Modeling anisotropic damage using a reconstruction by rational covariants of the elasticity tensor obtained by Discrete Element tests

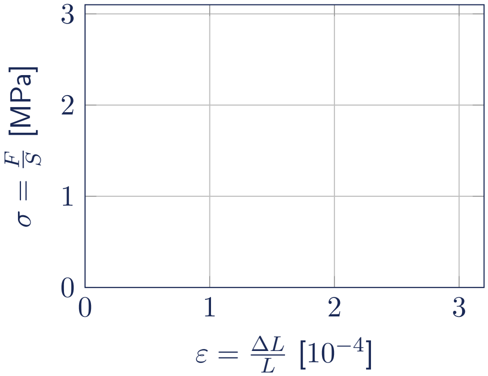
CFRAC 2023
Prague, Czech Republic, 21–23 June 2023.

Flavien Loiseau, Cécile Oliver-Leblond, Rodrigue Desmorat
flavien.loiseau@ens-paris-saclay.fr

Université Paris-Saclay, CentraleSupélec, ENS Paris-Saclay, CNRS, LMPS - Laboratoire de Mécanique Paris-Saclay, 91190, Gif-sur-Yvette, France.

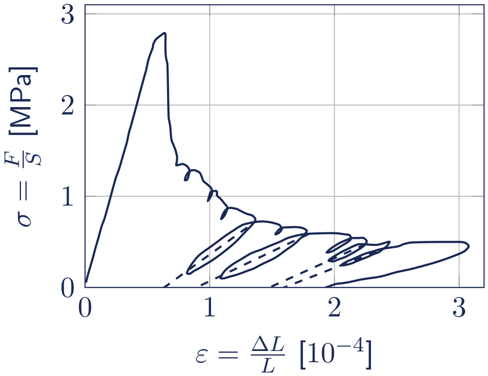
Context – Observations

Tensile test on concrete
(Terrien, 1980)



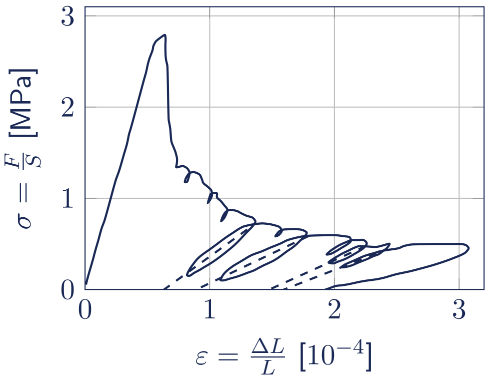
Context – Observations

Tensile test on concrete
(Terrien, 1980)

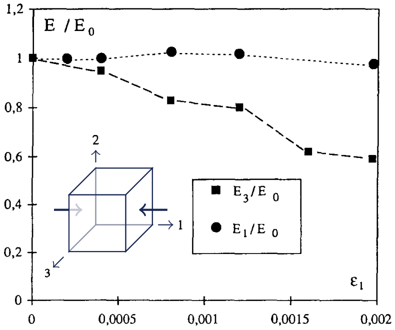


Context – Observations

Tensile test on concrete (Terrien, 1980)



Damage-induced anistropy (Berthaud, 1991)



Context – Objectives

Objective of the project

Formulating an anisotropic damage model for quasi-brittle materials in 2D

Context – Objectives

Objective of the project

Formulating an anisotropic damage model for quasi-brittle materials in 2D

Structure of a damage model

$$\mathcal{V} = \{\varepsilon, \mathbf{D}, ...\} \qquad \text{(State variables)}$$

$$\rho\psi = \frac{1}{2}\varepsilon{:}\mathbf{E}(\mathbf{D}){:}\varepsilon \qquad \text{(State potential)}$$

$$\dot{\mathbf{D}} = ... \qquad \text{(Damage evolution)}$$

Notations

- $\mathbf{D}$ damage variable
- $\mathbf{E}(\mathbf{D})$ effective elasticity tensor

Constraints

- $\mathbf{E}(\mathbf{D})$ is positive definite
- Positive dissipation

Context – Objectives

Objective of the project

Formulating an anisotropic damage model for quasi-brittle materials in 2D

Structure of a damage model

$$\mathcal{V} = \{\boldsymbol{\varepsilon}, \mathbf{D}, \ldots\} \qquad \text{(State variables)}$$

$$\rho\psi = \frac{1}{2}\boldsymbol{\varepsilon}:\mathbf{E}(\mathbf{D}):\boldsymbol{\varepsilon} \qquad \text{(State potential)}$$

$$\dot{\mathbf{D}} = \ldots \qquad \text{(Damage evolution)}$$

Notations

- $\mathbf{D}$ damage variable
- $\mathbf{E}(\mathbf{D})$ effective elasticity tensor

Constraints

- $\mathbf{E}(\mathbf{D})$ is positive definite
- Positive dissipation

Objectives of the presentation

- Generate a dataset of effective elasticity tensors
- Quantify micro-cracking (damage variable)
- Formulate a state model $\mathbf{E}(\mathbf{D})$

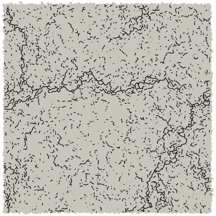
Context – Dataset of effective elasticity tensors

Virtual testing

Measure $\mathbb{E}$ from discrete model (Vassaux et al., 2016)

- 1 material,
- 36 meso-structures,
- 21 (prop and non-prop) loadings,
- 100 time steps,

for a total of ≈ 76 000 tensors.



Bi-tension

Shear

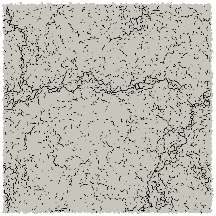
Context – Dataset of effective elasticity tensors

Virtual testing

Measure $\mathbb{E}$ from discrete model (Vassaux et al., 2016)

- 1 material,
- 36 meso-structures,
- 21 (prop and non-prop) loadings,
- 100 time steps,

for a total of $\approx$ 76 000 tensors.



Bi-tension

Shear

Objectives of the presentation

- ✓ Generate a dataset of effective elasticity tensors
- ○ Quantify micro-cracking (damage variable)
- ○ Formulate a state model

Damage variable – Micro-cracking induced anisotropy

Distance to orthotropy in 2D (Antonelli et al., 2022)

Question What tensorial order for the damage variable?

Damage variable – Micro-cracking induced anisotropy

Distance to orthotropy in 2D (Antonelli et al., 2022)

Question What tensorial order for the damage variable?

Tool Relative distance to a symmetry stratum Σ $\Delta_\Sigma(\mathbf{E}) = \min_{\mathbf{E}^* \in \Sigma} \frac{\|\mathbf{E}-\mathbf{E}^*\|}{\|\mathbf{E}\|}$

Damage variable – Micro-cracking induced anisotropy

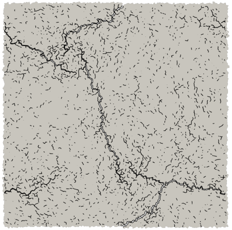
Distance to orthotropy in 2D (Antonelli et al., 2022)

Question What tensorial order for the damage variable?

Tool Relative distance to a symmetry stratum Σ

$$\Delta_\Sigma(\mathbf{E}) = \min_{\mathbf{E}^* \in \Sigma} \frac{\|\mathbf{E}-\mathbf{E}^*\|}{\|\mathbf{E}\|}$$

Illustration with periodic bi-tension



Damage variable – Micro-cracking induced anisotropy

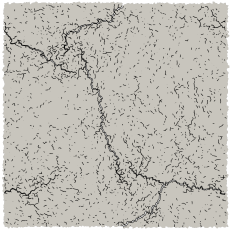
Distance to orthotropy in 2D (Antonelli et al., 2022)

Question What tensorial order for the damage variable?

Tool Relative distance to a symmetry stratum Σ

$$\Delta_\Sigma(\mathbf{E}) = \min_{\mathbf{E}^* \in \Sigma} \frac{\|\mathbf{E}-\mathbf{E}^*\|}{\|\mathbf{E}\|}$$

Illustration with periodic bi-tension



$\mathbf{E}$ [MPa]

$$\begin{bmatrix} 0.93 & -0.38 & -0.50 \\ -0.38 & 1.47 & 0.36 \\ -0.50 & 0.36 & 3.66 \end{bmatrix}$$

Damage variable – Micro-cracking induced anisotropy

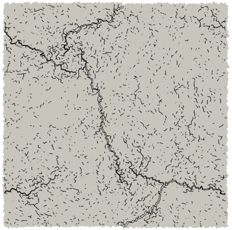
Distance to orthotropy in 2D (Antonelli et al., 2022)

Question What tensorial order for the damage variable?

