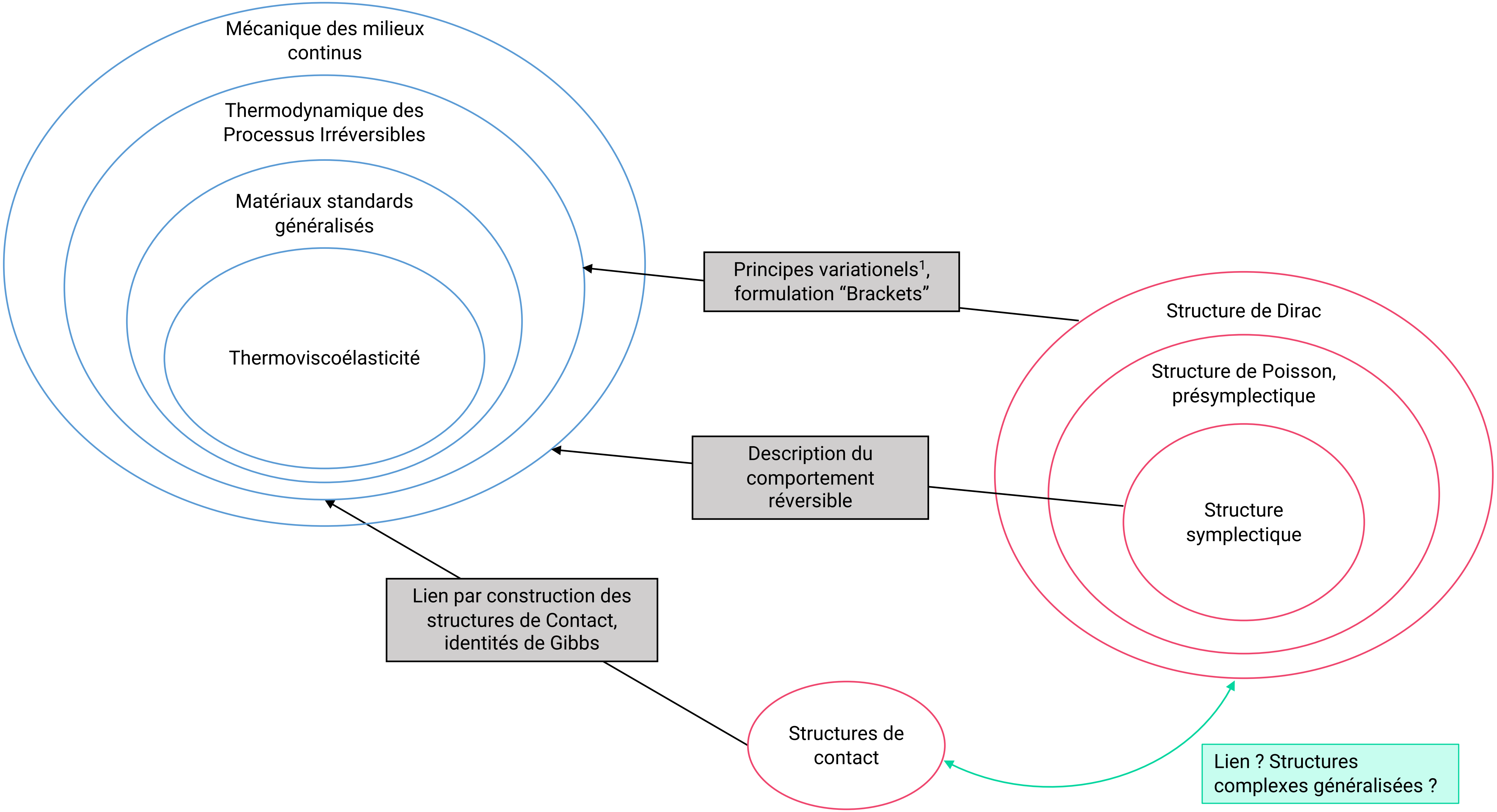
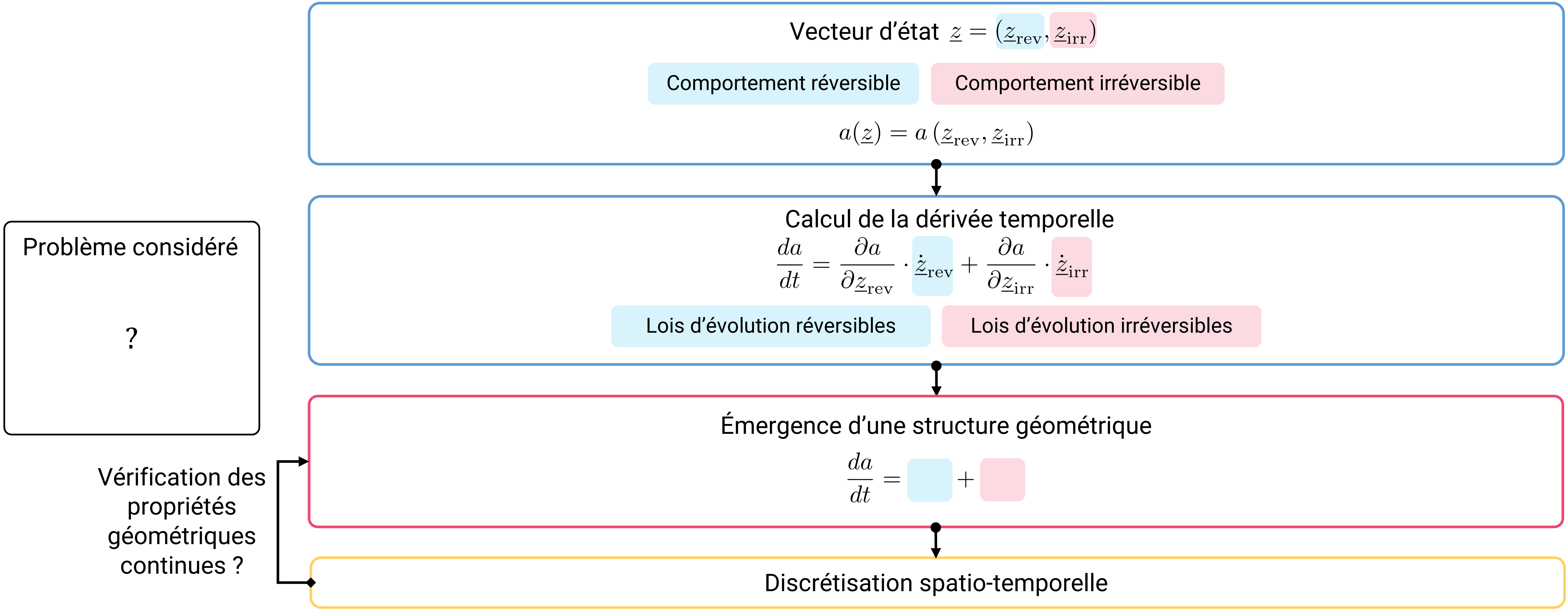

Mécanique des milieux continus
Thermodynamique des Processus Irréversibles
Matériaux standards généralisés
Thermoviscoélasticité
Structure mathématique ?



1. Gay-Balmaz, François, and Hiroaki Yoshimura. ‘Dirac Structures in Nonequilibrium Thermodynamics’. *Journal of Mathematical Physics* 59, no. 1 (1 January 2018): 012701

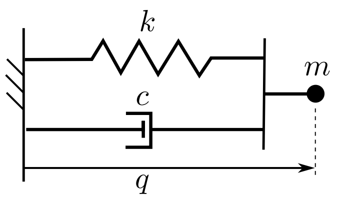
Méthodologie générale



Méthodologie générale

Un exemple : le système masse ressort amortisseur en parallèle

Problème considéré



Vecteur d'état $\underline{z} = (q, p, S)$

Comportement réversible | Comportement irréversible

$$a(\underline{z}) = a\,(q, p, S)$$

Calcul de la dérivée temporelle

$$\frac{da}{dt} = \frac{\partial a}{\partial q}\dot{q} + \frac{\partial a}{\partial p}\dot{p} + \frac{\partial a}{\partial S}\dot{S}$$

Lois d'évolution réversibles

$$\dot{q} = \frac{\partial l^\star}{\partial p}$$

$$\dot{p} = -\frac{\partial l^\star}{\partial q}$$

Lois d'évolution irréversibles

Utilisation d'un principe variationnel adapté pour la thermodynamique[1]

$$\dot{q} = \frac{\partial l^\star}{\partial p}$$

$$\dot{p} = -\frac{\partial l^\star}{\partial q} + F^{\mathrm{fr}} \quad \text{et} \quad F^{\mathrm{fr}} = -c\dot{q}$$

$$\dot{S} = -\left(\frac{\partial l^\star}{\partial S}\right)^{-1}\frac{\partial l^\star}{\partial p}F^{\mathrm{fr}}$$

k Raideur du ressort; c Coefficient d'amortissement; m Masse; q Position de la masse; $l^\star$ Hamiltonien, énergie totale; F^{ext} Effort extérieur; F^{fr} Effort dissipatif

1. Gay-Balmaz, François and Hiroaki Yoshimura (Jan. 2017). "A Lagrangian Variational Formulation for Nonequilibrium Thermodynamics. Part I: Discrete Systems". In: Journal of Geometry and Physics 111, pp. 169–193.

Méthodologie générale

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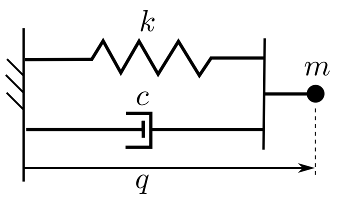

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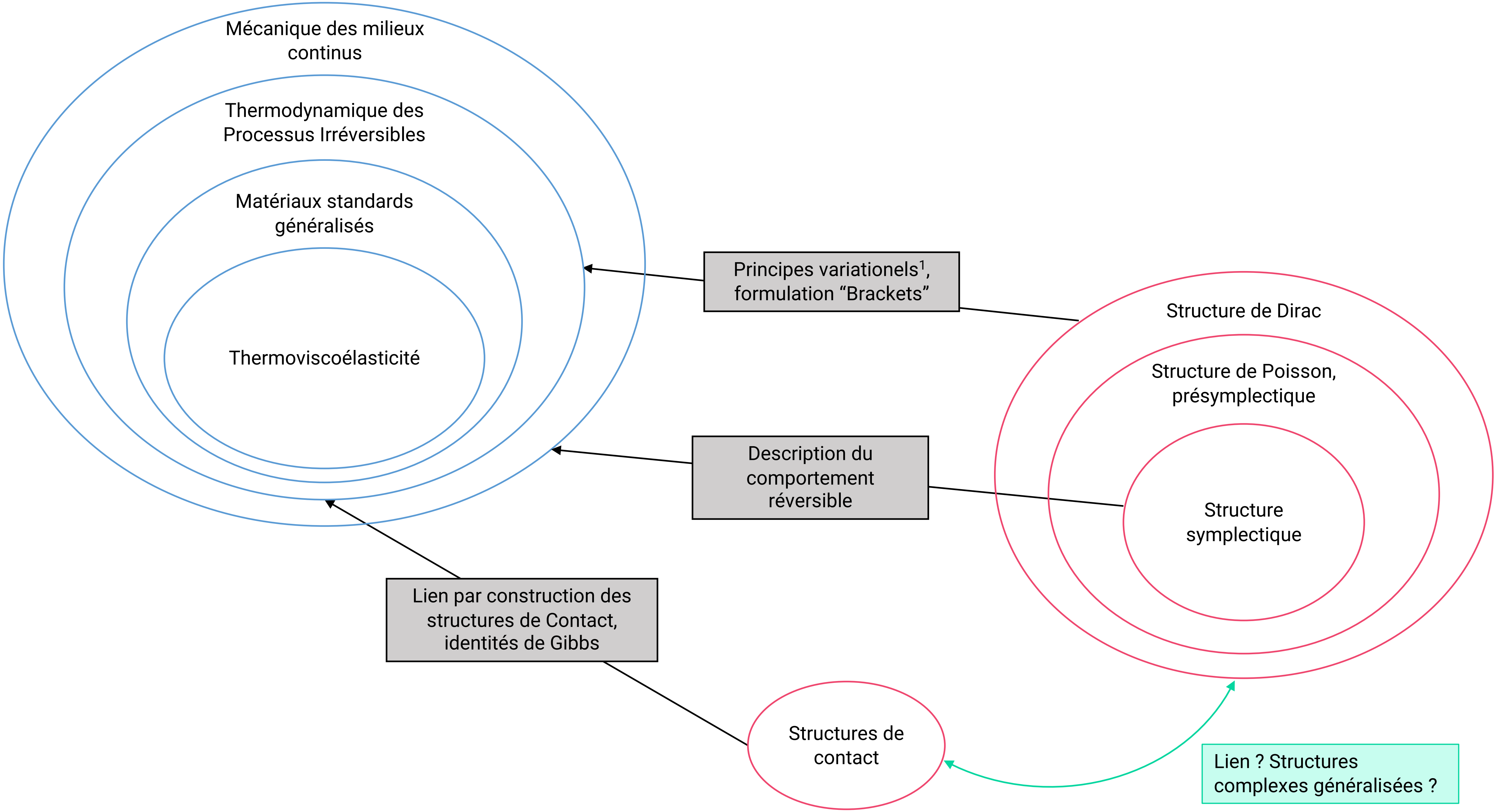
Émergence d'une structure géométrique

$$\frac{da}{dt} = \{a, l^\star\} + [a, l^\star], \text{avec}$$

$$\{a, l^\star\} = \frac{\partial a}{\partial q}\frac{\partial l^\star}{\partial p} - \frac{\partial a}{\partial p}\frac{\partial l^\star}{\partial q}$$

