Navier-Stokes symplectique et variationnel

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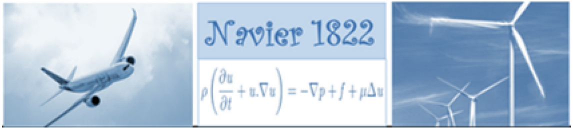

This work is presented to mark the occasion of the 200^{th} birthday of Navier's works that spearheaded the Navier-Stokes equation.

LaMCUBE Laboratoire de mécanique, multiphysique, multiéchelle

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Introduction

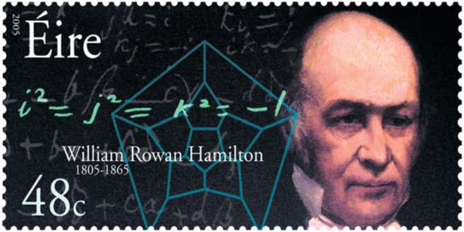
Unified frameworks for dissipative systems :

- **Metriplectic systems** Morrison 1986 and **GENERIC systems** Grmla Öttinger 1997
- **Port-Hamiltonian systems** Brockett 1977, van der Schaft 1984
- **Rate-independent systems** Mielke Theil 1999
- **Hamiltonian inclusions and BEN principle** Buliga 2009, Buliga de Saxcé 2016

Variational formulations of Navier-Stokes equations : (brief) State of the Art

- Pionnering works : Helmholtz 1869, Rayleigh 1913
- Based on Onsager's theory of the production of entropy (1931) :
  Glansdorff and Prigogine 1964, Lebon and Lambermont 1973
- Modification of Hamilton's principle : Fukagawa and Fujitani 2012
  Gay-Balmaz and Yoshimura 2017
- Razafindralandy and Hamdouni 2006 : bi-Lagrangian formalism
- The nearest formalism : anti-selfdual Lagrangians of Ghoussoub and Moameni 2005

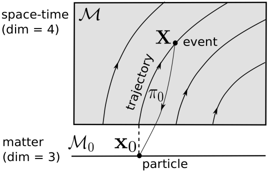
Symplectic formalism



Symplectic formalism : modelling the matter and its motion

Our convention : the intrinsic, coordinate-free objects are denoted by a **bold letter** while their representations in local charts are denoted by a normal letter

- An event $\boldsymbol{X}$ occuring at position x and at time t is represented by $X = \begin{bmatrix} t \\ x \end{bmatrix}$



- The matter and its motion is modelized by a line fiber bundle $\boldsymbol{\pi}_0 : \mathcal{M} \rightarrow \mathcal{M}_0$
- The fibers are the particle trajectories
- In local charts :
  - A material particle $\boldsymbol{x}_0 = \boldsymbol{\pi}_0(\boldsymbol{X})$ is represented by $x_0 \in \mathbb{R}^3$
  - the projection $\boldsymbol{\pi}_0$ is represented by $x_0 = \kappa(t, x) = \kappa(X)$

Symplectic formalism : calculus of variation by jet theory

- 1D heuristic : vary both the value y and the variable x of the function $y(x)$

$$\delta\left(\frac{dy}{dx}\right)=\frac{dx\,d(\delta y)-d(\delta x)\,dy}{(dx)^2}$$

$$\delta\left(\frac{dy}{dx}\right)=\frac{d}{dx}(\delta y)-\frac{dy}{dx}\,\frac{d}{dx}(\delta x)$$

Unlike the usual rule,
the variation of the derivative **is not equal to** the derivative of the variation

- the term in red provides an extra variational equation

- We shall be going to use this kind of variation to calculate variational derivatives of functionals for the Eulerian description

Hamiltonian formalism : variation by jet theory

- As $\mathcal{L}, \mathcal{H} = \pi \cdot v - \mathcal{L}$ are densities, for consistency, $\pi \cdot v$ so is, but v are the components of a 1-contravariant tensor,