Tool Relative distance to a symmetry stratum Σ

$$\Delta_\Sigma(\mathbf{E}) = \min_{\mathbf{E}^* \in \Sigma} \frac{\|\mathbf{E} - \mathbf{E}^*\|}{\|\mathbf{E}\|}$$

Illustration with periodic bi-tension



$\mathbf{E}$ [MPa]

$$\begin{bmatrix} 0.93 & -0.38 & -0.50 \\ -0.38 & 1.47 & 0.36 \\ -0.50 & 0.36 & 3.66 \end{bmatrix}$$

Isotropy $\Delta_{\mathcal{I}so} = 0.427$

$$\mathbf{E}_{\mathcal{I}so} = \begin{bmatrix} 1.68 & -0.91 & 0.00 \\ -0.91 & 1.68 & 0.00 \\ 0.00 & 0.00 & 2.59 \end{bmatrix}$$

Damage variable – Micro-cracking induced anisotropy

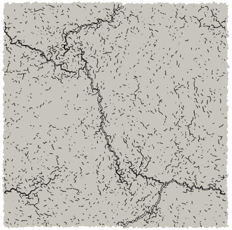
Distance to orthotropy in 2D (Antonelli et al., 2022)

Question What tensorial order for the damage variable?

Tool Relative distance to a symmetry stratum Σ

$$\Delta_\Sigma(\mathbf{E}) = \min_{\mathbf{E}^* \in \Sigma} \frac{\|\mathbf{E}-\mathbf{E}^*\|}{\|\mathbf{E}\|}$$

Illustration with periodic bi-tension



$\mathbf{E}$ [MPa]

$$\begin{bmatrix} 0.93 & -0.38 & -0.50 \\ -0.38 & 1.47 & 0.36 \\ -0.50 & 0.36 & 3.66 \end{bmatrix}$$

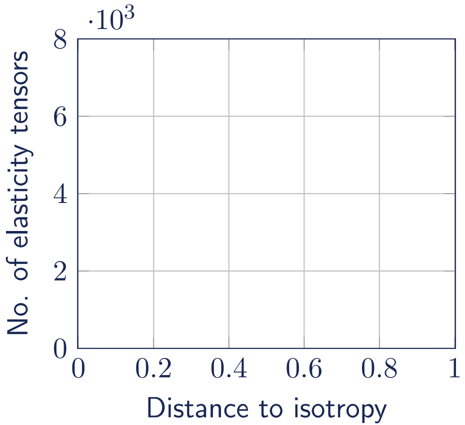
Isotropy $\Delta_{\mathcal{I}so} = 0.427$

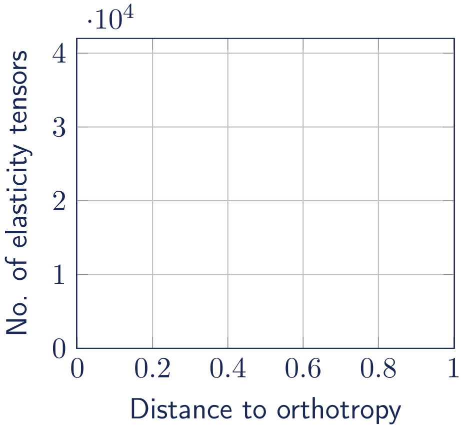
$$\mathbf{E}_{\mathcal{I}so} = \begin{bmatrix} 1.68 & -0.91 & 0.00 \\ -0.91 & 1.68 & 0.00 \\ 0.00 & 0.00 & 2.59 \end{bmatrix}$$

Orthotropy $\Delta_{\mathcal{O}rt} = 0.013$

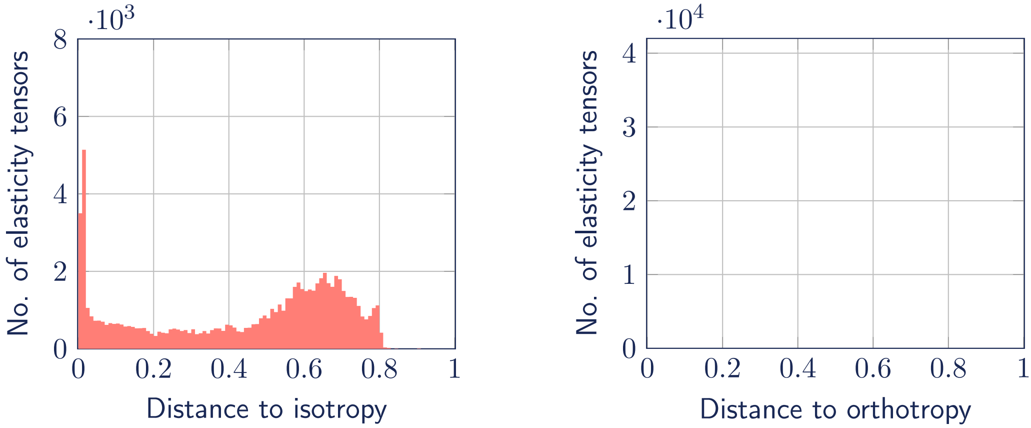
$$\mathbf{E}_{\mathcal{O}rt} = \begin{bmatrix} 0.92 & -0.38 & -0.48 \\ -0.38 & 1.38 & 0.39 \\ -0.48 & 0.39 & 3.66 \end{bmatrix}$$

Damage variable – Anisotropy of effective elasticity tensor

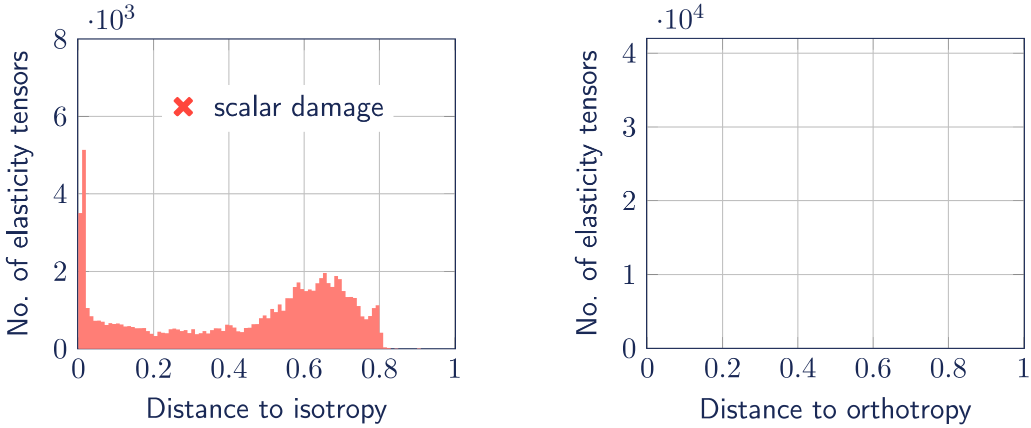




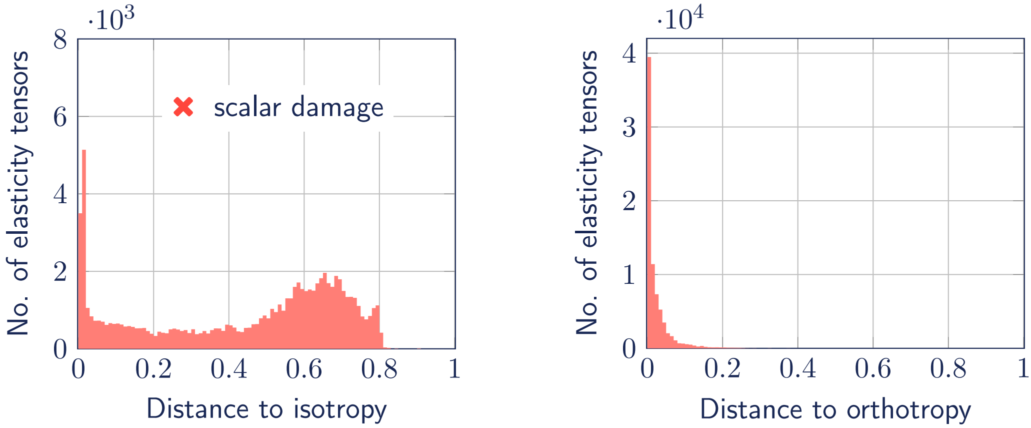
Damage variable – Anisotropy of effective elasticity tensor



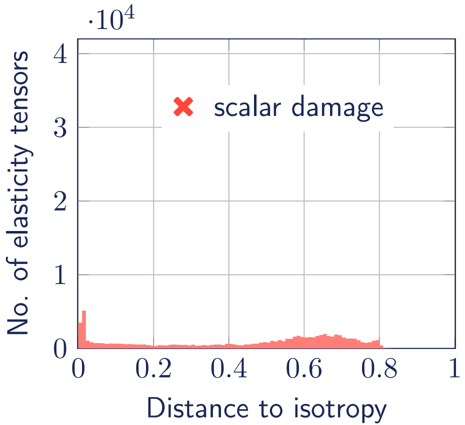
Damage variable – Anisotropy of effective elasticity tensor