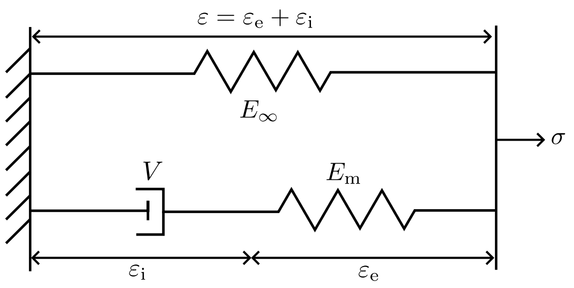
$$[a, l^\star] = -c\left(\frac{\partial a}{\partial p}\frac{\partial l^\star}{\partial p} - \frac{\partial a}{\partial S}\left(\frac{\partial l^\star}{\partial S}\right)^{-1}\frac{\partial l^\star}{\partial p}\frac{\partial l^\star}{\partial p}\right)$$



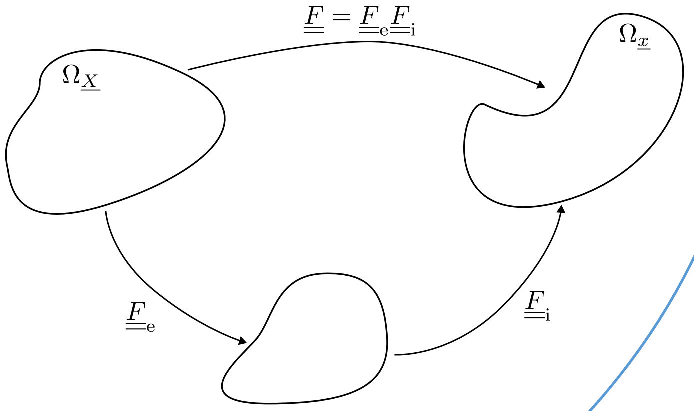
1. Gay-Balmaz, François, and Hiroaki Yoshimura. 'Dirac Structures in Nonequilibrium Thermodynamics'. *Journal of Mathematical Physics* 59, no. 1 (1 January 2018): 012701

Thermoviscoélasticité

Petites transformations
Unidimensionnel



Grandes transformations
Tridimensionnel



Thermoviscoélasticité

Petites transformations
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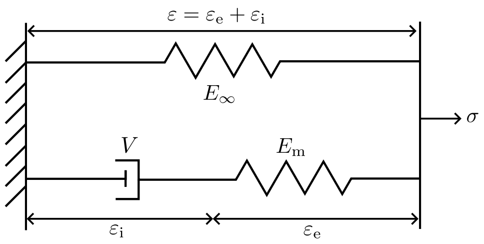
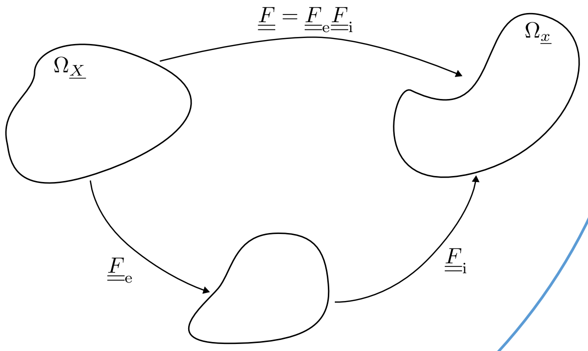


Grandes transformations
Tridimensionnel



Formalisme crochet à deux générateurs[1,2,3,4]

$$\frac{da}{dt} = \boxed{\{a, l^\star\}} + \boxed{(a, S)}$$

Crochet de Poisson

- Bilinéaire
- Antisymétrique
- Identité de Jacobi

Crochet dissipatif

- Bilinéaire
- Symétrique
- $(a, a) \geq 0$

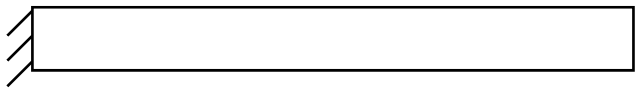
- Conditions de non interaction
$$\{S, l^\star\} = 0, (l^\star, S) = 0$$
- Vérification de la première et de la seconde loi de la thermodynamique
$$\frac{dl^\star}{dt} = 0, \frac{dS}{dt} \geq 0$$

1. Grmela, Miroslav. 'Bracket Formulation of Dissipative Fluid Mechanics Equations'. *Physics Letters A* 102, no. 8 (June 1984): 355–58.
2. Morrison, Philip J. 'Bracket Formulation for Irreversible Classical Fields'. *Physics Letters A* 100, no. 8 (February 1984): 423–27.
3. Öttinger, Hans Christian. *Beyond Equilibrium Thermodynamics*. 1st ed. Wiley, 2005.
4. Romero, Ignacio. 'Algorithms for Coupled Problems That Preserve Symmetries and the Laws of Thermodynamics'. *Computer Methods in Applied Mechanics and Engineering* 199, no. 25–28 (May 2010): 1841–58.

L'exemple de la thermo-visco-élasticité en HPP

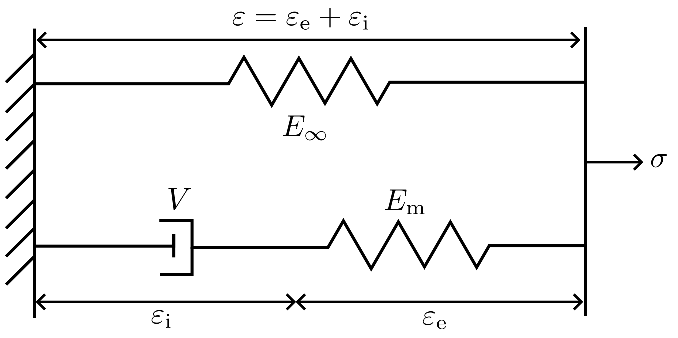
Problème considéré

Milieu continu 1D



$$\underline{\underline{\varepsilon}} = \frac{1}{2}\left(\frac{\partial \underline{u}}{\partial \underline{x}} + \frac{\partial \underline{u}}{\partial \underline{x}}^{\mathsf{T}}\right) \rightsquigarrow \varepsilon = \frac{\partial u}{\partial x}$$

Modèle rhéologique : Zener



- ε_{e} : Déformation réversible
- ε_{i} : Déformation irréversible
- E_∞ : Module élastique permanent
- E_{m} : Module élastique branche visqueuse
- V : Coefficient d'amortissement
- σ : Contrainte axiale

Choix des paramètres du vecteur d'état

- Déplacement u
- Quantité de mouvement p
- Déformation irréversible ε_{i}
- Température θ

Déplacement, quantité de mouvement → Lois d'évolution données par le principe d'Hamilton.

Déformation irréversible, température → Lois d'évolution données par les principes de la TPI.

Particularité du problème

On cherche, depuis les équations de la mécanique, à faire émerger une équation du type

$$\dot{\underline{z}} = \underline{\underline{L}}\frac{\partial l^\star}{\partial \underline{z}} + \underline{\underline{M}}\frac{\partial s}{\partial \underline{z}} \Leftrightarrow \frac{da}{dt} = \frac{\partial a}{\partial \underline{z}} \cdot \underline{\underline{L}}\frac{\partial l^\star}{\partial \underline{z}} + \frac{\partial a}{\partial \underline{z}} \cdot \underline{\underline{M}}\frac{\partial s}{\partial \underline{z}} \Leftrightarrow \frac{da}{dt} = \{a, l^\star\} + (a, s)$$

$\underline{\underline{L}}$: Matrice antisymétrique, représentative du crochet de Poisson

$\underline{\underline{M}}$: Matrice symétrique, signature de la dissipation (structure de Dirac)

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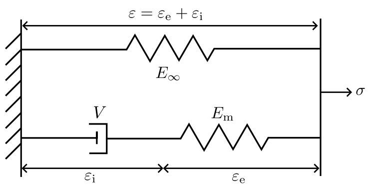

Vecteur d'état $\underline{z} = (u, p, \theta, \varepsilon_i)$

Comportement réversible | Comportement irréversible

$$a(\underline{z}) = a(u, p, \theta, \varepsilon_i)$$

Calcul de la dérivée temporelle

$$\frac{da}{dt} = \frac{\partial a}{\partial u}\dot{u} + \frac{\partial a}{\partial p}\dot{p} + \frac{\partial a}{\partial \theta}\dot{\theta} + \frac{\partial a}{\partial \varepsilon_i}\dot{\varepsilon}_i$$

Lois d'évolution réversibles

Lois d'évolution irréversibles

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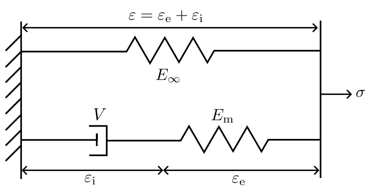
Problème considéré



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Lois d'évolution réversibles

Principe d'Hamilton – stationarisation de l'intégrale d'action

$$\mathcal{A}^\star[u,p] = \int_{\omega_t} \left[p\frac{\partial u}{\partial t} - l^\star\left(u, \frac{\partial u}{\partial t}\right)\right] dt$$

$$\delta\mathcal{A}^\star[u,p] = 0 \ \forall \delta u = 0 \text{ on } \delta\omega_t \Rightarrow \delta_u\mathcal{A}^\star + \delta_p\mathcal{A}^\star = 0 \ \forall \delta u = 0 \text{ on } \delta\omega_t$$

$$\dot{u} = \frac{\partial l^\star}{\partial p}$$

$$\dot{p} = -\frac{\partial l^\star}{\partial u}$$

Lois d'évolution irréversibles

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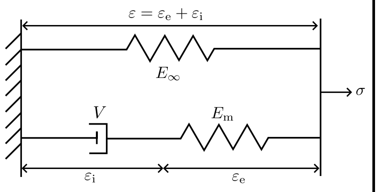

Calcul de la dérivée temporelle

$$\frac{da}{dt} = \frac{\partial a}{\partial u}\dot{u} + \frac{\partial a}{\partial p}\dot{p} + \frac{\partial a}{\partial \theta}\dot{\theta} + \frac{\partial a}{\partial \varepsilon_{\mathrm{i}}}\dot{\varepsilon}_{\mathrm{i}}$$

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$$\begin{bmatrix} \frac{\partial a}{\partial u}\dot{u} \\ \frac{\partial a}{\partial p}\dot{p} \\ \frac{\partial a}{\partial \theta}\dot{\theta} \\ \frac{\partial a}{\partial \varepsilon_{\mathrm{i}}}\dot{\varepsilon}_{\mathrm{i}} \end{bmatrix} = \begin{bmatrix} \frac{\partial a}{\partial u} \\ \frac{\partial a}{\partial p} \\ \frac{\partial a}{\partial \theta} \\ \frac{\partial a}{\partial \varepsilon_{\mathrm{i}}} \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ \square & \square & \square & \square \\ \square & \square & \square & \square \end{bmatrix} \begin{bmatrix} \frac{\partial l^*}{\partial u} \\ \frac{\partial l^*}{\partial p} \\ \frac{\partial l^*}{\partial \theta} \\ \frac{\partial l^*}{\partial \varepsilon_{\mathrm{i}}} \end{bmatrix} + \begin{bmatrix} \frac{\partial a}{\partial u} \\ \frac{\partial a}{\partial p} \\ \frac{\partial a}{\partial \theta} \\ \frac{\partial a}{\partial \varepsilon_{\mathrm{i}}} \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \square & \square & \square & \square \\ \square & \square & \square & \square \end{bmatrix} \begin{bmatrix} \frac{\partial s}{\partial u} \\ \frac{\partial s}{\partial p} \\ \frac{\partial s}{\partial \theta} \\ \frac{\partial s}{\partial \varepsilon_{\mathrm{i}}} \end{bmatrix}$$