  Then π are the components of a 1-covariant and antisymmetric 3-contravariant tensor

- Hamiltonian of the system at time t : $H[x_0, \pi] = \int_{\Omega_t} \mathcal{H}(t, x, x_0, \nabla x_0, \pi)\, d^3x$

- We claim that the motion of the continuum is described by the canonical equations

$$\zeta = \left(\frac{dx}{dt}, \frac{\partial \pi}{\partial t} \right) = \left(v, \frac{\partial \pi}{\partial t} \right) = X_H$$

- where X_H is the Hamiltonian vector field for the canonical symplectic form

$$\omega(\zeta, \zeta') = \int_{\Omega_t} \left(\frac{dx}{dt} \cdot \frac{\partial \pi'}{\partial t} - \frac{\partial \pi}{\partial t} \cdot \frac{dx'}{dt} \right) \mathrm{d}^3 x$$

- Variation by jet theory : new parameterization $x = \psi(y)$

- Calculate the symplectic variational derivative X_H of the Hamitonian $H[x, x_0, \pi'] = \int_{\Omega'_t} \mathcal{H}\left(t, \psi(y), x_0, \nabla_y x_0 \cdot \nabla y, \det(\nabla y)\, (\nabla y)^T \cdot \pi'\right) \det(\nabla_y x)\, d^3y$

- Consider $y = x$

Hamiltonian formalism :
Hamiltonian vector field

- The corresponding **canonical equations** are

$$\frac{dx}{dt} = \nabla_\pi \mathcal{H}$$

$$\frac{\partial \pi}{\partial t} = -\nabla \mathcal{H}$$

$$-\nabla \cdot \left[\nabla_{\nabla x_0} \mathcal{H} \cdot \nabla x_0 - (\mathcal{H} - \nabla_\pi \mathcal{H} \cdot \pi)\, I + \nabla_\pi \mathcal{H} \otimes \pi\right]$$

  with the extra terms of the jet theory in red

- For a classical Hamiltonian, we recover
  the definition of the linear momentum and its equation of conservation

$$\frac{dx}{dt} = \frac{\pi}{\rho} - A, \qquad -\frac{\partial \pi}{\partial t} + \nabla \cdot (\sigma_R - v \otimes \pi) + \rho\,((\nabla A) \cdot v - \nabla \phi) = 0$$

  where A, ϕ are the potentials of the Galilean gravitation

  For a barotropic fluid $-\rho\, \frac{Dv}{Dt} - \nabla p + \rho\, (g - 2\,\Omega \times v) = 0$ (Euler's equation)

  where occur the gravity $g = -\nabla \phi - \frac{\partial A}{\partial t}$ and Coriolis' vector $\Omega = \frac{1}{2}\, \nabla \times A$

A symplectic minimum principle for dissipative media

Dissipation potential

- **Decomposition** of the evolution into reversible and irreversible parts

$$\zeta = \zeta_R + \zeta_I, \qquad \zeta_R = X_H, \qquad \zeta_I = \zeta - X_H$$

- **Dissipative constitutive law** $\zeta_I = X_\Phi$ where X_Φ is such that

$$\forall \zeta', \qquad \omega(X_\Phi, \zeta') = \lim_{\epsilon \to 0} \frac{1}{\epsilon} (\Phi(\zeta + \epsilon\, \zeta') - \Phi(\zeta))$$

- **Convex dissipation potential** Φ such that

$$\forall \zeta', \quad \Phi(\zeta + \zeta') - \Phi(\zeta) \geq \omega(X_\Phi, \zeta')$$

- **Symplectic polar** (or conjugate) function $\Phi^{*\omega}(\zeta_I) = \sup_\zeta \left(\omega(\zeta_I, \zeta) - \Phi(\zeta)\right)$
- Satisfying a symplectic Fenchel inequality

$$\forall \zeta', \forall \zeta'_I, \qquad \Phi(\zeta') + \Phi^{*\omega}(\zeta'_I) - \omega(\zeta'_I, \zeta') \geq 0$$