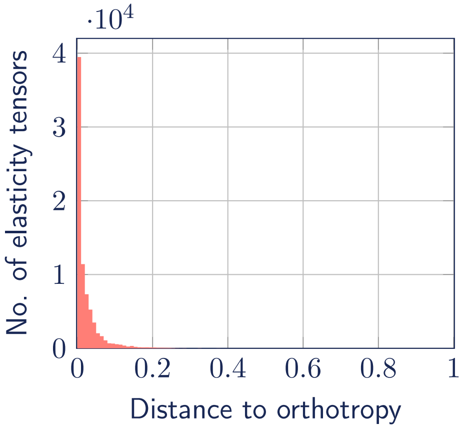


Damage variable – Anisotropy of effective elasticity tensor

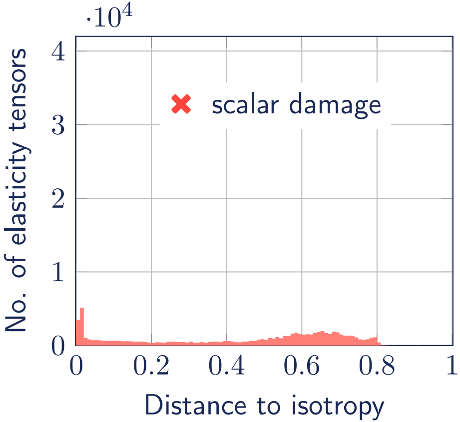


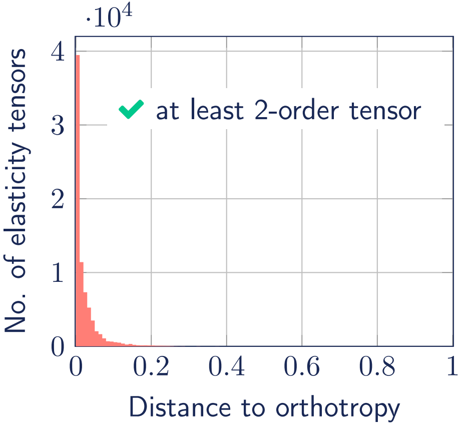
Damage variable – Anisotropy of effective elasticity tensor





Damage variable – Anisotropy of effective elasticity tensor





Damage variable – 2D Harmonic decomposition

Applications to elasticity tensor: 3D (Backus, 1970), 2D (Blinowski et al., 1996)

$$\mathbf{E} \cong (\mu, \kappa, \mathbf{d}', \mathbf{H}) \in \mathbb{H}^1(\mathbb{R}^2) \oplus \mathbb{H}^1(\mathbb{R}^2) \oplus \mathbb{H}^2(\mathbb{R}^2) \oplus \mathbb{H}^4(\mathbb{R}^2)$$

Damage variable – 2D Harmonic decomposition

Applications to elasticity tensor: 3D (Backus, 1970), 2D (Blinowski et al., 1996)

$$\mathbf{E} \cong (\mu, \kappa, \mathbf{d}', \mathbf{H}) \in \mathbb{H}^1(\mathbb{R}^2) \oplus \mathbb{H}^1(\mathbb{R}^2) \oplus \mathbb{H}^2(\mathbb{R}^2) \oplus \mathbb{H}^4(\mathbb{R}^2)$$

Properties

- μ and κ are invariants

Damage variable – 2D Harmonic decomposition

Applications to elasticity tensor: 3D (Backus, 1970), 2D (Blinowski et al., 1996)

$$\mathbf{E} \cong (\mu, \kappa, \mathbf{d}', \mathbf{H}) \in \mathbb{H}^1(\mathbb{R}^2) \oplus \mathbb{H}^1(\mathbb{R}^2) \oplus \mathbb{H}^2(\mathbb{R}^2) \oplus \mathbb{H}^4(\mathbb{R}^2)$$

Properties

- μ and κ are invariants
- $\mathbf{d}'$ and $\mathbf{H}$ are covariants

Damage variable – 2D Harmonic decomposition

Applications to elasticity tensor: 3D (Backus, 1970), 2D (Blinowski et al., 1996)

$$\mathbf{E} \cong (\mu, \kappa, \mathbf{d}', \mathbf{H}) \in \mathbb{H}^1(\mathbb{R}^2) \oplus \mathbb{H}^1(\mathbb{R}^2) \oplus \mathbb{H}^2(\mathbb{R}^2) \oplus \mathbb{H}^4(\mathbb{R}^2)$$

Properties

- μ and κ are invariants
- $\mathbf{d}'$ and $\mathbf{H}$ are covariants

Reconstruction $(\mu, \kappa, \mathbf{d}', \mathbf{H}) \mapsto \mathbf{E} = \underbrace{2\mu\mathbf{J} + \kappa \mathbf{1}_2 \otimes \mathbf{1}_2}_{\textbf{Iso}} + \underbrace{\frac{1}{2}\left(\mathbf{d}' \otimes \mathbf{1}_2 + \mathbf{1}_2 \otimes \mathbf{d}'\right)}_{\textbf{Dil}} + \mathbf{H}$

Damage variable – 2D Harmonic decomposition

Applications to elasticity tensor: 3D (Backus, 1970), 2D (Blinowski et al., 1996)

$$\mathbf{E} \cong (\mu, \kappa, \mathbf{d}', \mathbf{H}) \in \mathbb{H}^1(\mathbb{R}^2) \oplus \mathbb{H}^1(\mathbb{R}^2) \oplus \mathbb{H}^2(\mathbb{R}^2) \oplus \mathbb{H}^4(\mathbb{R}^2)$$

Properties

- μ and κ are invariants
- $\mathbf{d}'$ and $\mathbf{H}$ are covariants

Reconstruction $(\mu, \kappa, \mathbf{d}', \mathbf{H}) \mapsto \mathbf{E} = \underbrace{2\mu\mathbf{J} + \kappa\mathbf{1}_2 \otimes \mathbf{1}_2}_{\textbf{Iso}} + \underbrace{\frac{1}{2}\left(\mathbf{d}' \otimes \mathbf{1}_2 + \mathbf{1}_2 \otimes \mathbf{d}'\right)}_{\textbf{Dil}} + \mathbf{H}$

Decomposition $\mathbf{E} \mapsto (\mu, \kappa, \mathbf{d}', \mathbf{H})$ where $\mathbf{d} = \mathrm{tr}_{12}\, \mathbf{E}$, $\mathbf{v} = \mathrm{tr}_{13}\, \mathbf{E}$,

Damage variable – 2D Harmonic decomposition

Applications to elasticity tensor: 3D (Backus, 1970), 2D (Blinowski et al., 1996)

$$\mathbf{E} \cong (\mu, \kappa, \mathbf{d}', \mathbf{H}) \in \mathbb{H}^1(\mathbb{R}^2) \oplus \mathbb{H}^1(\mathbb{R}^2) \oplus \mathbb{H}^2(\mathbb{R}^2) \oplus \mathbb{H}^4(\mathbb{R}^2)$$

Properties

- μ and κ are invariants
- $\mathbf{d}'$ and $\mathbf{H}$ are covariants

Reconstruction $(\mu, \kappa, \mathbf{d}', \mathbf{H}) \mapsto \mathbf{E} = \underbrace{2\mu\mathbf{J} + \kappa\mathbf{1}_2 \otimes \mathbf{1}_2}_{\textbf{Iso}} + \underbrace{\frac{1}{2}\left(\mathbf{d}' \otimes \mathbf{1}_2 + \mathbf{1}_2 \otimes \mathbf{d}'\right)}_{\textbf{Dil}} + \mathbf{H}$

Decomposition $\mathbf{E} \mapsto (\mu, \kappa, \mathbf{d}', \mathbf{H})$ where $\mathbf{d} = \operatorname{tr}_{12}\mathbf{E}$, $\mathbf{v} = \operatorname{tr}_{13}\mathbf{E}$,

$$\mu = \frac{1}{8}\left(2\operatorname{tr}\mathbf{v} - \operatorname{tr}\mathbf{d}\right)$$

$$\kappa = \frac{1}{4}\operatorname{tr}\mathbf{d}$$

Damage variable – 2D Harmonic decomposition

Applications to elasticity tensor: 3D (Backus, 1970), 2D (Blinowski et al., 1996)

$$\mathbf{E} \cong (\mu, \kappa, \mathbf{d}', \mathbf{H}) \in \mathbb{H}^1(\mathbb{R}^2) \oplus \mathbb{H}^1(\mathbb{R}^2) \oplus \mathbb{H}^2(\mathbb{R}^2) \oplus \mathbb{H}^4(\mathbb{R}^2)$$

Properties

- μ and κ are invariants
- $\mathbf{d}'$ and $\mathbf{H}$ are covariants

Reconstruction $(\mu, \kappa, \mathbf{d}', \mathbf{H}) \mapsto \mathbf{E} = \underbrace{2\mu\mathbf{J} + \kappa \mathbf{1}_2 \otimes \mathbf{1}_2}_{\textbf{Iso}} + \underbrace{\frac{1}{2}\left(\mathbf{d}' \otimes \mathbf{1}_2 + \mathbf{1}_2 \otimes \mathbf{d}'\right)}_{\textbf{Dil}} + \mathbf{H}$

Decomposition $\mathbf{E} \mapsto (\mu, \kappa, \mathbf{d}', \mathbf{H})$ where $\mathbf{d} = \mathrm{tr}_{12}\,\mathbf{E}$, $\mathbf{v} = \mathrm{tr}_{13}\,\mathbf{E}$,

$$\mu = \frac{1}{8}\left(2\,\mathrm{tr}\,\mathbf{v} - \mathrm{tr}\,\mathbf{d}\right) \qquad \mathbf{d}' = \mathbf{d} - \frac{1}{2}\,\mathrm{tr}\,\mathbf{d}\ \mathbf{1}_2$$

$$\kappa = \frac{1}{4}\,\mathrm{tr}\,\mathbf{d} \qquad \mathbf{H} = \mathbf{E} - \mathbf{Iso} - \mathbf{Dil}$$

Damage variable – State model basis and damage definition

Knowing isotropic $\mathbf{E}_0 \cong (\mu_0, \kappa_0, \mathbf{0}_2, \mathbf{0}_4)$ and $\mathbf{D}$, we want to model

$$\mathbf{E}(\mathbf{D}) = 2\mu(\mathbf{D})\mathbf{J} + \kappa(\mathbf{D})\mathbf{1}_2 \otimes \mathbf{1}_2 + \frac{1}{2}\left(\mathbf{d}'(\mathbf{D}) \otimes \mathbf{1}_2 + \mathbf{1}_2 \otimes \mathbf{d}'(\mathbf{D})\right) + \mathbf{H}(\mathbf{D})$$

How to define damage?