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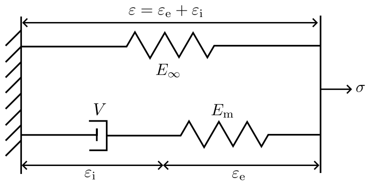
Problème considéré



Lois d'évolution réversibles

$$\dot{u} = \frac{\partial l^\star}{\partial p}$$
$$\dot{p} = -\frac{\partial l^\star}{\partial u}$$

Lois d'évolution irréversibles – Déformation inélastique[1]

Potentiel de dissipation dual $\dot{\varepsilon}_{\mathrm{i}} = \frac{\partial \varphi^\star}{\partial \sigma}, \dot{\varepsilon}_{\mathrm{i}} = V\sigma, \Rightarrow \varphi^\star = \frac{1}{2}V^{-1}\sigma\sigma$

Potentiel de dissipation $\varphi(\dot{\varepsilon}_{\mathrm{i}}) = \sigma\dot{\varepsilon}_{\mathrm{i}} - \varphi^\star(\sigma) \Rightarrow \varphi(\dot{\varepsilon}_{\mathrm{i}}) = \frac{1}{2}V\dot{\varepsilon}_{\mathrm{i}}\dot{\varepsilon}_{\mathrm{i}}$

Loi de Biot[2] $\frac{\partial \varphi}{\partial \dot{\varepsilon}_{\mathrm{i}}} + \rho\frac{\partial \psi}{\partial \varepsilon_{\mathrm{i}}} = 0 \Rightarrow \dot{\varepsilon}_{\mathrm{i}} = -\rho V^{-1}\frac{\partial \psi}{\partial \varepsilon_{\mathrm{i}}}$ $\quad \psi = e - \theta s$

$$\dot{\varepsilon}_{\mathrm{i}} = -\rho c^{-1}\theta V^{-1}\frac{\partial e}{\partial \varepsilon_{\mathrm{i}}}\frac{\partial s}{\partial \theta} + \rho\theta V^{-1}\frac{\partial s}{\partial \varepsilon_{\mathrm{i}}}$$

$$c = \frac{\partial e}{\partial \theta} = \theta\frac{\partial s}{\partial \theta}$$

ψ : Énergie libre de Helmholtz, s : Entropie, e : Énergie interne, c : Capacité thermique

1. Lemaitre, Jean, and Jean-Louis Chaboche. *Mécanique des matériaux solides*. 2e éd. Sciences sup. Paris: Dunod, 2001.
2. Biot, Maurice A. *Mechanics of Incremental Deformations*. London: John Wiley & Sons, 1965.

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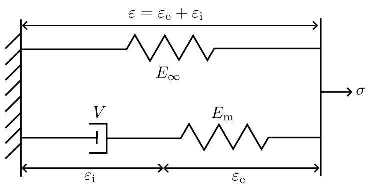

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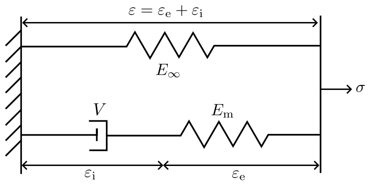
Lois d'évolution irréversibles – Thermique

Problème considéré



L’exemple de la thermo-visco-élasticité en HPP

Lois d’évolution irréversibles - Thermique

$$\frac{da}{dt} = \frac{\partial a}{\partial u}\dot{u} + \frac{\partial a}{\partial p}\dot{p} + \frac{\partial a}{\partial \theta}\dot{\theta} + \frac{\partial a}{\partial \varepsilon_{\mathrm{i}}}\dot{\varepsilon}_{\mathrm{i}}$$

- Principes de la thermodynamique $\rho\dot{e} = \sigma\dot{\varepsilon} - \frac{\partial q}{\partial x},\ \rho\dot{s} - \frac{\partial q/\theta}{\partial x} \geq 0$
- Définition d’un potentiel d’énergie libre de Helmholtz $\psi = \hat{\psi}(\varepsilon, \theta, \varepsilon_{\mathrm{i}})$

$$\rho\frac{\partial \hat{\psi}}{\partial \varepsilon} = \sigma, \quad \frac{\partial \hat{\psi}}{\partial \theta} = -s$$

- Définition du potentiel d’énergie libre de Helmholtz en fonction de l’énergie interne, de la température et de l’entropie $\psi = e - \theta s$
- Dérivation temporelle $\dot{e} = \dot{\psi} - \dot{\theta}\frac{\partial \psi}{\partial \theta} - \theta\frac{d}{dt}\frac{\partial \psi}{\partial \theta}$
- Définition de la capacité thermique $c = \frac{\partial e}{\partial \theta} = \theta\frac{\partial s}{\partial \theta}$

$$\dot{\theta} = \underbrace{\theta c^{-1}\frac{\partial s}{\partial \varepsilon}\dot{\varepsilon}}_{\mathrm{I}} + \underbrace{c^{-1}\rho V^{-1}\frac{\partial e}{\partial \varepsilon_{\mathrm{i}}}\frac{\partial \psi}{\partial \varepsilon_{\mathrm{i}}}}_{\mathrm{II}} + \underbrace{c^{-1}\frac{\partial}{\partial x}\left(K\frac{\partial \theta}{\partial x}\right)}_{\mathrm{III}}$$

L’exemple de la thermo-visco-élasticité en HPP

Lois d’évolution irréversibles - Thermique

$$\frac{da}{dt} = \frac{\partial a}{\partial u}\dot{u} + \frac{\partial a}{\partial p}\dot{p} + \frac{\partial a}{\partial \theta}\dot{\theta} + \frac{\partial a}{\partial \varepsilon_{\mathrm{i}}}\dot{\varepsilon}_{\mathrm{i}}$$

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Partie I – couplage thermomécanique

- En se souvenant de la loi d’évolution de la dérivée du déplacement, on obtient

$$\theta c^{-1}\frac{\partial s}{\partial \varepsilon}\dot{\varepsilon} = \theta c^{-1}\frac{\partial s}{\partial \varepsilon}\frac{\partial}{\partial x}\left(\frac{\partial l^\star}{\partial p}\right)$$

Évolution réversible de la température[1,2]

$$\begin{bmatrix} \frac{\partial a}{\partial u}\dot{u} \\ \frac{\partial a}{\partial p}\dot{p} \\ \frac{\partial a}{\partial \theta}\dot{\theta} \\ \frac{\partial a}{\partial \varepsilon_{\mathrm{i}}}\dot{\varepsilon}_{\mathrm{i}} \end{bmatrix} = \begin{bmatrix} \frac{\partial a}{\partial u} \\ \frac{\partial a}{\partial p} \\ \frac{\partial a}{\partial \theta} \\ \frac{\partial a}{\partial \varepsilon_{\mathrm{i}}} \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ \square & \theta c^{-1}\frac{\partial s}{\partial \varepsilon}\frac{\partial}{\partial x}(\circ) & \square & \square \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial l^\star}{\partial u} \\ \frac{\partial l^\star}{\partial p} \\ \frac{\partial l^\star}{\partial \theta} \\ \frac{\partial l^\star}{\partial \varepsilon_{\mathrm{i}}} \end{bmatrix} + \begin{bmatrix} \frac{\partial a}{\partial u} \\ \frac{\partial a}{\partial p} \\ \frac{\partial a}{\partial \theta} \\ \frac{\partial a}{\partial \varepsilon_{\mathrm{i}}} \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \square & \square & \square & \square \\ 0 & 0 & -\rho V^{-1}\frac{\partial e}{\partial \varepsilon_{\mathrm{i}}}\theta c^{-1} & \theta\rho V^{-1} \end{bmatrix} \begin{bmatrix} \frac{\partial s}{\partial u} \\ \frac{\partial s}{\partial p} \\ \frac{\partial s}{\partial \theta} \\ \frac{\partial s}{\partial \varepsilon_{\mathrm{i}}} \end{bmatrix}$$

Modification de la seconde loi issue du principe d’Hamilton ?

1. Green, A. E., and P. M. Naghdi. ‘Thermoelasticity without Energy Dissipation’. *Journal of Elasticity* 31, no. 3 (June 1993): 189–208.
2. Maugin, G. A., and V. K. Kalpakides. ‘A Hamiltonian Formulation for Elasticity and Thermoelasticity’. *Journal of Physics A: Mathematical and General* 35, no. 50 (2002): 10775.

L'exemple de la thermo-visco-élasticité en HPP

Lois d'évolution irréversibles - Thermique

$$\frac{da}{dt} = \frac{\partial a}{\partial u}\dot{u} + \frac{\partial a}{\partial p}\dot{p} + \frac{\partial a}{\partial \theta}\dot{\theta} + \frac{\partial a}{\partial \varepsilon_\mathrm{i}}\dot{\varepsilon}_\mathrm{i}$$

$$\dot{\theta} = \underbrace{\theta c^{-1}\frac{\partial s}{\partial \varepsilon}\dot{\varepsilon}}_{\mathrm{I}} + \underbrace{c^{-1}\rho V^{-1}\frac{\partial e}{\partial \varepsilon_\mathrm{i}}\frac{\partial \psi}{\partial \varepsilon_\mathrm{i}}}_{\mathrm{II}} + \underbrace{c^{-1}\frac{\partial}{\partial x}\left(K\frac{\partial \theta}{\partial x}\right)}_{\mathrm{III}}$$

Partie II – couplage thermomécanique dissipatif

$$c^{-1}\rho V^{-1}\frac{\partial e}{\partial \varepsilon_\mathrm{i}}\frac{\partial \psi}{\partial \varepsilon_\mathrm{i}} = c^{-1}\rho V^{-1}\theta\frac{\partial e}{\partial \varepsilon_\mathrm{i}}\frac{\partial e}{\partial \varepsilon_\mathrm{i}}c^{-1}\frac{\partial s}{\partial \theta} - c^{-1}\rho V^{-1}\theta\frac{\partial e}{\partial \varepsilon_\mathrm{i}}\frac{\partial s}{\partial \varepsilon_\mathrm{i}}$$

$$\begin{bmatrix} \frac{\partial a}{\partial u}\dot{u} \\ \frac{\partial a}{\partial p}\dot{p} \\ \frac{\partial a}{\partial \theta}\dot{\theta} \\ \frac{\partial a}{\partial \varepsilon_\mathrm{i}}\dot{\varepsilon}_\mathrm{i} \end{bmatrix} = \begin{bmatrix} \frac{\partial a}{\partial u} \\ \frac{\partial a}{\partial p} \\ \frac{\partial a}{\partial \theta} \\ \frac{\partial a}{\partial \varepsilon_\mathrm{i}} \end{bmatrix}^\mathsf{T} \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & ? & 0 \\ \square & \theta c^{-1}\frac{\partial s}{\partial \varepsilon}\frac{\partial}{\partial x}(\circ) & \square & \square \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial l^\star}{\partial u} \\ \frac{\partial l^\star}{\partial p} \\ \frac{\partial l^\star}{\partial \theta} \\ \frac{\partial l^\star}{\partial \varepsilon_\mathrm{i}} \end{bmatrix} + \begin{bmatrix} \frac{\partial a}{\partial u} \\ \frac{\partial a}{\partial p} \\ \frac{\partial a}{\partial \theta} \\ \frac{\partial a}{\partial \varepsilon_\mathrm{i}} \end{bmatrix}^\mathsf{T} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & c^{-1}\rho V^{-1}\theta\frac{\partial e}{\partial \varepsilon_\mathrm{i}}\frac{\partial e}{\partial \varepsilon_\mathrm{i}}c^{-1} & -c^{-1}\rho V^{-1}\theta\frac{\partial e}{\partial \varepsilon_\mathrm{i}} \\ 0 & 0 & -\rho V^{-1}\frac{\partial e}{\partial \varepsilon_\mathrm{i}}\theta c^{-1} & \theta\rho V^{-1} \end{bmatrix} \begin{bmatrix} \frac{\partial s}{\partial u} \\ \frac{\partial s}{\partial p} \\ \frac{\partial s}{\partial \theta} \\ \frac{\partial s}{\partial \varepsilon_\mathrm{i}} \end{bmatrix}$$