- and the equality is reached for the constitutive law

$$\zeta_I = X_\Phi \qquad \Leftrightarrow \qquad \Phi(\zeta) + \Phi^{*\omega}(\zeta_I) - \omega(\zeta_I, \zeta) = 0$$

(extremality condition)

A symplectic minimum principle for dissipative media

- Original idea [Brezis & Ekeland CRAS 1976, Nayroles CRAS 1976]
- Symplectic version [Buliga & de Saxcé MMS 2016]
- An evolution path $t \mapsto (\kappa_t, \zeta)$ is said **admissible** if it satisfies the initial and boundary conditions
- **Symplectic Brezis-Ekeland-Nayroles principle (SBEN) :** *the natural evolution path $t \mapsto (\kappa_t, \zeta)$ minimizes the functional*

$$\Pi[\kappa, \zeta] = \int_0^T \{\Phi(\zeta) + \Phi^{*\omega}(\zeta - X_H) - \omega(\zeta - X_H, \zeta)\} \, \mathrm{d}t \tag{1}$$

among all the admissible evolution paths, and the minimum is zero.

SBEN principle for compressible Navier-Stokes equation

- the canonical equations lead to $\zeta_I = \zeta - X_H = (v_I, \pi_I)$
  with $v_I = v - \frac{\pi}{\rho} + A, \qquad \pi_I = \rho\,\frac{Dv}{Dt} + \nabla p - \rho\,(g - 2\,\Omega \times v)$
- **Hypothesis 1 :** $\frac{\partial \pi}{\partial t}$ is **ignorable** in Φ : $\Phi(\zeta) = \varphi(v)$
  then the symplectic Fenchel polar function has a finite value
  $\Phi^{*\omega}(\zeta_I) = \Phi^{*\omega}(v_I, \pi_I) = \varphi^*(-\pi_I) \quad$ if $v_I = 0$
- the last term in the functional becomes
  $-\omega(\zeta - X_H, \zeta) = \int_{\Omega_t} \left(\pi_I \cdot v - v_I \cdot \frac{\partial \pi}{\partial t}\right) \mathrm{d}^3x = \int_{\Omega_t} \pi_I \cdot v\, \mathrm{d}^3x$
- Then the SBEN functional becomes
  $\Pi[\kappa, \zeta] = \int_0^T \{\varphi(v) + \varphi^*(-\pi_I) + \int_{\Omega_t} \pi_I \cdot v\, \mathrm{d}^3x\}\, \mathrm{d}t$
- **Remark :** $\Phi(\zeta) + \Phi^{*\omega}(\zeta_I) = \varphi(v) + \varphi^*(-\pi_I)$ is Ghoussoub's anti-selfdual Lagrangian. This reveals its symplectic origin

SBEN principle for compressible Navier-Stokes equation

- **SBEN principle for compressible Navier-Stokes equation :**
  the natural evolution path $t \mapsto (\kappa_t, v)$ minimizes the functional

  $$\Pi[\kappa, v] = \int_0^T \{\varphi(v) + \varphi^* \left(-\rho \frac{Dv}{Dt} - \nabla p + \rho\,(g - 2\,\Omega \times v)\right) + \int_{\Omega_t} \left[\rho \frac{Dv}{Dt} + \nabla p - \rho\, g\right] \cdot v \,\mathrm{d}^3 x\} \,\mathrm{d}t$$

  among all the admissible evolution paths, and the minimum is zero.

- **Remark :** For the limit case of inviscid flows, the potential of dissipation φ vanishes and its polar function φ^* has a finite value equal to zero if $\pi_I = 0$, *i.e.* Euler's equations,

  then the SBEN principle claims that **the total head loss is zero**, that is the expression of **Bernoulli's principle**.