Damage variable – State model basis and damage definition

Knowing isotropic $\mathbf{E}_0 \cong (\mu_0, \kappa_0, \mathbf{0}_2, \mathbf{0}_4)$ and $\mathbf{D}$, we want to model

$$\mathbf{E}(\mathbf{D}) = 2\mu(\mathbf{D})\mathbf{J} + \kappa(\mathbf{D})\mathbf{1}_2 \otimes \mathbf{1}_2 + \frac{1}{2}\left(\mathbf{d}'(\mathbf{D}) \otimes \mathbf{1}_2 + \mathbf{1}_2 \otimes \mathbf{d}'(\mathbf{D})\right) + \mathbf{H}(\mathbf{D})$$

How to define damage? Using the harmonic decomposition

$$\mu = \frac{1}{8}\left(2\operatorname{tr}\mathbf{v} - \operatorname{tr}\mathbf{d}\right) \qquad \mathbf{d}' = \mathbf{d} - \frac{1}{2}\operatorname{tr}\mathbf{d}\ \mathbf{1}_2$$

$$\kappa = \frac{1}{4}\operatorname{tr}\mathbf{d} \qquad \mathbf{H} = \mathbf{E} - \mathbf{Iso} - \mathbf{Dil}$$

Damage variable – State model basis and damage definition

Knowing isotropic $\mathbf{E}_0 \cong (\mu_0, \kappa_0, \mathbf{0}_2, \mathbf{0}_4)$ and $\mathbf{D}$, we want to model

$$\mathbf{E}(\mathbf{D}) = 2\mu(\mathbf{D})\mathbf{J} + \kappa(\mathbf{D})\mathbf{1}_2 \otimes \mathbf{1}_2 + \frac{1}{2}\left(\mathbf{d}'(\mathbf{D}) \otimes \mathbf{1}_2 + \mathbf{1}_2 \otimes \mathbf{d}'(\mathbf{D})\right) + \mathbf{H}(\mathbf{D})$$

How to define damage? Using the harmonic decomposition

$$\mu = \frac{1}{8}(2\,\mathrm{tr}\,\mathbf{v} - \mathrm{tr}\,\mathbf{d}) \qquad \mathbf{d}' = \mathbf{d} - \frac{1}{2}\,\mathrm{tr}\,\mathbf{d}\;\mathbf{1}_2$$

$$\kappa = \frac{1}{4}\,\mathrm{tr}\,\mathbf{d} \qquad \mathbf{H} = \mathbf{E} - \mathbf{Iso} - \mathbf{Dil}$$

Damage variable – State model basis and damage definition

Knowing isotropic $\mathbf{E}_0 \cong (\mu_0, \kappa_0, \mathbf{0}_2, \mathbf{0}_4)$ and $\mathbf{D}$, we want to model

$$\mathbf{E}(\mathbf{D}) = 2\mu(\mathbf{D})\mathbf{J} + \kappa(\mathbf{D})\mathbf{1}_2 \otimes \mathbf{1}_2 + \frac{1}{2}\left(\mathbf{d}'(\mathbf{D}) \otimes \mathbf{1}_2 + \mathbf{1}_2 \otimes \mathbf{d}'(\mathbf{D})\right) + \mathbf{H}(\mathbf{D})$$

How to define damage? Using the harmonic decomposition

$$\mu = \frac{1}{8}(2\,\mathrm{tr}\,\mathbf{v} - \mathrm{tr}\,\mathbf{d}) \qquad \mathbf{d}' = \mathbf{d} - \frac{1}{2}\,\mathrm{tr}\,\mathbf{d}\;\mathbf{1}_2$$

$$\kappa = \frac{1}{4}\,\mathrm{tr}\,\mathbf{d} \qquad \mathbf{H} = \mathbf{E} - \mathbf{Iso} - \mathbf{Dil}$$

Damage variable

$$\mathbf{D} = (\mathbf{d}_0 - \mathbf{d}) \cdot \mathbf{d}_0^{-1}$$

Damage variable – State model basis and damage definition

Knowing isotropic $\mathbf{E}_0 \cong (\mu_0, \kappa_0, \mathbf{0}_2, \mathbf{0}_4)$ and $\mathbf{D}$, we want to model

$$\mathbf{E}(\mathbf{D}) = 2\mu(\mathbf{D})\mathbf{J} + \kappa(\mathbf{D})\mathbf{1}_2 \otimes \mathbf{1}_2 + \frac{1}{2}\left(\mathbf{d}'(\mathbf{D}) \otimes \mathbf{1}_2 + \mathbf{1}_2 \otimes \mathbf{d}'(\mathbf{D})\right) + \mathbf{H}(\mathbf{D})$$

How to define damage? Using the harmonic decomposition

$$\mu = \frac{1}{8}(2\,\mathrm{tr}\,\mathbf{v} - \mathrm{tr}\,\mathbf{d}) \qquad \mathbf{d}' = \mathbf{d} - \frac{1}{2}\mathrm{tr}\,\mathbf{d}\;\mathbf{1}_2$$

$$\kappa = \frac{1}{4}\mathrm{tr}\,\mathbf{d} \qquad \mathbf{H} = \mathbf{E} - \mathbf{Iso} - \mathbf{Dil}$$

Damage variable

$$\mathbf{D} = (\mathbf{d}_0 - \mathbf{d}) \cdot \mathbf{d}_0^{-1} = \mathbf{1}_2 - \frac{1}{2\kappa_0}\mathbf{d}$$

Damage variable – State model basis and damage definition

Knowing isotropic $\mathbf{E}_0 \cong (\mu_0, \kappa_0, \mathbf{0}_2, \mathbf{0}_4)$ and $\mathbf{D}$, we want to model

$$\mathbf{E}(\mathbf{D}) = 2\mu(\mathbf{D})\mathbf{J} + \kappa(\mathbf{D})\mathbf{1}_2 \otimes \mathbf{1}_2 + \frac{1}{2}\left(\mathbf{d}'(\mathbf{D}) \otimes \mathbf{1}_2 + \mathbf{1}_2 \otimes \mathbf{d}'(\mathbf{D})\right) + \mathbf{H}(\mathbf{D})$$

How to define damage? Using the harmonic decomposition

$$\mu = \frac{1}{8}\left(2\,\mathrm{tr}\,\mathbf{v} - \mathrm{tr}\,\mathbf{d}\right) \qquad \mathbf{d}' = \mathbf{d} - \frac{1}{2}\,\mathrm{tr}\,\mathbf{d}\;\mathbf{1}_2$$

$$\kappa = \frac{1}{4}\,\mathrm{tr}\,\mathbf{d} \qquad \mathbf{H} = \mathbf{E} - \mathbf{Iso} - \mathbf{Dil}$$

Damage variable

$$\mathbf{D} = (\mathbf{d}_0 - \mathbf{d}) \cdot \mathbf{d}_0^{-1} = \mathbf{1}_2 - \frac{1}{2\kappa_0}\mathbf{d} \iff \mathbf{d} = 2\kappa_0(\mathbf{1}_2 - \mathbf{D})$$

State model – Advantages of the damage definition

$$\mathbf{D} = \mathbf{1}_2 - \frac{1}{2\kappa_0}\mathbf{d} \iff \mathbf{d} = 2\kappa_0(\mathbf{1}_2 - \mathbf{D})$$

State model – Advantages of the damage definition

$$\mathbf{D} = \mathbf{1}_2 - \frac{1}{2\kappa_0}\mathbf{d} \iff \mathbf{d} = 2\kappa_0(\mathbf{1}_2 - \mathbf{D})$$

Expression of $\kappa(\mathbf{D})$

$$\mathbf{D} \xleftrightarrow{\ \mathbf{D}\text{ def}\ } \mathbf{d} \xrightarrow{\ \frac{1}{4}\operatorname{tr}\bullet\ } \kappa$$

State model – Advantages of the damage definition

$$\mathbf{D} = \mathbf{1}_2 - \frac{1}{2\kappa_0}\mathbf{d} \iff \mathbf{d} = 2\kappa_0(\mathbf{1}_2 - \mathbf{D})$$