L’exemple de la thermo-visco-élasticité en HPP

Lois d’évolution irréversibles - Thermique

$$\frac{da}{dt} = \frac{\partial a}{\partial u}\dot{u} + \frac{\partial a}{\partial p}\dot{p} + \frac{\partial a}{\partial \theta}\dot{\theta} + \frac{\partial a}{\partial \varepsilon_{\mathrm{i}}}\dot{\varepsilon}_{\mathrm{i}}$$

$$\dot{\theta} = \underbrace{\theta c^{-1}\frac{\partial s}{\partial \varepsilon}\dot{\varepsilon}}_{\mathrm{I}} + \underbrace{c^{-1}\rho V^{-1}\frac{\partial e}{\partial \varepsilon_{\mathrm{i}}}\frac{\partial \psi}{\partial \varepsilon_{\mathrm{i}}}}_{\mathrm{II}} + \underbrace{c^{-1}\frac{\partial}{\partial x}\left(K\frac{\partial \theta}{\partial x}\right)}_{\mathrm{III}}$$

Partie III – conduction thermique

$$\frac{\partial a}{\partial \theta}c^{-1}\frac{\partial}{\partial x}\left(K\frac{\partial \theta}{\partial x}\right) = -\frac{\partial a}{\partial \theta}c^{-1}\frac{\partial}{\partial x}\left(Kc^{-1}\theta^2\frac{\partial}{\partial x}\left(\frac{\partial s}{\partial \theta}\right)\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial a}{\partial \theta}\right)K\theta^2c^{-2}\frac{\partial}{\partial x}\left(\frac{\partial s}{\partial \theta}\right) - \frac{\partial}{\partial x}\left(\frac{\partial a}{\partial \theta}K\theta^2c^{-2}\frac{\partial s}{\partial \theta}\right)$$

Terme qui, après intégration de la densité sur le volume, est négligé dans les modèles de la littérature[1]

$$\begin{bmatrix} \frac{\partial a}{\partial u}\dot{u} \\ \frac{\partial a}{\partial p}\dot{p} \\ \frac{\partial a}{\partial \theta}\dot{\theta} \\ \frac{\partial a}{\partial \varepsilon_{\mathrm{i}}}\dot{\varepsilon}_{\mathrm{i}} \end{bmatrix} = \begin{bmatrix} \frac{\partial a}{\partial u} \\ \frac{\partial a}{\partial p} \\ \frac{\partial a}{\partial \theta} \\ \frac{\partial a}{\partial \varepsilon_{\mathrm{i}}} \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & ? & 0 \\ 0 & \theta c^{-1}\frac{\partial s}{\partial \varepsilon}\frac{\partial}{\partial x}(\circ) & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial l^\star}{\partial u} \\ \frac{\partial l^\star}{\partial p} \\ \frac{\partial l^\star}{\partial \theta} \\ \frac{\partial l^\star}{\partial \varepsilon_{\mathrm{i}}} \end{bmatrix} + \begin{bmatrix} \frac{\partial a}{\partial u} \\ \frac{\partial a}{\partial p} \\ \frac{\partial a}{\partial \theta} \\ \frac{\partial a}{\partial \varepsilon_{\mathrm{i}}} \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & c^{-1}\rho V^{-1}\theta\frac{\partial e}{\partial \varepsilon_{\mathrm{i}}}\frac{\partial e}{\partial \varepsilon_{\mathrm{i}}}c^{-1} + \frac{\partial}{\partial x}(\circ)\left(K\theta^2c^{-2}\right)\frac{\partial}{\partial x}(\circ) & -c^{-1}\rho V^{-1}\theta\frac{\partial e}{\partial \varepsilon_{\mathrm{i}}} \\ 0 & 0 & -\rho V^{-1}\frac{\partial e}{\partial \varepsilon_{\mathrm{i}}}\theta c^{-1} & \theta\rho V^{-1} \end{bmatrix} \begin{bmatrix} \frac{\partial s}{\partial u} \\ \frac{\partial s}{\partial p} \\ \frac{\partial s}{\partial \theta} \\ \frac{\partial s}{\partial \varepsilon_{\mathrm{i}}} \end{bmatrix}$$

1. Hütter, Markus, and Bob Svendsen. ‘Thermodynamic Model Formulation for Viscoplastic Solids as General Equations for Non-Equilibrium Reversible–Irreversible Coupling’. *Continuum Mechanics and Thermodynamics* 24, no. 3 (May 2012): 211–27.

L’exemple de la thermo-visco-élasticité en HPP

Vecteur d’état $\underline{z} = (u, p, \theta, \varepsilon_\mathrm{i})$

Comportement réversible | Comportement irréversible

$$a(\underline{z}) = a(u, p, \theta, \varepsilon_\mathrm{i})$$

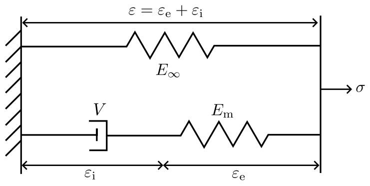
Calcul de la dérivée temporelle

$$\frac{da}{dt} = \frac{\partial a}{\partial u}\dot{u} + \frac{\partial a}{\partial p}\dot{p} + \frac{\partial a}{\partial \theta}\dot{\theta} + \frac{\partial a}{\partial \varepsilon_\mathrm{i}}\dot{\varepsilon}_\mathrm{i}$$

Lois d’évolution réversibles

Lois d’évolution irréversibles

Problème considéré

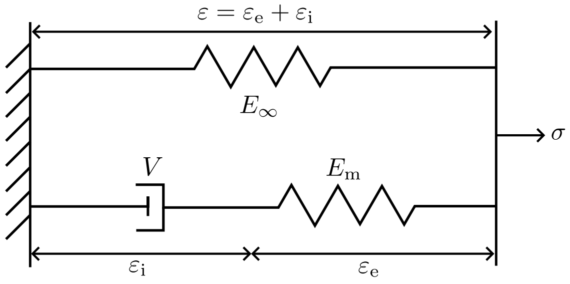


Émergence d’une structure géométrique

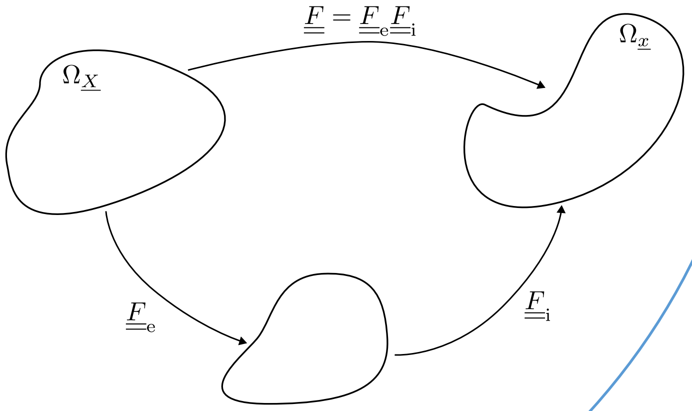
$$\begin{bmatrix} \frac{\partial a}{\partial u}\dot{u} \\ \frac{\partial a}{\partial p}\dot{p} \\ \frac{\partial a}{\partial \theta}\dot{\theta} \\ \frac{\partial a}{\partial \varepsilon_\mathrm{i}}\dot{\varepsilon}_\mathrm{i} \end{bmatrix} = \begin{bmatrix} \frac{\partial a}{\partial u} \\ \frac{\partial a}{\partial p} \\ \frac{\partial a}{\partial \theta} \\ \frac{\partial a}{\partial \varepsilon_\mathrm{i}} \end{bmatrix}^\mathsf{T} \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & ? & 0 \\ 0 & \theta c^{-1} \frac{\partial s}{\partial \varepsilon} \frac{\partial}{\partial x}(\circ) & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial l^\star}{\partial u} \\ \frac{\partial l^\star}{\partial p} \\ \frac{\partial l^\star}{\partial \theta} \\ \frac{\partial l^\star}{\partial \varepsilon_\mathrm{i}} \end{bmatrix} + \begin{bmatrix} \frac{\partial a}{\partial u} \\ \frac{\partial a}{\partial p} \\ \frac{\partial a}{\partial \theta} \\ \frac{\partial a}{\partial \varepsilon_\mathrm{i}} \end{bmatrix}^\mathsf{T} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & c^{-1}\rho V^{-1}\theta \frac{\partial e}{\partial \varepsilon_\mathrm{i}} \frac{\partial e}{\partial \varepsilon_\mathrm{i}} c^{-1} + \frac{\partial}{\partial x}(\circ)\left(K\theta^2 c^{-2}\right)\frac{\partial}{\partial x}(\circ) & -c^{-1}\rho V^{-1}\theta \frac{\partial e}{\partial \varepsilon_\mathrm{i}} \\ 0 & 0 & -\rho V^{-1}\frac{\partial e}{\partial \varepsilon_\mathrm{i}}\theta c^{-1} & \theta\rho V^{-1} \end{bmatrix} \begin{bmatrix} \frac{\partial s}{\partial u} \\ \frac{\partial s}{\partial p} \\ \frac{\partial s}{\partial \theta} \\ \frac{\partial s}{\partial \varepsilon_\mathrm{i}} \end{bmatrix}$$

Thermoviscoélasticité

Petites transformations
Unidimensionnel

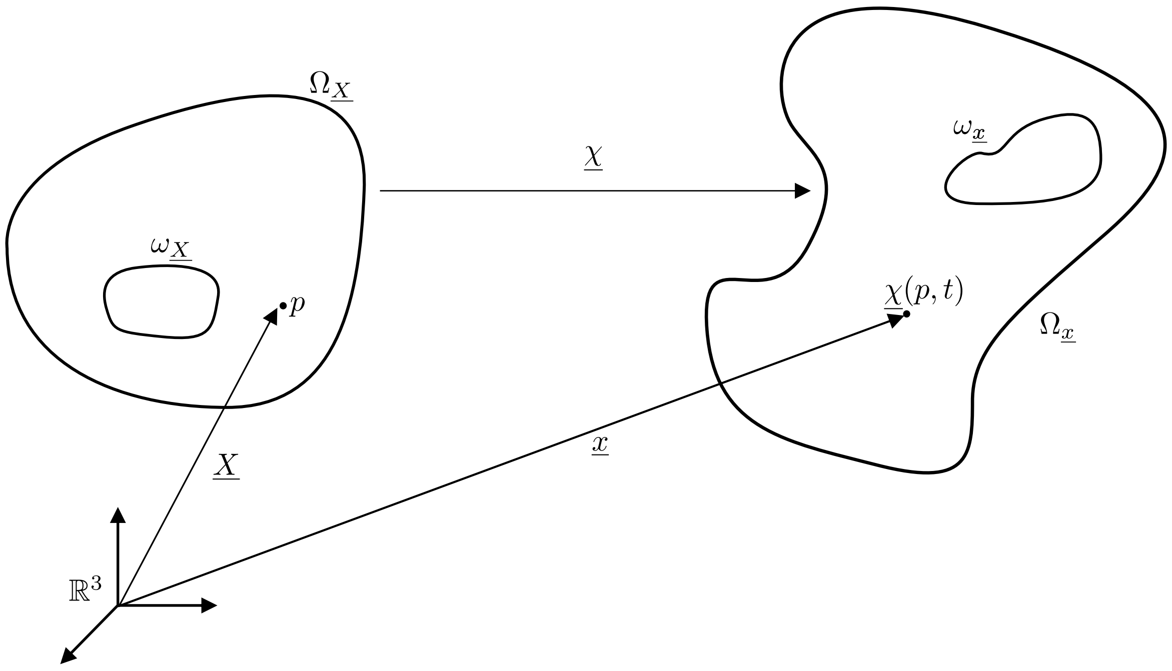


Grandes transformations
Tridimensionnel



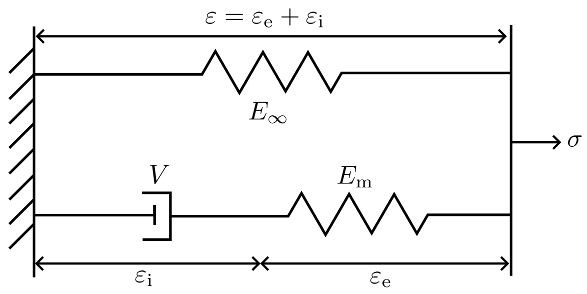
L'exemple de la thermo-visco-élasticité en grandes transformations
Probléme considéré