  It is worth to notice that in this limit case
  the SBEN principle does not degenerate into Hamilton's principle.

SBEN principle for compressible Navier-Stokes equation

- **Hypothesis 2 :** φ depends on v through its symmetric gradient
  $D = \mathcal{D}(v) = \nabla_s v = 1/2\,(\nabla v + (\nabla v)^T)$
  and is quadratic with respect to v of the form

  $$\varphi(v) = \int_{\Omega_t} W(\mathcal{D}(v))\, d^3x = \int_{\Omega_t} \mu\, \left[Tr(D^2) - \frac{1}{3}\,(Tr(D))^2 \right]\, \mathrm{d}^3 x$$

- then the viscous part of the stress tensor is traceless (Stokes hypothesis)
  $\sigma_I = \nabla_D W(\mathcal{D}(v)) = 2\mu\, \left(D - \frac{1}{3}\, Tr(D)\, I \right)$

- **Proof that the principle of minimum restitues Navier-Stokes equation**
  Indeed, if the minimum equal to zero is reached, we have

  a.e. in $[0, T]$, $\qquad \varphi(v) + \varphi^*(-\pi_I) + \int_{\Omega_t} \pi_I \cdot v\, \mathrm{d}^3 x = 0$

  that is equivalent to the dissipative constitutive law

  $-\pi_I = \nabla_v \varphi(v) = -\nabla \cdot \sigma_I$

  Owing to Stokes hypothesis, we recover Navier-Stokes equation

  $$\rho\, \frac{Dv}{Dt} = -\nabla p + \mu \triangle v + \frac{\mu}{3}\, \nabla\, (\nabla \cdot v) + \rho\, (g - 2\,\Omega \times v)$$

SBEN principle for incompressible Navier-Stokes equation

- For this limit case, $\nabla \cdot v = 0$
  and the pressure p becomes a free variable independent of κ.
  Navier-Stokes equation is reduced to
  $\rho \, \frac{Dv}{Dt} = -\nabla p + \mu \, \triangle v + \rho \, (g - 2 \, \Omega \times v)$

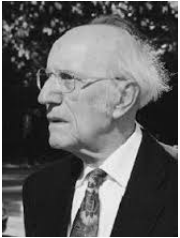
- To obtain the corresponding SBEN principle, the internal energy is cancelled in the functional and the Hamiltonian.
  The incompressibility condition is introduced as a constraint in the minimization.
  The pressure disappears of the functional and reappears as a Lagrange multiplier of this constraint

- **SBEN principle for incompressible Navier-Stokes equation :**
  the natural evolution path $t \mapsto (\kappa_t, v)$ minimizes the functional

  $$\Pi[\kappa, v] = \int_0^T \left\{ \varphi(v) + \varphi^* \left(-\rho \, \frac{Dv}{Dt} + \rho \, (g - 2 \, \Omega \times v) \right) + \int_{\Omega_t} \rho \, \left(\frac{Dv}{Dt} - g \right) \cdot v \, d^3x \right\} \, dt$$

  among all the admissible evolution paths such that $\nabla \cdot v = 0$,
  and the minimum is zero.

Extension to NonSmooth Mechanics



Extension to NonSmooth Mechanics

- For **set-valued dissipative laws**, we consider convex but not differentiable potentials of dissipation Φ
- **Symplectic subdifferential** of Φ at ζ [Buliga 2009]

$$\partial^\omega \Phi(\zeta) = \{\zeta_I \ \text{ such that } \ \forall \zeta', \quad \Phi(\zeta + \zeta') - \Phi(\zeta) \geq \omega(\zeta_I, \zeta')\}$$

- Then the dissipative constitutive law is given by the **Hamiltonian inclusion**

$$\zeta_I \in \partial^\omega \Phi(\zeta)$$

- Everything else remains identical (symplectic polar, SBEN principle)