Expression of $\kappa(\mathbf{D})$

$$\mathbf{D} \xleftrightarrow{\ \mathbf{D}\text{ def}\ } \mathbf{d} \xrightarrow{\ \frac{1}{4}\operatorname{tr}\bullet\ } \kappa$$

$$\Downarrow$$

$$\kappa(\mathbf{D}) = \kappa_0\left(1 - \frac{1}{2}\operatorname{tr}\mathbf{D}\right)$$

State model – Advantages of the damage definition

$$\mathbf{D} = \mathbf{1}_2 - \frac{1}{2\kappa_0}\mathbf{d} \iff \mathbf{d} = 2\kappa_0(\mathbf{1}_2 - \mathbf{D})$$

Expression of $\kappa(\mathbf{D})$

$$\mathbf{D} \xleftrightarrow{\mathbf{D}\text{ def}} \mathbf{d} \xrightarrow{\frac{1}{4}\operatorname{tr}\bullet} \kappa$$

$$\Downarrow$$

$$\kappa(\mathbf{D}) = \kappa_0\left(1 - \frac{1}{2}\operatorname{tr}\mathbf{D}\right)$$

Expression of $\mathbf{d}'(\mathbf{D})$

$$\mathbf{D} \xleftrightarrow{\mathbf{D}\text{ def}} \mathbf{d} \xrightarrow{\text{Dev }\bullet'} \mathbf{d}'$$

State model – Advantages of the damage definition

$$\mathbf{D} = \mathbf{1}_2 - \frac{1}{2\kappa_0}\mathbf{d} \iff \mathbf{d} = 2\kappa_0(\mathbf{1}_2 - \mathbf{D})$$

Expression of $\kappa(\mathbf{D})$

$$\mathbf{D} \xleftrightarrow{\mathbf{D}\text{ def}} \mathbf{d} \xrightarrow{\frac{1}{4}\operatorname{tr}\bullet} \kappa$$

$$\Downarrow$$

$$\kappa(\mathbf{D}) = \kappa_0\left(1 - \frac{1}{2}\operatorname{tr}\mathbf{D}\right)$$

Expression of $\mathbf{d}'(\mathbf{D})$

$$\mathbf{D} \xleftrightarrow{\mathbf{D}\text{ def}} \mathbf{d} \xrightarrow{\text{Dev }\bullet'} \mathbf{d}'$$

$$\Downarrow$$

$$\mathbf{d}'(\mathbf{D}) = -2\kappa_0\mathbf{D}'$$

State model – Basis of the model

Objectives of the presentation

- ✓ Generate a dataset of effective elasticity tensors
- ✓ Quantify micro-cracking (damage variable)
- ○ Formulate a state model $\mathbf{E}(\mathbf{D})$

State model – Basis of the model

Objectives of the presentation

- ✓ Generate a dataset of effective elasticity tensors
- ✓ Quantify micro-cracking (damage variable)
- ○ Formulate a state model $\mathbf{E}(\mathbf{D})$

Knowing isotropic $\mathbf{E}_0 \cong (\mu_0, \kappa_0, \mathbf{0}_2, \mathbf{0}_4)$ and $\mathbf{D}$, elasticity tensor $\mathbf{E}$ can be modelled as

$$\mathbf{E}(\mathbf{D}) = 2\mu(\mathbf{D})\mathbf{J} + \kappa(\mathbf{D})\mathbf{1}_2 \otimes \mathbf{1}_2 + \frac{1}{2}\left(\mathbf{d}'(\mathbf{D}) \otimes \mathbf{1}_2 + \mathbf{1}_2 \otimes \mathbf{d}'(\mathbf{D})\right) + \mathbf{H}(\mathbf{D})$$

where decomposition $\mathbf{E} \mapsto (\mu, \kappa, \mathbf{d}', \mathbf{H})$ and damage definition $\mathbf{d} \mapsto \mathbf{D}$ give

$$\mu(\mathbf{D}) = \frac{1}{8}\left(2 \operatorname{tr} \mathbf{v}(\mathbf{D}) - \operatorname{tr} \mathbf{d}(\mathbf{D})\right) \qquad \mathbf{d}'(\mathbf{D}) = -2\kappa_0 \mathbf{D}'$$

$$\kappa(\mathbf{D}) = \kappa_0 \left(1 - \frac{1}{2} \operatorname{tr} \mathbf{D}\right) \qquad \mathbf{H}(\mathbf{D}) = \mathbf{E} - \mathbf{Iso} - \mathbf{Dil}$$

State model – Basis of the model

Objectives of the presentation

- ✓ Generate a dataset of effective elasticity tensors
- ✓ Quantify micro-cracking (damage variable)
- ○ Formulate a state model $\mathbf{E}(\mathbf{D})$

Knowing isotropic $\mathbf{E}_0 \cong (\mu_0, \kappa_0, \mathbf{0}_2, \mathbf{0}_4)$ and $\mathbf{D}$, elasticity tensor $\mathbf{E}$ can be modelled as

$$\mathbf{E}(\mathbf{D}) = 2\mu(\mathbf{D})\mathbf{J} + \kappa(\mathbf{D})\mathbf{1}_2 \otimes \mathbf{1}_2 + \frac{1}{2}\left(\mathbf{d}'(\mathbf{D}) \otimes \mathbf{1}_2 + \mathbf{1}_2 \otimes \mathbf{d}'(\mathbf{D})\right) + \mathbf{H}(\mathbf{D})$$

where decomposition $\mathbf{E} \mapsto (\mu, \kappa, \mathbf{d}', \mathbf{H})$ and damage definition $\mathbf{d} \mapsto \mathbf{D}$ give

$$\mu(\mathbf{D}) = \frac{1}{8}(2 \operatorname{tr} \mathbf{v}(\mathbf{D}) - \operatorname{tr} \mathbf{d}(\mathbf{D})) \qquad \mathbf{d}'(\mathbf{D}) = -2\kappa_0 \mathbf{D}'$$

$$\kappa(\mathbf{D}) = \kappa_0 \left(1 - \frac{1}{2}\operatorname{tr} \mathbf{D}\right) \qquad \mathbf{H}(\mathbf{D}) = \mathbf{E} - \mathbf{Iso} - \mathbf{Dil}$$

State model – Basis of the model

Objectives of the presentation

- ✓ Generate a dataset of effective elasticity tensors
- ✓ Quantify micro-cracking (damage variable)
- ○ Formulate a state model $\mathbf{E}(\mathbf{D})$

Knowing isotropic $\mathbf{E}_0 \cong (\mu_0, \kappa_0, \mathbf{0}_2, \mathbf{0}_4)$ and $\mathbf{D}$, elasticity tensor $\mathbf{E}$ can be modelled as

$$\mathbf{E}(\mathbf{D}) = 2\mu(\mathbf{D})\mathbf{J} + \kappa(\mathbf{D})\mathbf{1}_2 \otimes \mathbf{1}_2 + \frac{1}{2}\left(\mathbf{d}'(\mathbf{D}) \otimes \mathbf{1}_2 + \mathbf{1}_2 \otimes \mathbf{d}'(\mathbf{D})\right) + \mathbf{H}(\mathbf{D})$$

where decomposition $\mathbf{E} \mapsto (\mu, \kappa, \mathbf{d}', \mathbf{H})$ and damage definition $\mathbf{d} \mapsto \mathbf{D}$ give

$$\mu(\mathbf{D}) = \frac{1}{8}\left(2 \operatorname{tr} \mathbf{v}(\mathbf{D}) - \operatorname{tr} \mathbf{d}(\mathbf{D})\right) \qquad \mathbf{d}'(\mathbf{D}) = -2\kappa_0 \mathbf{D}'$$

$$\kappa(\mathbf{D}) = \kappa_0 \left(1 - \frac{1}{2}\operatorname{tr} \mathbf{D}\right) \qquad \mathbf{H}(\mathbf{D}) = \mathbf{E} - \mathbf{Iso} - \mathbf{Dil}$$

Questions How to model ○ shear modulus $\mu(\mathbf{D})$? ○ harmonic part $\mathbf{H}(\mathbf{D})$?