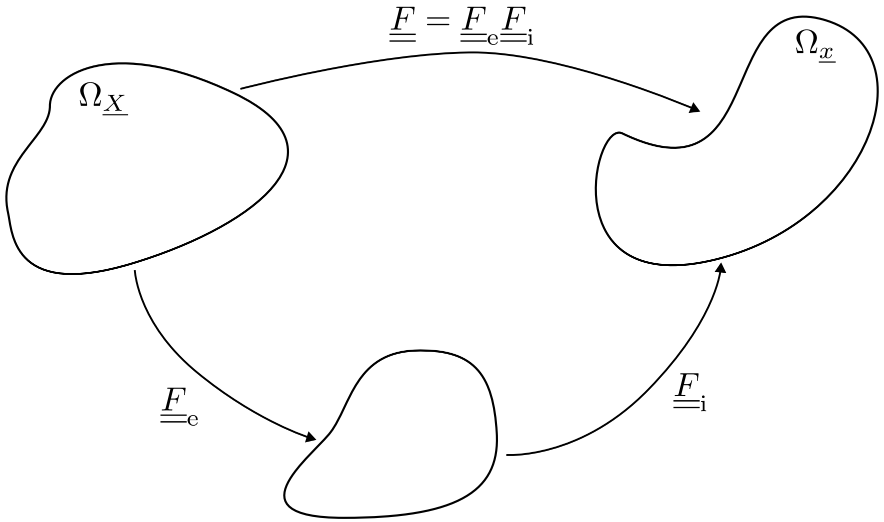
Milieu continu 3D



Comportement viscoélastique en grandes transformations

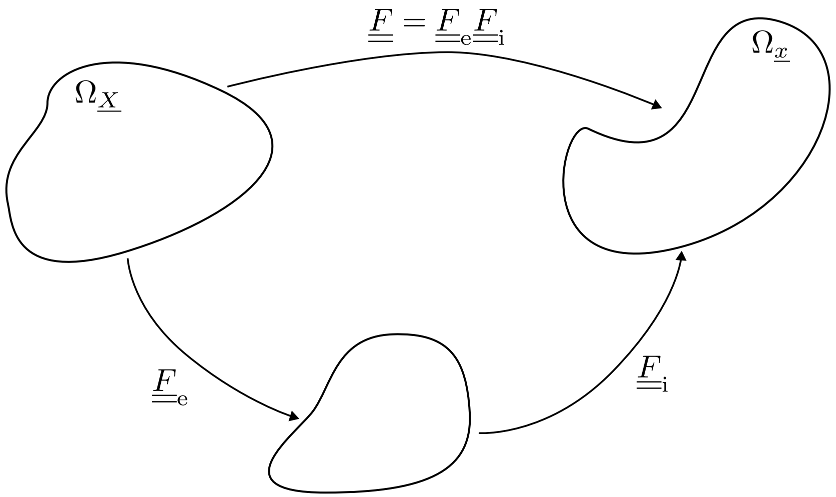
Décomposition équivalente au modèle HPP





L'exemple de la thermo-visco-élasticité en grandes transformations

Problème considéré



Vecteur d'état

$$\underline{z} = (\underline{\chi}, \underline{p}, \underline{\underline{\Pi}}, \Theta, \underline{\underline{C}}_{\mathrm{i}}^{-1})$$

Comportement réversible | Comportement irréversible

$$a(\underline{z}) = a(\underline{\chi}, \underline{p}, \underline{\underline{\Pi}}, \Theta, \underline{\underline{C}}_{\mathrm{i}}^{-1})$$

Calcul de la dérivée temporelle

$$\frac{da}{dt} = \frac{\partial a}{\partial \underline{\chi}} \cdot \dot{\underline{\chi}} + \frac{\partial a}{\partial \underline{p}} \cdot \dot{\underline{p}} + \frac{\partial a}{\partial \underline{\underline{\Pi}}} : \dot{\underline{\underline{\Pi}}} + \frac{\partial a}{\partial \Theta} \dot{\Theta} + \frac{\partial a}{\partial \underline{\underline{C}}_{\mathrm{i}}^{-1}} : \dot{\overline{\underline{\underline{C}}_{\mathrm{i}}^{-1}}}$$

Lois d'évolution réversibles | Lois d'évolution irréversibles

Émergence d'une structure géométrique

$$\frac{da}{dt} = \square + \square$$

L’exemple de la thermo-visco-élasticité en grandes transformations

Partie réversible – crochet de Poisson multisymplectique

$$\frac{da}{dt} = \frac{\partial a}{\partial \underline{\chi}} \cdot \dot{\underline{\chi}} + \frac{\partial a}{\partial \underline{p}} \cdot \dot{\underline{p}} + \frac{\partial a}{\partial \underline{\underline{\Pi}}} : \dot{\underline{\underline{\Pi}}} + \frac{\partial a}{\partial \Theta}\dot{\Theta} + \frac{\partial a}{\partial \underline{\underline{C}}_{\mathrm{i}}^{-1}} : \dot{\overline{\underline{\underline{C}}_{\mathrm{i}}^{-1}}}$$

- Définition d’un **Hamiltonien**, obtenu par deux dualisations partielles du Lagrangien

$$l^\star(\underline{\chi}, \underline{p}, \underline{\underline{\Pi}}) = \underline{p} \cdot \frac{\partial \underline{\chi}}{\partial t} - \underline{\underline{\Pi}} : \frac{\partial \underline{\chi}}{\partial \underline{X}} - l\left(\underline{\chi}, \frac{\partial \underline{\chi}}{\partial t}, \frac{\partial \underline{\chi}}{\partial \underline{X}}\right), \; \underline{p} = \frac{\partial l}{\partial\left(\frac{\partial \underline{\chi}}{\partial t}\right)}, \; \underline{\underline{\Pi}} = -\frac{\partial l}{\partial\left(\frac{\partial \underline{\chi}}{\partial \underline{X}}\right)}$$

$\underline{p}$: Quantité de mouvement

$\underline{\underline{\Pi}}$: Tenseur de contraintes de Piola Kirchhoff 1

- Définition d’une **action** duale $\mathcal{A}^\star\left[\underline{\chi}, \underline{p}, \underline{\underline{\Pi}}\right] = \int_{\omega_t}\int_{\omega_{\underline{X}}} \left\{\underline{p} \cdot \frac{\partial \underline{\chi}}{\partial t} - \underline{\underline{\Pi}} : \frac{\partial \underline{\chi}}{\partial \underline{X}} - l^\star(\underline{\chi}, \underline{p}, \underline{\underline{\Pi}})\right\} dV \; dt$

- Application du **principe d’Hamilton** $\delta\mathcal{A}^\star[\underline{\chi}, \underline{p}, \underline{\underline{\Pi}}] = 0 \; \forall \delta\underline{\chi} = \underline{0} \text{ on } \partial\omega_t \times \partial\omega_{\underline{X}}, \delta\underline{p}, \delta\underline{\underline{\Pi}}$

$$\begin{pmatrix} \frac{\partial l^\star}{\partial \underline{\chi}} \\ \frac{\partial l^\star}{\partial \underline{p}} \\ \frac{\partial l^\star}{\partial \underline{\underline{\Pi}}} \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \frac{\partial}{\partial t} \begin{pmatrix} \underline{\chi} \\ \underline{p} \\ \underline{\underline{\Pi}} \end{pmatrix} + \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \frac{\partial}{\partial \underline{X}} \begin{pmatrix} \underline{\chi} \\ \underline{p} \\ \underline{\underline{\Pi}} \end{pmatrix}$$

- Pour un Lagrangien tel que $l\left(\underline{\chi}, \frac{\partial \underline{\chi}}{\partial t}, \frac{\partial \underline{\chi}}{\partial \underline{X}}\right) = \frac{1}{2}\rho \frac{\partial \underline{\chi}}{\partial t} \cdot \frac{\partial \underline{\chi}}{\partial t} - \Psi\left(\frac{\partial \underline{\chi}}{\partial \underline{X}}\right) + \underline{F}_{\mathrm{d}} \cdot \underline{\chi}$

  on retrouve l’équation matérielle de conservation de la quantité de mouvement $\frac{d\underline{p}}{dt} = \frac{\partial}{\partial \underline{X}} \cdot \underline{\underline{\Pi}} + \underline{F}_{\mathrm{d}}$

L'exemple de la thermo-visco-élasticité en grandes transformations

Partie réversible – crochet de Poisson multisymplectique

$$\frac{da}{dt} = \frac{\partial a}{\partial \underline{\chi}} \cdot \dot{\underline{\chi}} + \frac{\partial a}{\partial \underline{p}} \cdot \dot{\underline{p}} + \frac{\partial a}{\partial \underline{\underline{\Pi}}} : \dot{\underline{\underline{\Pi}}} + \frac{\partial a}{\partial \Theta} \dot{\Theta} + \frac{\partial a}{\partial \underline{\underline{C}}_{\mathrm{i}}^{-1}} : \dot{\overline{\underline{\underline{C}}_{\mathrm{i}}^{-1}}}$$

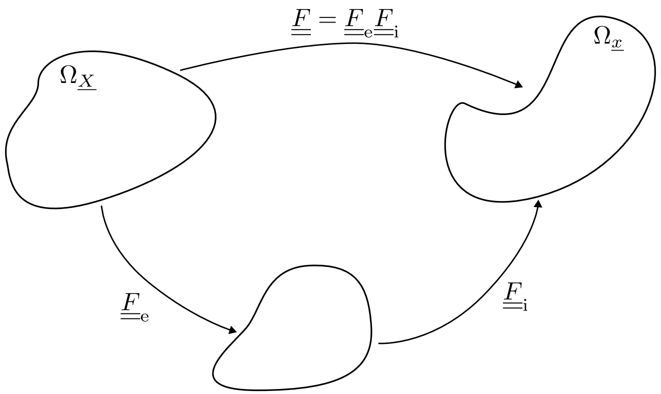
Définition d'un **crochet de Poisson multisymplectique**[1,2]

$$\begin{aligned}
\frac{d}{dt} a(\underline{\chi}, \underline{p}, \underline{\underline{\Pi}}) &= \frac{\partial a}{\partial \underline{\chi}} \frac{\partial l^\star}{\partial \underline{p}} + \frac{\partial a}{\partial \underline{p}} \left(-\frac{\partial l^\star}{\partial \underline{\chi}} + \frac{\partial \underline{\underline{\Pi}}}{\partial \underline{X}} \right) + \frac{\partial a}{\partial \underline{\underline{\Pi}}} : \dot{\underline{\underline{\Pi}}} \\
&= \frac{\partial a}{\partial \underline{\chi}} \frac{\partial l^\star}{\partial \underline{p}} - \frac{\partial a}{\partial \underline{p}} \frac{\partial l^\star}{\partial \underline{\chi}} + \frac{\partial a}{\partial \underline{p}} \frac{\partial \underline{\underline{\Pi}}}{\partial \underline{X}} + \frac{\partial a}{\partial \underline{\underline{\Pi}}} : \left(\frac{\partial \underline{\underline{\Pi}}}{\partial \underline{X}} \left(\frac{\partial \underline{\chi}}{\partial \underline{X}} \right)^{-1} \frac{\partial \underline{\chi}}{\partial t} \right) \\
&= \frac{\partial a}{\partial \underline{\chi}} \frac{\partial l^\star}{\partial \underline{p}} - \frac{\partial a}{\partial \underline{p}} \frac{\partial l^\star}{\partial \underline{\chi}} + \frac{\partial \underline{\underline{\Pi}}}{\partial \underline{X}} \left(\frac{\partial \underline{\chi}}{\partial \underline{X}} \right)^{-1} \left(\frac{\partial a}{\partial \underline{\underline{\Pi}}} \frac{\partial l^\star}{\partial \underline{p}} - \frac{\partial a}{\partial \underline{p}} \frac{\partial l^\star}{\partial \underline{\underline{\Pi}}} \right) \\
&:= \{a, l^\star\} \\
&= \begin{bmatrix} \frac{\partial a}{\partial \underline{\chi}} \\ \frac{\partial a}{\partial \underline{p}} \\ \frac{\partial a}{\partial \underline{\underline{\Pi}}} \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & \frac{\partial \underline{\underline{\Pi}}}{\partial \underline{X}} \left(\frac{\partial \underline{\chi}}{\partial \underline{X}} \right)^{-1} \\ 0 & -\frac{\partial \underline{\underline{\Pi}}}{\partial \underline{X}} \left(\frac{\partial \underline{\chi}}{\partial \underline{X}} \right)^{-1} & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial l^\star}{\partial \underline{\chi}} \\ \frac{\partial l^\star}{\partial \underline{p}} \\ \frac{\partial l^\star}{\partial \underline{\underline{\Pi}}} \end{bmatrix}
\end{aligned}$$

1. Marsden, Jerrold E., George W. Patrick, and Steve Shkoller. 'Multisymplectic Geometry, Variational Integrators, and Nonlinear PDEs'. *Communications in Mathematical Physics* 199, no. 2 (1 December 1998): 351–95.
2. Gay-Balmaz, François, Juan C. Marrero, and Nicolás Martínez Alba. 'A New Canonical Affine Bracket Formulation of Hamiltonian Classical Field Theories of First Order'. *Revista de La Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas* 118, no. 3 (July 2024): 103.