Applications

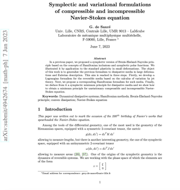
- **Plasticity :** DOF $= (u, \varepsilon^p)$ with numerical applications [Cao, Oueslati, An Danh & de Saxcé *Comput. Mech.* 2020], [Cao et al. *Appl. Math. Model.* 2021], [Cao et al. CMAME 2021]
- **Fracture Mechanics :** DOF $= (u, \psi)$ [de Saxcé IJSS 2022]
- **Bingham fluids**
- **Extension to the non associated plasticity** using the symplectic bipotential (ANR Project "BigBen", in start-up phase)

Conclusion

- **Advantages of the present formulation**
  - The present variational approach covers a large class of problems including Navier-Stokes equations
  - the expression of the functional is independent of the boundary conditions that appear only as constraints of the minimization.
  - The functional is not convex but there is (at least partial) convexity, that is favourable for the convergence of the minimization algorithm.
  - It paves the way to provide variational approximations of the solutions

Perspective

- Analytical exemples
- Numerical applications
  - Develop **symplectic integrators**
  - Construct **variational schemes** based on the Lagrangian of the SBEN principle
- Functional analysis aspects



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Symplectic and variational formulations of compressible and incompressible Navier-Stokes equation

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Abstract

In a previous paper, we proposed a symplectic version of Brezis-Ekeland-Nayroles principle based on the concepts of Hamiltonian inclusions and symplectic polar functions. We illustrated it by application to the standard plasticity in small deformations. The object of this work is to generalize the previous formalism to dissipative media in large deformations and Eulerian description. This aim is reached in three steps. Firstly, we develop a Lagrangian formalism for the reversible media based on the calculus of variation by jet theory. Next, we propose a corresponding Hamiltonian formalism for such media. Finally, we deduce from it a symplectic minimum principle for dissipative media and we show how to obtain a minimum principle for unstationary compressible and incompressible Navier-Stokes equation.

Keywords: Dynamical dissipative systems; Hamiltonian methods; Brezis-Ekeland-Nayroles principle; convex dissipation; Navier-Stokes equation

1 Introduction

This paper was written out to mark the occasion of the 200th birthday of Navier's works that spearheaded the Navier-Stokes equation.

Among the tools of the differential geometry, one of the most used is the geometry of the Riemannian spaces, equipped with a symmetric 2-covariant tensor, the metric

$$g(dz, dz') = dz^T K \, dz'$$

allowing to measure lengths, but there is another interesting geometry, the one of the symplectic space, equipped with an antisymmetric 2-covariant tensor

$$\omega(dz, dz') = dz^T J \, dz'$$

allowing to measure areas ([35], [57]). One of the origins of the symplectic geometry is the dynamics of reversible systems. We are working with the phase space of which the elements are of the form

$$z = \begin{bmatrix} \xi \\ \eta \end{bmatrix}$$

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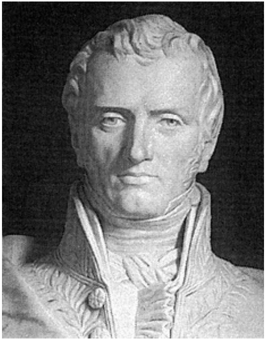
FIGURE – Arxiv publication

Cooperations with researchers in fluid mechanics and mathematics are welcome

Thank you !



Claude-Louis NAVIER

Georges Gabriel STOKES

Calculus of the Fenchel polar function of φ

- if W is quadratic
  and if the velocity or the dissipative stress vector is null on the boundary, Fenchel polar function of φ is :

$$\varphi^*(f) = \int_{\Omega_t} W(\mathcal{D}(K^{-1}(f)))\, d^3x$$

- where the linear operator K is define by $f = K(v) = -\nabla \cdot (\nabla_D W(\mathcal{D}(v)))$