State model – Modelling shear modulus and harmonic part – Overview

Shear modulus

$$\mu(\mathbf{D}) = \frac{1}{8}\left(2\ \operatorname{tr}\mathbf{v}(\mathbf{D}) - \operatorname{tr}\mathbf{d}(\mathbf{D})\right)$$

Assumptions

$$\mu(\mathbf{D} = \mathbf{0}_2) = \mu_0 \qquad \text{(Initial)}$$

$$\mu(\mathbf{D} = \mathbf{1}_2) = 0 \qquad \text{(Full damage)}$$

$$\text{If } \mathbf{D} \approx \mathbf{0}_2,\ \ \operatorname{tr}\mathbf{d} = \operatorname{tr}\mathbf{v} \qquad \text{(Early}^{1}\text{)}$$

Model μ as a linear comb. of invariants

$$\mu(\mathbf{D}) = \mu_0 - \frac{\kappa_0}{4}\operatorname{tr}\mathbf{D} + \frac{\kappa_0 - 2\mu_0}{4}\mathbf{D}{:}\mathbf{D}$$

[1] Early damage $\implies$ non-interacting cracks $\implies$ Tot sym stiffness loss (Kachanov, 1992)

State model – Modelling shear modulus and harmonic part – Overview

Shear modulus

$$\mu(\mathbf{D}) = \frac{1}{8}\left(2\ \operatorname{tr}\mathbf{v}(\mathbf{D}) - \operatorname{tr}\mathbf{d}(\mathbf{D})\right)$$

Assumptions

$\mu(\mathbf{D} = \mathbf{0}_2) = \mu_0$ (Initial)

$\mu(\mathbf{D} = \mathbf{1}_2) = 0$ (Full damage)

If $\mathbf{D} \approx \mathbf{0}_2,\ \operatorname{tr}\mathbf{d} = \operatorname{tr}\mathbf{v}$ (Early[1])

Model μ as a linear comb. of invariants

$$\mu(\mathbf{D}) = \mu_0 - \frac{\kappa_0}{4}\operatorname{tr}\mathbf{D} + \frac{\kappa_0 - 2\mu_0}{4}\mathbf{D}:\mathbf{D}$$

Harmonic part

(Desmorat & Desmorat, 2015)

$$\mathbf{H} = \pm\|\mathbf{H}\|\frac{\mathbf{d}' * \mathbf{d}'}{\|\mathbf{d}' * \mathbf{d}'\|} \qquad \text{(Orthotropy)}$$

where $\mathbf{d}' * \mathbf{d}' = \mathbf{d}' \otimes \mathbf{d}' - \frac{1}{2}(\mathbf{d}':\mathbf{d}')\mathbf{J}$.

Model $\|\mathbf{H}\|$ as a polynomial of invariants and apply sparse regression on the dataset

$$\|\mathbf{H}\| \approx \mathbf{H}^{\mathrm{m}}(\mathbf{D}) = h(\operatorname{tr}\mathbf{D})^4\mathbf{D}' * \mathbf{D}'$$

where h is the harmonic parameter.

[1] Early damage $\Longrightarrow$ non-interacting cracks $\Longrightarrow$ Tot sym stiffness loss (Kachanov, 1992)

State model – Conclusion

Knowing κ_0, μ_0 and $\mathbf{D}$, the elasticity tensor can be modelled as

$$\mathbf{E}(\mathbf{D}) = 2\mu(\mathbf{D})\mathbf{J} + \kappa(\mathbf{D})\mathbf{1}_2 \otimes \mathbf{1}_2 + \frac{1}{2}\left(\mathbf{d}'(\mathbf{D}) \otimes \mathbf{1}_2 + \mathbf{1}_2 \otimes \mathbf{d}'(\mathbf{D})\right) + \mathbf{H}(\mathbf{D})$$

where the invariants and covariants models are

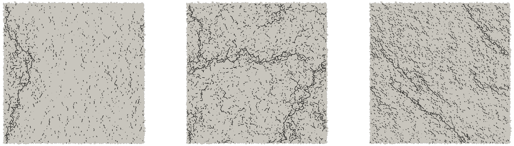
$$\mu(\mathbf{D}) = \mu_0 - \frac{\kappa_0}{4}(\operatorname{tr}\mathbf{D}) + \frac{\kappa_0 - 2\mu_0}{4}(\mathbf{D}:\mathbf{D}) \qquad \mathbf{d}'(\mathbf{D}) = -2\kappa_0\mathbf{D}'$$

$$\kappa(\mathbf{D}) = \kappa_0\left(1 - \frac{1}{2}\operatorname{tr}\mathbf{D}\right) \qquad \mathbf{H}^{\mathrm{m}}(\mathbf{D}) = h(\operatorname{tr}\mathbf{D})^4\mathbf{D}' * \mathbf{D}'$$

Conclusion – Summary

Objectives of the presentation

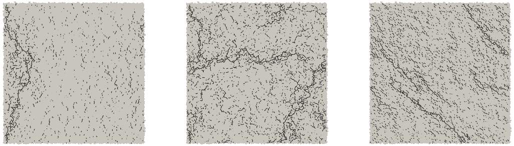
1. Gather data on the behavior of quasi-brittle material
   - Virtual testing



Conclusion – Summary

Objectives of the presentation

1. Gather data on the behavior of quasi-brittle material
   - Virtual testing



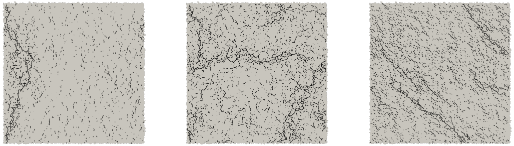
$$\mathbf{D} = \mathbf{1}_2 - \frac{1}{2\kappa_0} \operatorname{tr}_{12} \mathbf{E}$$

2. Quantify the micro-cracking
   - Definition of a damage variable

Conclusion – Summary

Objectives of the presentation

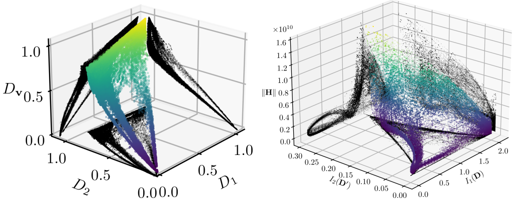
1. Gather data on the behavior of quasi-brittle material
   - Virtual testing



$$\mathbf{D} = \mathbf{1}_2 - \frac{1}{2\kappa_0} \operatorname{tr}_{12} \mathbf{E}$$

2. Quantify the micro-cracking
   - Definition of a damage variable

3. Model the impact of micro-cracking on the relation between $\boldsymbol{\varepsilon}$ and $\boldsymbol{\sigma}$
   - Exact model of $\kappa(\mathbf{D})$, $\mathbf{d}'(\mathbf{D})$
   - Model of $\mu(\mathbf{D})$, $\mathbf{H}(\mathbf{D})$



Thank you for your attention!

CFRAC 2023

Prague, Czech Republic, 21–23 June 2023.

Flavien Loiseau, Cécile Oliver-Leblond, Rodrigue Desmorat

flavien.loiseau@ens-paris-saclay.fr

Université Paris-Saclay, CentraleSupélec, ENS Paris-Saclay, CNRS, LMPS - Laboratoire de Mécanique Paris-Saclay, 91190, Gif-sur-Yvette, France.

Conclusion – References I

Backus, G. (1970). A geometrical picture of anisotropic elastic tensors. *Reviews of Geophysics*, *8*(3), 633–671. https://doi.org/10.1029/RG008i003p00633

Terrien, M. (1980). Emission acoustique et comportement mécanique post-critique d'un béton sollicité en traction. *Bulletin de liaison des laboratoires des ponts et chaussees*, *1980*(105), 65–71.

Berthaud, Y. (1991). Damage measurements in concrete via an ultrasonic technique. part i experiment. *Cement and Concrete Research*, *21*(1), 73–82. https://doi.org/10.1016/0008-8846(91)90033-E

Kachanov, M. (1992). Effective elastic properties of cracked solids: Critical review of some basic concepts [Publisher: American Society of Mechanical Engineers Digital Collection]. *Applied Mechanics Reviews*, *45*(8), 304–335. https://doi.org/10.1115/1.3119761

Blinowski, A., Ostrowska-Maciejewska, J., & Rychlewski, J. (1996). Two-dimensional hooke's tensors - isotropic decomposition, effective symmetry criteria [Number: 2]. *Archives of Mechanics*, *48*(2), 325–345. https://doi.org/10.24423/aom.1345

D'Addetta, G. A., Kun, F., & Ramm, E. (2002). On the application of a discrete model to the fracture process of cohesive granular materials. *Granular Matter*, *4*(2), 77–90. https://doi.org/10.1007/s10035-002-0103-9

Conclusion – References II

Vannucci, P. (2005). Plane Anisotropy by the Polar Method*. *Meccanica*, *40*(4), 437–454. https://doi.org/10.1007/s11012-005-2132-z

Delaplace, A. (2008). Modélisation discrète appliquée au comportement des matériaux et des structures.

Desmorat, B., & Desmorat, R. (2015). Tensorial Polar Decomposition of 2D fourth-order tensors. *Comptes Rendus Mécanique*, *343*(9), 471–475. https://doi.org/10.1016/j.crme.2015.07.002

Vassaux, M., Oliver-Leblond, C., Richard, B., & Ragueneau, F. (2016). Beam-particle approach to model cracking and energy dissipation in concrete: Identification strategy and validation. *Cement and Concrete Composites*, *70*, 1–14. https://doi.org/10.1016/j.cemconcomp.2016.03.011

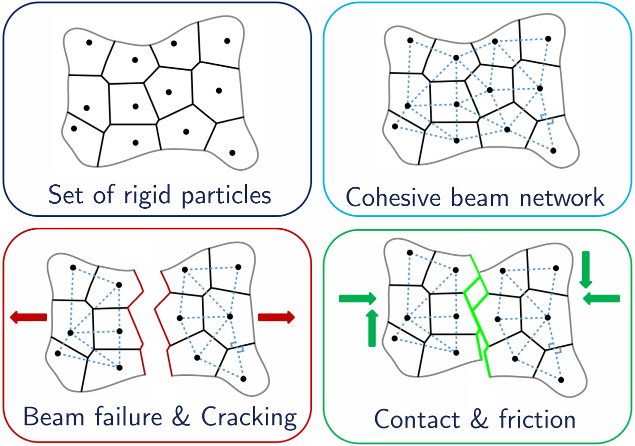
Oliver-Leblond, C. (2019). Discontinuous crack growth and toughening mechanisms in concrete: A numerical study based on the beam-particle approach. *Engineering Fracture Mechanics*, *207*, 1–22. https://doi.org/10.1016/j.engfracmech.2018.11.050

Antonelli, A., Desmorat, B., Kolev, B., & Desmorat, R. (2022). Distance to plane elasticity orthotropy by Euler–Lagrange method. *Comptes Rendus. Mécanique*, *350*(G2), 413–430. https://doi.org/10.5802/crmeca.122

Appendices

Virtual testing – Discrete (beam-particle) model

(Vassaux et al., 2016) on the basis of (D'Addetta et al., 2002), (Delaplace, 2008), ...