L'exemple de la thermo-visco-élasticité en grandes transformations

Problème considéré



Vecteur d'état

$$\underline{z} = (\underline{\chi}, \underline{p}, \underline{\underline{\Pi}}, \Theta, \underline{\underline{C}}_{\mathrm{i}}^{-1})$$

Comportement réversible | Comportement irréversible

$$a(\underline{z}) = a(\underline{\chi}, \underline{p}, \underline{\underline{\Pi}}, \Theta, \underline{\underline{C}}_{\mathrm{i}}^{-1})$$

Calcul de la dérivée temporelle

$$\frac{da}{dt} = \frac{\partial a}{\partial \underline{\chi}} \cdot \dot{\underline{\chi}} + \frac{\partial a}{\partial \underline{p}} \cdot \dot{\underline{p}} + \frac{\partial a}{\partial \underline{\underline{\Pi}}} : \dot{\underline{\underline{\Pi}}} + \frac{\partial a}{\partial \Theta} \dot{\Theta} + \frac{\partial a}{\partial \underline{\underline{C}}_{\mathrm{i}}^{-1}} : \overline{\underline{\underline{C}}_{\mathrm{i}}^{-1}}^{\cdot}$$

Lois d'évolution réversibles | Lois d'évolution irréversibles

Émergence d'une structure géométrique

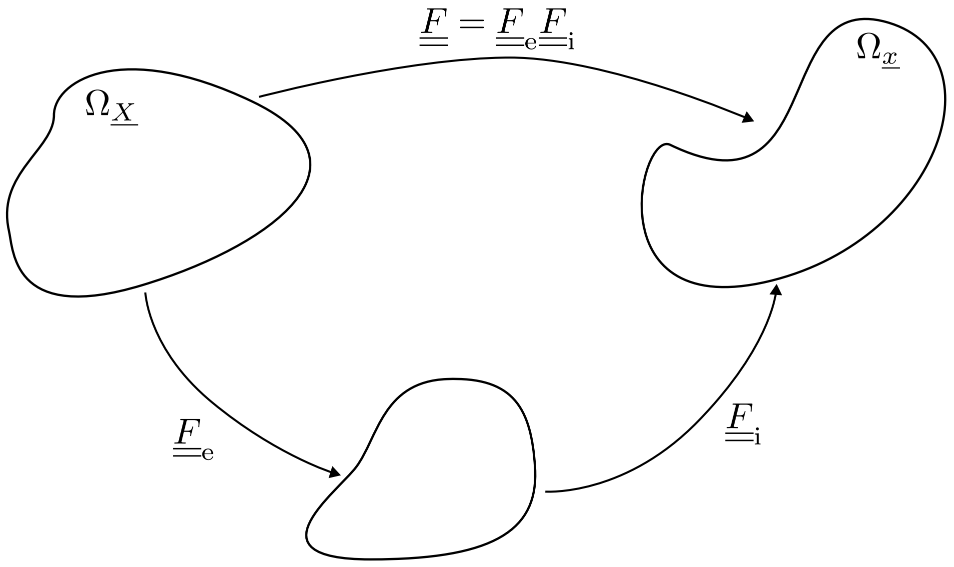
$$\frac{da}{dt} = \{a, l^{\star}\} + \square$$

L’exemple de la thermo-visco-élasticité en grandes transformations

Partie irréversible – dissipation due aux effets visqueux

$$\frac{da}{dt} = \frac{\partial a}{\partial \underline{\chi}} \cdot \dot{\underline{\chi}} + \frac{\partial a}{\partial \underline{p}} \cdot \dot{\underline{p}} + \frac{\partial a}{\partial \underline{\underline{\Pi}}} : \dot{\underline{\underline{\Pi}}} + \frac{\partial a}{\partial \Theta}\dot{\Theta} + \frac{\partial a}{\partial \underline{\underline{C}}_{\mathrm{i}}^{-1}} : \dot{\overline{\underline{\underline{C}}_{\mathrm{i}}^{-1}}}$$

Décomposition de Sidoroff[1]



$\underline{\underline{F}}_{\mathrm{e}}$: Transformation réversible

$\underline{\underline{F}}_{\mathrm{i}}$: Transformation visqueuse inélastique

Définition d’un potentiel thermodynamique[2]

$$\Psi = \Psi(\underline{\underline{C}}, \underline{\underline{Q}}_1, \ldots, \underline{\underline{Q}}_k, \Theta)$$

vérifiant la seconde loi de la Thermodynamique

$$\begin{aligned}\mathcal{D}_{\mathrm{mech}} &= \underline{\underline{S}} : \dot{\underline{\underline{E}}} - \rho_{\mathrm{ref}}(\dot{\Theta}S + \dot{\Psi}) \geq 0 \\ &= \left(\underline{\underline{S}} - 2\rho_{\mathrm{ref}}\frac{\partial \Psi}{\partial \underline{\underline{C}}}\right) : \dot{\underline{\underline{E}}} + \left(-\rho_{\mathrm{ref}}S - \rho_{\mathrm{ref}}\frac{\partial \Psi}{\partial \Theta}\right) : \dot{\Theta} - \rho_{\mathrm{ref}}\sum_{i=1}^{k}\frac{\partial \Psi}{\partial \underline{\underline{Q}}_i} : \dot{\underline{\underline{Q}}}_i \geq 0\end{aligned}$$

$$S = -\frac{\partial \Psi}{\partial \Theta}, \underline{\underline{S}} = 2\rho_{\mathrm{ref}}\frac{\partial \Psi}{\partial \underline{\underline{C}}}$$

S : Entropie

$\underline{\underline{S}}$: Tenseur de contraintes de Piola Kirchhoff 2

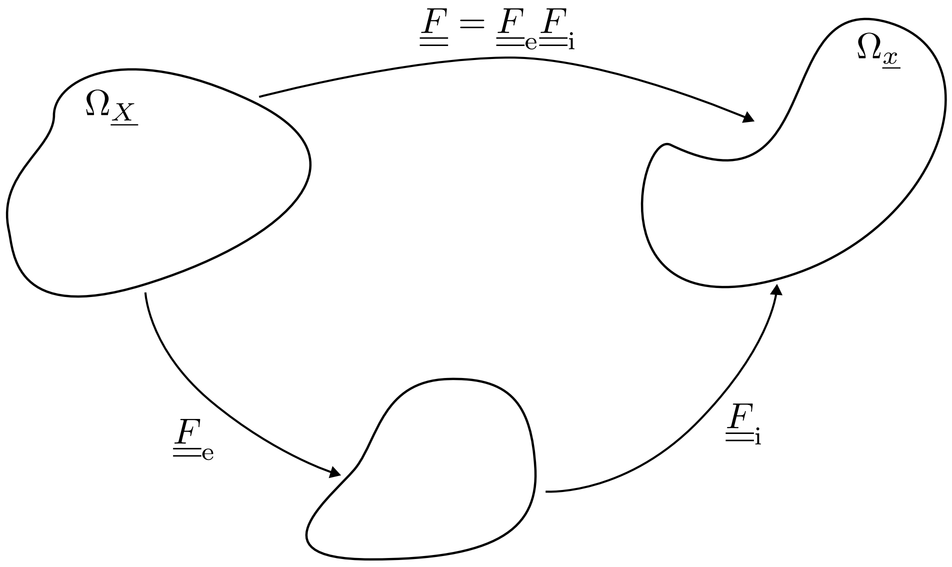
1. Sidoroff, F. ‘Un Modèle Viscoélastique Non Linéaire Avec Configuration Intermédiaire.’ Journal de Mécanique 13 (1974): 679–713.
2. Coleman, Bernard D., and Morton E. Gurtin. ‘Thermodynamics with Internal State Variables’. The Journal of Chemical Physics 47, no. 2 (15 July 1967): 597–613.

L'exemple de la thermo-visco-élasticité en grandes transformations

Partie irréversible – dissipation due aux effets visqueux

$$\frac{da}{dt} = \frac{\partial a}{\partial \underline{\chi}} \cdot \dot{\underline{\chi}} + \frac{\partial a}{\partial \underline{p}} \cdot \dot{\underline{p}} + \frac{\partial a}{\partial \underline{\underline{\Pi}}} : \dot{\underline{\underline{\Pi}}} + \frac{\partial a}{\partial \Theta}\dot{\Theta} + \frac{\partial a}{\partial \underline{\underline{C}}_{\mathrm{i}}^{-1}} : \dot{\overline{\underline{\underline{C}}_{\mathrm{i}}^{-1}}}$$

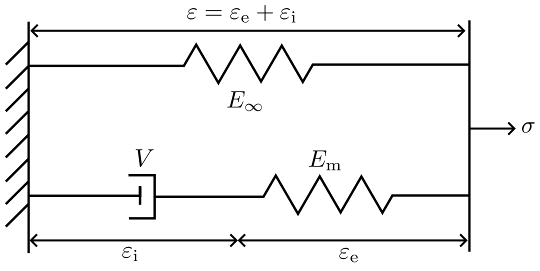
Décomposition de Sidoroff



$\underline{\underline{F}}_{\mathrm{e}}$: Transformation réversible

$\underline{\underline{F}}_{\mathrm{i}}$: Transformation visqueuse inélastique

À l'image du problème en HPP, on decompose le potentiel thermodynamique en deux parties[1]



$$\Psi = \Psi_{\mathrm{EQ}}(\underline{\underline{C}}) + \Psi_{\mathrm{NEQ}}(\underline{\underline{C}}_{\mathrm{e}})$$

Par cette analyse, on a $\underline{\underline{Q}}_i = \underline{\underline{F}}_{\mathrm{i}}$

Cette forme de potentiel dans la seconde loi de la thermodynamique mène à l'équation suivante

$$-\frac{\partial \Psi}{\partial \underline{\underline{C}}_{\mathrm{e}}} : \frac{\partial \underline{\underline{C}}_{\mathrm{e}}}{\partial \underline{\underline{F}}_{\mathrm{i}}} : \dot{\underline{\underline{F}}}_{\mathrm{i}} \geq 0$$

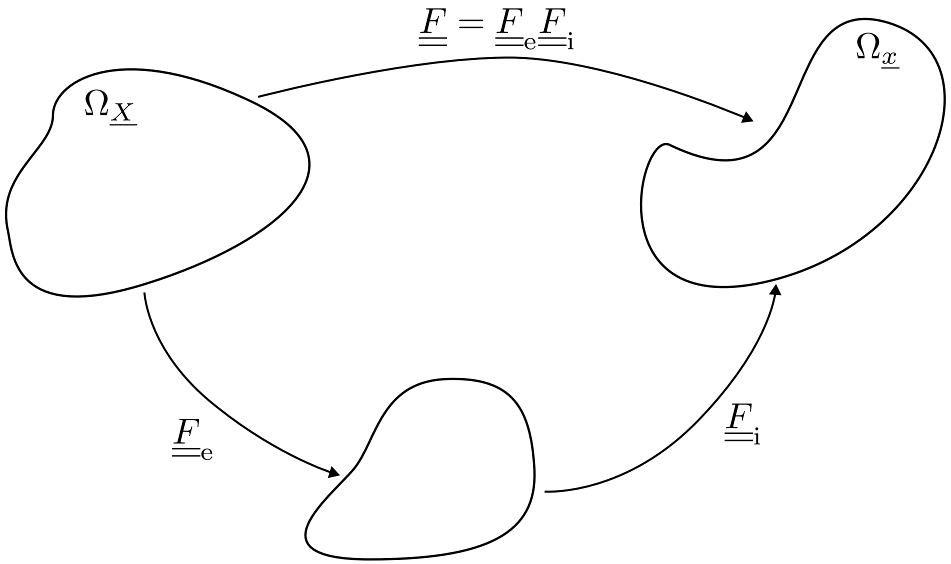
1. Reese, Stefanie, and Sanjay Govindjee. 'A Theory of Finite Viscoelasticity and Numerical Aspects'. *International Journal of Solids and Structures* 35, no. 26–27 (September 1998): 3455–82.