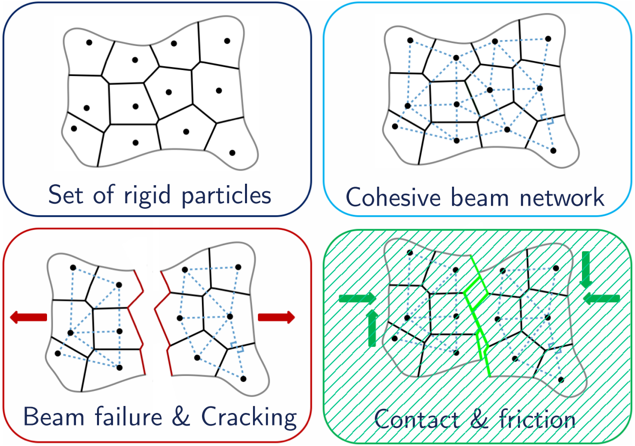


Features

- ✔ Heterogeneous
- ✔ Explicit cracking
- ✔ Accurate fracture process (Oliver-Leblond, 2019)
- ✖ Computational cost

Virtual testing – Discrete (beam-particle) model

(Vassaux et al., 2016) on the basis of (D'Addetta et al., 2002), (Delaplace, 2008), ...



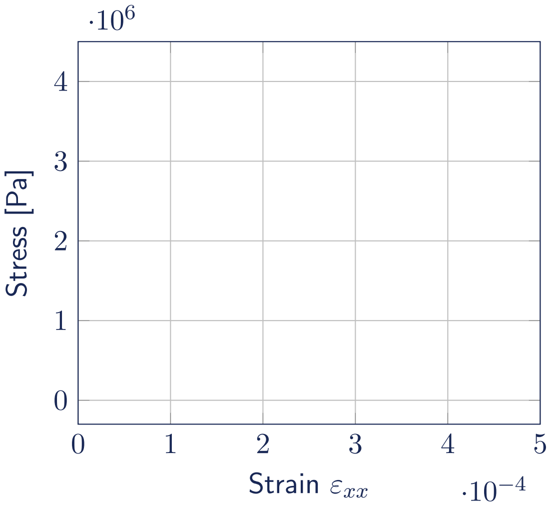
Features

- ✔ Heterogeneous
- ✔ Explicit cracking
- ✔ Accurate fracture process (Oliver-Leblond, 2019)
- ✖ Computational cost

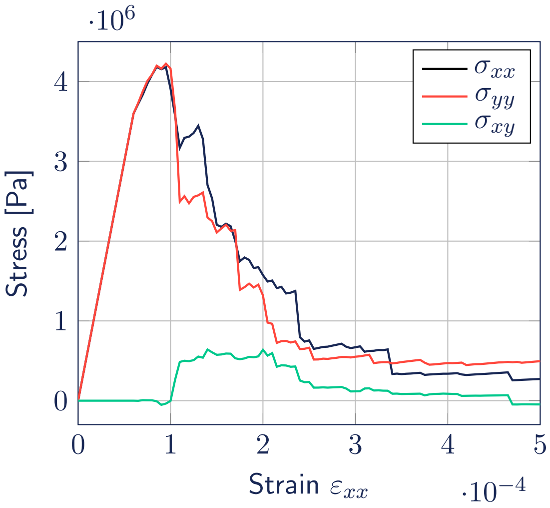
Remark

Contact/friction disabled

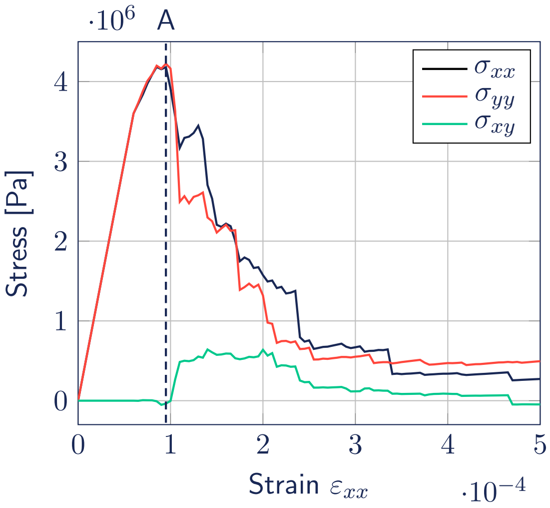
Virtual testing – Illustration of beam-particle model – Periodic bi-tension



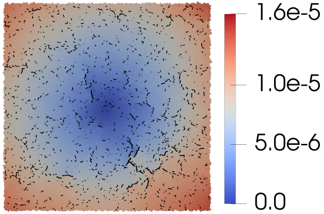
Virtual testing – Illustration of beam-particle model – Periodic bi-tension



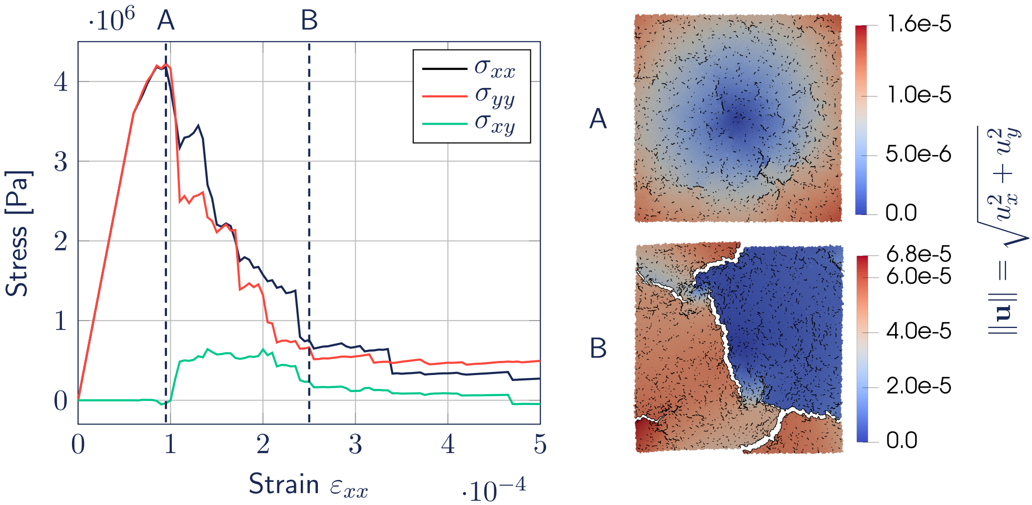
Virtual testing – Illustration of beam-particle model – Periodic bi-tension



A



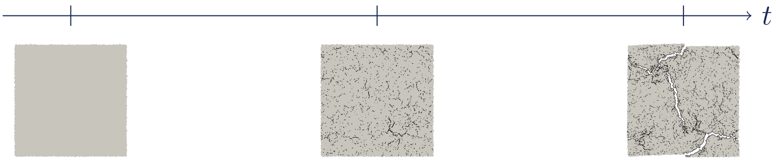
Virtual testing – Illustration of beam-particle model – Periodic bi-tension