L'exemple de la thermo-visco-élasticité en grandes transformations

Partie irréversible – dissipation due aux effets visqueux

$$\frac{da}{dt} = \frac{\partial a}{\partial \underline{\chi}} \cdot \dot{\underline{\chi}} + \frac{\partial a}{\partial \underline{p}} \cdot \dot{\underline{p}} + \frac{\partial a}{\partial \underline{\underline{\Pi}}} : \dot{\underline{\underline{\Pi}}} + \frac{\partial a}{\partial \Theta} \dot{\Theta} + \frac{\partial a}{\partial \underline{\underline{C}}_{\mathrm{i}}^{-1}} : \dot{\overline{\underline{\underline{C}}_{\mathrm{i}}^{-1}}}$$

Décomposition de Sidoroff



$\underline{\underline{F}}_{\mathrm{e}}$: Transformation réversible

$\underline{\underline{F}}_{\mathrm{i}}$: Transformation visqueuse inélastique

Cette forme de potentiel dans la seconde loi de la thermodynamique mène à l'équation suivante

$$-\frac{\partial \Psi}{\partial \underline{\underline{C}}_{\mathrm{e}}} : \frac{\partial \underline{\underline{C}}_{\mathrm{e}}}{\partial \underline{\underline{F}}_{\mathrm{i}}} : \dot{\underline{\underline{F}}}_{\mathrm{i}} \geq 0$$

On peut montrer que cette équation mène à la loi de comportement suivante

$$\frac{1}{2} \mathcal{L}_{\mathrm{v}}(\underline{\underline{b}}_{\mathrm{e}}) \underline{\underline{b}}_{\mathrm{e}}^{-1} = -\underline{\underline{\underline{\underline{V}}}}^{-1} : \underline{\underline{\tau}}_{\mathrm{NEQ}}$$

$\underline{\underline{b}}_{\mathrm{e}} = \underline{\underline{F}}_{\mathrm{e}} \underline{\underline{F}}_{\mathrm{e}}^{\mathsf{T}}$: Tenseur de Cauchy Green droit

$\mathcal{L}_{\mathrm{v}}(\underline{\underline{b}}_{\mathrm{e}})$: Dérivée de Lie du tenseur de Cauchy Green droit

$\underline{\underline{\underline{\underline{V}}}}^{-1}$: Tenseur inélastique d'ordre 4

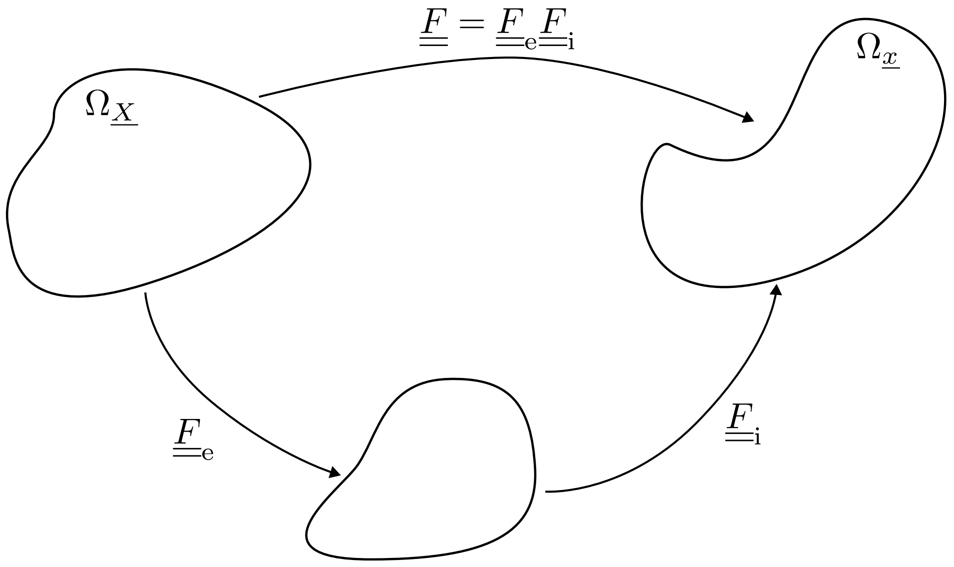
$\underline{\underline{\tau}}_{\mathrm{NEQ}}$: Tenseur de contraintes de Kirchhoff, partie hors équilibre

L’exemple de la thermo-visco-élasticité en grandes transformations

Partie irréversible – dissipation due aux effets visqueux

$$\frac{da}{dt} = \frac{\partial a}{\partial \underline{\chi}} \cdot \dot{\underline{\chi}} + \frac{\partial a}{\partial \underline{p}} \cdot \dot{\underline{p}} + \frac{\partial a}{\partial \underline{\underline{\Pi}}} : \dot{\underline{\underline{\Pi}}} + \frac{\partial a}{\partial \Theta}\dot{\Theta} + \frac{\partial a}{\partial \underline{\underline{C}}_{\mathrm{i}}^{-1}} : \dot{\overline{\underline{\underline{C}}_{\mathrm{i}}^{-1}}}$$

Décomposition de Sidoroff



$\underline{\underline{F}}_{\mathrm{e}}$: Transformation réversible

$\underline{\underline{F}}_{\mathrm{i}}$: Transformation visqueuse inélastique

Cette forme de potentiel dans la seconde loi de la thermodynamique mène à l’équation suivante

$$-\frac{\partial \Psi}{\partial \underline{\underline{C}}_{\mathrm{e}}} : \frac{\partial \underline{\underline{C}}_{\mathrm{e}}}{\partial \underline{\underline{F}}_{\mathrm{i}}} : \dot{\underline{\underline{F}}}_{\mathrm{i}} \geq 0$$

On peut montrer que cette équation mène à la loi de comportement suivante[1,2]

$$\frac{1}{2}\mathcal{L}_{\mathrm{v}}(\underline{\underline{b}}_{\mathrm{e}})\underline{\underline{b}}_{\mathrm{e}}^{-1} = -\underline{\underline{\underline{\underline{V}}}}^{-1} : \underline{\underline{\tau}}_{\mathrm{NEQ}}$$

$$\Leftrightarrow \dot{\overline{\underline{\underline{C}}_{\mathrm{i}}^{-1}}} = -4\underline{\underline{F}}^{-1}\left(\underline{\underline{\underline{\underline{V}}}}^{-1} : \underline{\underline{F}}^{-\mathsf{T}}\underline{\underline{C}}\frac{\partial \Psi_{\mathrm{NEQ}}}{\partial \underline{\underline{C}}}\underline{\underline{F}}^{\mathsf{T}}\right)\underline{\underline{F}}\,\underline{\underline{C}}_{\mathrm{i}}^{-1}$$

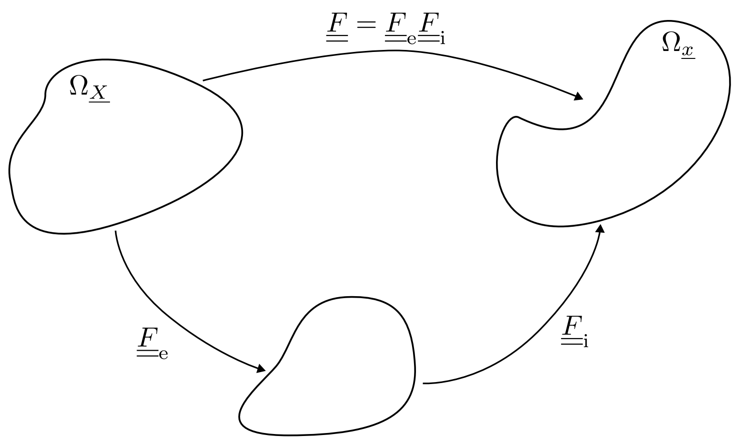
$\underline{\underline{C}}_{\mathrm{i}} = \underline{\underline{F}}_{\mathrm{i}}^{\mathsf{T}}\underline{\underline{F}}_{\mathrm{i}}$: Tenseur de Cauchy Green inélastique

1. Reese, Stefanie, and Sanjay Govindjee. ‘A Theory of Finite Viscoelasticity and Numerical Aspects’. *International Journal of Solids and Structures* 35, no. 26–27 (September 1998): 3455–82.
2. Betsch, Peter, and Mark Schiebl. ‘GENERIC-Based Formulation and Discretization of Initial Boundary Value Problems for Finite Strain Thermoelasticity’. *Computational Mechanics* 65, no. 2 (February 2020): 503–31.

L'exemple de la thermo-visco-élasticité en grandes transformations

Partie irréversible – dissipation due aux effets visqueux

$$\frac{da}{dt} = \frac{\partial a}{\partial \underline{\chi}} \cdot \underline{\dot{\chi}} + \frac{\partial a}{\partial \underline{p}} \cdot \underline{\dot{p}} + \frac{\partial a}{\partial \underline{\underline{\Pi}}} : \underline{\underline{\dot{\Pi}}} + \frac{\partial a}{\partial \Theta}\dot{\Theta} + \frac{\partial a}{\partial \underline{\underline{C}}_{\mathrm{i}}^{-1}} : \dot{\overline{\underline{\underline{C}}_{\mathrm{i}}^{-1}}}$$



À partir de la loi de comportement précédente, on peut montrer que le produit contracté de la dérivée de la fonction d'état choisie par l'inverse du tenseur de Cauchy-Green inélastique donne

$$\begin{aligned}
\frac{\partial a}{\partial \underline{\underline{C}}_{\mathrm{i}}^{-1}} : \dot{\overline{\underline{\underline{C}}_{\mathrm{i}}^{-1}}} = & -4\left[\frac{\partial a}{\partial \underline{\underline{C}}_{\mathrm{i}}^{-1}}\underline{\underline{C}}_{\mathrm{i}}^{-1}\right] : \left[\underline{\underline{F}}^{-1}\left(\underline{\underline{F}}^{-1}\underline{\underline{\underline{\underline{V}}}}^{-1}\underline{\underline{F}}\right)^{\mathsf{T}}\underline{\underline{F}}\right] : \left[\frac{\partial \Psi_{\mathrm{NEQ}}}{\partial \underline{\underline{C}}_{\mathrm{i}}^{-1}}\underline{\underline{C}}_{\mathrm{i}}^{-1}\right] \\
&= 4\left[\frac{\partial a}{\partial \underline{\underline{C}}_{\mathrm{i}}^{-1}}\underline{\underline{C}}_{\mathrm{i}}^{-1}\right] : \Theta\left[\underline{\underline{F}}^{-1}\left(\underline{\underline{F}}^{-1}\underline{\underline{\underline{\underline{V}}}}^{-1}\underline{\underline{F}}\right)^{\mathsf{T}}\underline{\underline{F}}\right] : \left[\frac{\partial S}{\partial \underline{\underline{C}}_{\mathrm{i}}^{-1}}\underline{\underline{C}}_{\mathrm{i}}^{-1}\right] + \\
&\quad - 4\left[\frac{\partial a}{\partial \underline{\underline{C}}_{\mathrm{i}}^{-1}}\underline{\underline{C}}_{\mathrm{i}}^{-1}\right] : c^{-1}\Theta\left[\underline{\underline{F}}^{-1}\left(\underline{\underline{F}}^{-1}\underline{\underline{\underline{\underline{V}}}}^{-1}\underline{\underline{F}}\right)^{\mathsf{T}}\underline{\underline{F}}\right] : \left[\frac{\partial E}{\partial \underline{\underline{C}}_{\mathrm{i}}^{-1}}\underline{\underline{C}}_{\mathrm{i}}^{-1}\right]\frac{\partial S}{\partial \Theta} \\
&:= (a, S)_{\mathrm{visqueux}}
\end{aligned}$$

Possibilité de retrouver les résultats obtenus en petites perturbations

$$\dot{\varepsilon}_{\mathrm{i}} = -\rho c^{-1}\theta V^{-1}\frac{\partial e}{\partial \varepsilon_{\mathrm{i}}}\frac{\partial s}{\partial \theta} + \rho\theta V^{-1}\frac{\partial s}{\partial \varepsilon_{\mathrm{i}}}$$

L’exemple de la thermo-visco-élasticité en grandes transformations

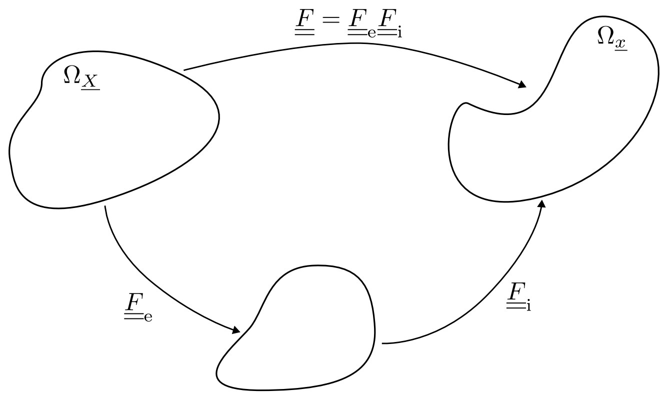
Partie irréversible – dissipation thermique

$$\frac{da}{dt} = \frac{\partial a}{\partial \underline{\chi}} \cdot \dot{\underline{\chi}} + \frac{\partial a}{\partial \underline{p}} \cdot \dot{\underline{p}} + \frac{\partial a}{\partial \underline{\underline{\Pi}}} : \dot{\underline{\underline{\Pi}}} + \frac{\partial a}{\partial \Theta} \dot{\Theta} + \frac{\partial a}{\partial \underline{\underline{C}}_{\mathrm{i}}^{-1}} : \dot{\overline{\underline{\underline{C}}_{\mathrm{i}}^{-1}}}$$

Encore en cours de construction !

L'exemple de la thermo-visco-élasticité en grandes transformations

Problème considéré



Vecteur d'état

$$\underline{z} = (\underline{\chi}, \underline{p}, \underline{\underline{\Pi}}, \Theta, \underline{\underline{C}}_{\mathrm{i}}^{-1})$$

Comportement réversible | Comportement irréversible

$$a(\underline{z}) = a(\underline{\chi}, \underline{p}, \underline{\underline{\Pi}}, \Theta, \underline{\underline{C}}_{\mathrm{i}}^{-1})$$

Calcul de la dérivée temporelle

$$\frac{da}{dt} = \frac{\partial a}{\partial \underline{\chi}} \cdot \dot{\underline{\chi}} + \frac{\partial a}{\partial \underline{p}} \cdot \dot{\underline{p}} + \frac{\partial a}{\partial \underline{\underline{\Pi}}} : \dot{\underline{\underline{\Pi}}} + \frac{\partial a}{\partial \Theta}\dot{\Theta} + \frac{\partial a}{\partial \underline{\underline{C}}_{\mathrm{i}}^{-1}} : \dot{\overline{\underline{\underline{C}}_{\mathrm{i}}^{-1}}}$$

Lois d'évolution réversibles | Lois d'évolution irréversibles

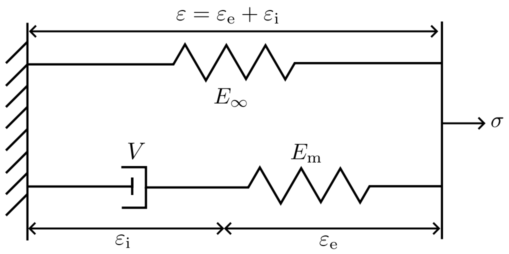
Émergence d'une structure géométrique

$$\frac{da}{dt} = \{a, l^{\star}\} + (a, S)_{\text{visqueux}} + \dots$$

Conclusion

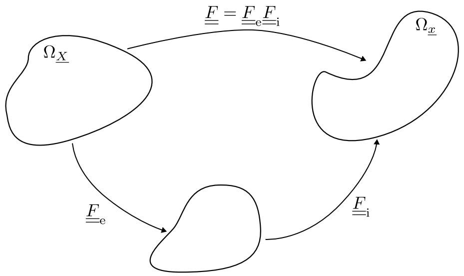
Présentation d'une méthodologie générale pour placer la TPI dans le cadre des structures de Dirac, via deux exemples

- Thermoviscoélasticité unidimensionnelle sous l'hypothèse des petites perturbations



$$\begin{bmatrix} \frac{\partial a}{\partial u}\dot{u} \\ \frac{\partial a}{\partial p}\dot{p} \\ \frac{\partial a}{\partial \theta}\dot{\theta} \\ \frac{\partial a}{\partial \varepsilon_\mathrm{i}}\dot{\varepsilon}_\mathrm{i} \end{bmatrix} = \begin{bmatrix} \frac{\partial a}{\partial u} \\ \frac{\partial a}{\partial p} \\ \frac{\partial a}{\partial \theta} \\ \frac{\partial a}{\partial \varepsilon_\mathrm{i}} \end{bmatrix}^\mathsf{T} \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & ? & 0 \\ 0 & \theta c^{-1}\frac{\partial s}{\partial \varepsilon}\frac{\partial}{\partial x}\left(\frac{\partial \circ}{\partial p}\right) & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial l^\star}{\partial u} \\ \frac{\partial l^\star}{\partial p} \\ \frac{\partial l^\star}{\partial \theta} \\ \frac{\partial l^\star}{\partial \varepsilon_\mathrm{i}} \end{bmatrix} + \begin{bmatrix} \frac{\partial a}{\partial u} \\ \frac{\partial a}{\partial p} \\ \frac{\partial a}{\partial \theta} \\ \frac{\partial a}{\partial \varepsilon_\mathrm{i}} \end{bmatrix}^\mathsf{T} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & c^{-1}\rho V^{-1}\theta\frac{\partial e}{\partial \varepsilon_\mathrm{i}}\frac{\partial e}{\partial \varepsilon_\mathrm{i}}c^{-1} + \frac{\partial}{\partial x}\left(\frac{\partial \circ}{\partial \theta}\right)\left(Kc^{-2}\theta^2\right)\frac{\partial}{\partial x}\left(\frac{\partial \circ}{\partial \theta}\right) & -c^{-1}\rho V^{-1}\theta\frac{\partial e}{\partial \varepsilon_\mathrm{i}} \\ 0 & 0 & -\rho V^{-1}\frac{\partial e}{\partial \varepsilon_\mathrm{i}}\theta c^{-1} & \theta\rho V^{-1} \end{bmatrix} \begin{bmatrix} \frac{\partial s}{\partial u} \\ \frac{\partial s}{\partial p} \\ \frac{\partial s}{\partial \theta} \\ \frac{\partial s}{\partial \varepsilon_\mathrm{i}} \end{bmatrix}$$

- Thermoviscoélasticité tridimensionnelle en grandes transformations



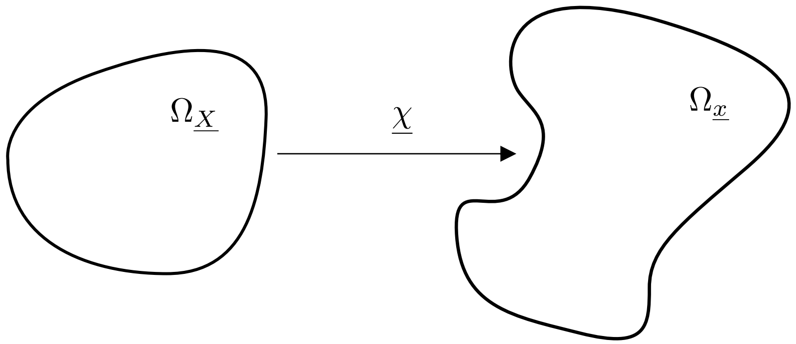
$$\frac{da}{dt} = \{a, l^\star\} + (a, S)_\mathrm{visqueux} + \dots$$

$$\{a, l^\star\} = \begin{bmatrix} \frac{\partial a}{\partial \underline{\chi}} \\ \frac{\partial a}{\partial \underline{p}} \\ \frac{\partial a}{\partial \underline{\underline{\Pi}}} \\ \frac{\partial a}{\partial \underline{\underline{C}}_\mathrm{i}^{-1}} \\ \frac{\partial a}{\partial \Theta} \end{bmatrix}^\mathsf{T} \begin{bmatrix} 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & \frac{\partial \underline{\underline{\Pi}}}{\partial \underline{X}}\left(\frac{\partial \underline{\chi}}{\partial \underline{X}}\right)^{-1} & 0 & 0 \\ 0 & -\frac{\partial \underline{\underline{\Pi}}}{\partial \underline{X}}\left(\frac{\partial \underline{\chi}}{\partial \underline{X}}\right)^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \square & \square & \square & \square & \square \end{bmatrix} \begin{bmatrix} \frac{\partial l^\star}{\partial \underline{\chi}} \\ \frac{\partial l^\star}{\partial \underline{p}} \\ \frac{\partial l^\star}{\partial \underline{\underline{\Pi}}} \\ \frac{\partial l^\star}{\partial \underline{\underline{C}}_\mathrm{i}^{-1}} \\ \frac{\partial l^\star}{\partial \Theta} \end{bmatrix}$$

$$\begin{aligned}(a, S)_\mathrm{visqueux} = 4\left[\frac{\partial a}{\partial \underline{\underline{C}}_\mathrm{i}^{-1}}\underline{\underline{C}}_\mathrm{i}^{-1}\right] : \Theta\left[\underline{\underline{F}}^{-1}\left(\underline{\underline{F}}^{-1}\underline{\underline{\underline{V}}}^{-1}\underline{\underline{F}}\right)^\mathsf{T}\underline{\underline{F}}\right] : \left[\frac{\partial S}{\partial \underline{\underline{C}}_\mathrm{i}^{-1}}\underline{\underline{C}}_\mathrm{i}^{-1}\right] + \\ -4\left[\frac{\partial a}{\partial \underline{\underline{C}}_\mathrm{i}^{-1}}\underline{\underline{C}}_\mathrm{i}^{-1}\right] : c^{-1}\Theta\left[\underline{\underline{F}}^{-1}\left(\underline{\underline{F}}^{-1}\underline{\underline{\underline{V}}}^{-1}\underline{\underline{F}}\right)^\mathsf{T}\underline{\underline{F}}\right] : \left[\frac{\partial E}{\partial \underline{\underline{C}}_\mathrm{i}^{-1}}\underline{\underline{C}}_\mathrm{i}^{-1}\right]\frac{\partial S}{\partial \Theta}\end{aligned}$$

Perspectives

Problème considéré



Vecteur d'état

$$\underline{z} = (\underline{\chi}, \underline{p}, \underline{\underline{\Pi}}, \Theta, \underline{\underline{C}}_{\mathrm{i}}^{-1})$$

Comportement réversible | Comportement irréversible

$$a(\underline{z}) = a(\underline{\chi}, \underline{p}, \underline{\underline{\Pi}}, \Theta, \underline{\underline{C}}_{\mathrm{i}}^{-1})$$

Calcul de la dérivée temporelle

$$\frac{da}{dt} = \frac{\partial a}{\partial \underline{\chi}} \cdot \dot{\underline{\chi}} + \frac{\partial a}{\partial \underline{p}} \cdot \dot{\underline{p}} + \frac{\partial a}{\partial \underline{\underline{\Pi}}} : \dot{\underline{\underline{\Pi}}} + \frac{\partial a}{\partial \Theta} \dot{\Theta} + \frac{\partial a}{\partial \underline{\underline{C}}_{\mathrm{i}}^{-1}} : \dot{\overline{\underline{\underline{C}}_{\mathrm{i}}^{-1}}}$$

Lois d'évolution réversibles | Lois d'évolution irréversibles

Émergence d'une structure géométrique

$$\frac{da}{dt} = \square + \square$$

Vérification des propriétés géométriques continues ?

Discrétisation spatio-temporelle

De la Thermodynamique des Processus Irréversibles aux structures de Dirac

L'exemple de la thermo-visco-élasticité en grandes transformations

Rencontre du GDR-GDM 2024

Benjamin GEORGETTE (benjamin.georgette@insa-lyon.fr)
Sous la direction des Pr. Anthony GRAVOUIL et David DUREISSEIX

LaMCoS
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