Virtual testing – Measurement of effective elasticity tensors

Virtual testing – Measurement of effective elasticity tensors

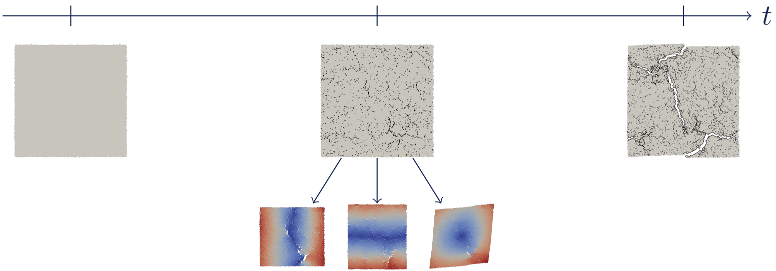
Damaging loading



Virtual testing – Measurement of effective elasticity tensors

Damaging loading

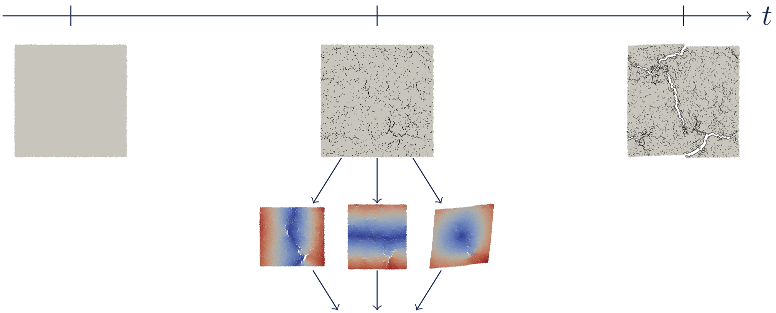
Measurement loadings



Virtual testing – Measurement of effective elasticity tensors

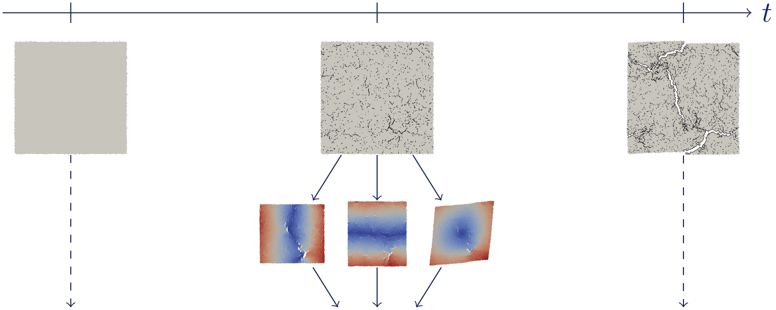
Damaging loading

Measurement loadings



$$\begin{bmatrix} 33.5 & 5.62 & -0.39 \\ 5.62 & 36.0 & 0.34 \\ -0.39 & 0.34 & 28.0 \end{bmatrix}$$

Virtual testing – Measurement of effective elasticity tensors



Damaging loading

Measurement loadings

$\mathbf{E}$ in Kelvin notation [MPa]

$$\begin{bmatrix} 49.8 & 10.2 & -0.01 \\ 10.2 & 49.8 & 0.01 \\ -0.01 & 0.01 & 38.4 \end{bmatrix} \quad \begin{bmatrix} 33.5 & 5.62 & -0.39 \\ 5.62 & 36.0 & 0.34 \\ -0.39 & 0.34 & 28.0 \end{bmatrix} \quad \begin{bmatrix} 3.82 & -1.27 & 0.14 \\ -1.27 & 3.30 & 0.82 \\ 0.14 & 0.82 & 6.84 \end{bmatrix}$$

Virtual testing – Dataset of effective elasticity tensors

Repeat procedure for

- 1 material,
- 36 particle distributions,
- 21 loadings,
- 100 time steps,

for a total of

- ≈ 76 000 tensors.

Bi-tension Shear Tension

Virtual testing – Dataset of effective elasticity tensors

Repeat procedure for

- 1 material,
- 36 particle distributions,
- 21 loadings,
- 100 time steps,

for a total of

- ≈ 76 000 tensors.

Objectives of the presentation

- ✓ Generate a dataset of effective elasticity tensors
- ○ Quantify micro-cracking (damage variable)
- ○ Formulate a state model

LASSO – Polynomial of invariants → linear relationship

The polynomial can be rewritten as a linear relationship

$$p(\mathbf{D}) = \begin{bmatrix} I_1(\mathbf{D}) & I_2(\mathbf{D}') & \dots & I_1(\mathbf{D})^{n_1} I_2(\mathbf{D}')^{n_2} \end{bmatrix} \begin{bmatrix} c_{1,0} \\ c_{0,1} \\ \vdots \\ c_{n_1,n_2} \end{bmatrix}.$$

Remark – Numerous parameters

New question – How to fit the model?

LASSO – Regression

Notations

$\mathbf{y} = y_j$ outcomes

$\mathbf{X} = (x_j)_i$ input variables

$\mathbf{c} = c_i$ coefficients

Least-square linear regression

$$\mathbf{c}^* = \underset{\mathbf{c}\in\mathbb{R}^{N_c}}{\operatorname{argmin}} \left(\frac{1}{N} \|\mathbf{y} - \mathbf{X}\cdot\mathbf{c}\|_2^2 \right)$$

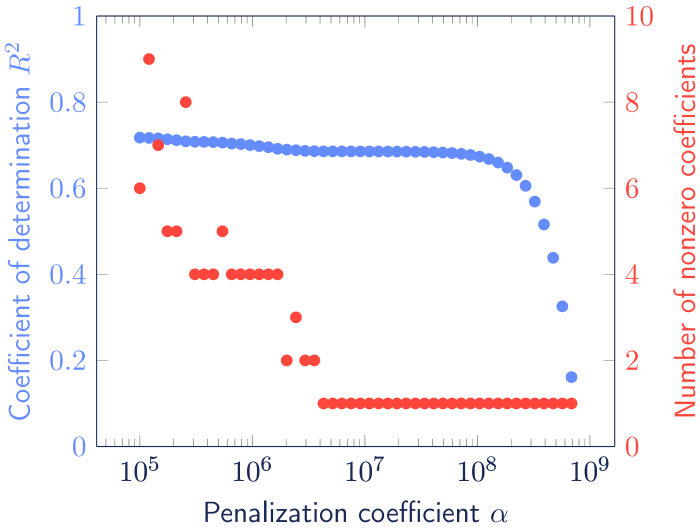
Sparse regression (LASSO)

$$\mathbf{c}^* = \underset{\mathbf{c}\in\mathbb{R}^{N_c}}{\operatorname{argmin}} \left(\frac{1}{N} \|\mathbf{y} - \mathbf{X}\cdot\mathbf{c}\|_2^2 + \alpha\|\mathbf{c}\|_1 \right)$$

Features

- Penalize nonzero parameters
- Linear convex optimization problem, easy linear constraints
- Arbitrary penalization coefficient

LASSO – Choosing the penalization coefficient ($N = 6$)




```
=== Non-zero parameters: 1 | Coefficient of determination 0.686
||H|| =
     +1.23e+10 x I_1(D) x I_1(D) x I_1(D) x I_1(D) x I_2(D')

=== Non-zero parameters: 2 | Coefficient of determination 0.690
||H|| =
     +1.29e+10 x I_1(D) x I_1(D) x I_1(D) x I_1(D) x I_2(D')
     -6.06e+09 x I_1(D) x I_1(D) x I_2(D') x I_2(D')

=== Non-zero parameters: 3 | Coefficient of determination 0.688
||H|| =
     -6.11e+08 x I_1(D) x I_1(D) x I_2(D')
     +1.29e+10 x I_1(D) x I_1(D) x I_1(D) x I_1(D) x I_2(D')
     -2.62e+09 x I_1(D) x I_1(D) x I_2(D') x I_2(D')

=== Non-zero parameters: 4 | Coefficient of determination 0.708
||H|| =
     +3.23e+10 x I_2(D')
     -3.09e+10 x I_1(D) x I_2(D')
     -2.74e+10 x I_2(D') x I_2(D')
     +1.66e+10 x I_1(D) x I_1(D) x I_1(D) x I_1(D) x I_2(D')

=== Non-zero parameters: 5 | Coefficient of determination 0.714
||H|| =
     +4.28e+10 x I_2(D')
     -3.74e+10 x I_1(D) x I_2(D')
     -8.45e+10 x I_2(D') x I_2(D')
     +1.61e+10 x I_1(D) x I_1(D) x I_1(D) x I_1(D) x I_2(D')
     +3.32e+10 x I_1(D) x I_1(D) x I_2(D') x I_2(D')

=== Non-zero parameters: 6 | Coefficient of determination 0.717
||H|| =
     +5.21e+10 x I_2(D')
     -4.20e+10 x I_1(D) x I_2(D')
     -1.48e+11 x I_2(D') x I_2(D')
     +4.55e+05 x I_1(D) x I_2(D') x I_2(D')
     +1.52e+10 x I_1(D) x I_1(D) x I_1(D) x I_1(D) x I_2(D')
     +6.97e+10 x I_1(D) x I_1(D) x I_2(D') x I_2(D')

=== Non-zero parameters: 7 | Coefficient of determination 0.716
||H|| =
     +4.69e+10 x I_2(D')
     -3.92e+10 x I_1(D) x I_2(D')
     +1.96e+05 x I_1(D) x I_1(D) x I_2(D')
     -1.14e+11 x I_2(D') x I_2(D')
     +2.16e+05 x I_1(D) x I_1(D) x I_1(D) x I_2(D')
     +1.56e+10 x I_1(D) x I_1(D) x I_1(D) x I_1(D) x I_2(D')
     +5.03e+10 x I_1(D) x I_1(D) x I_2(D') x I_2(D')
```

LASSO – Conclusion

Models for the harmonic part

The harmonic part $\mathbf{H}$ can be modelled as

$$\mathbf{H}^{\mathrm{m}}(\mathbf{D}) = +H^{\mathrm{m,i}}(\mathbf{D}) \frac{\mathbf{d}'(\mathbf{D}) * \mathbf{d}'(\mathbf{D})}{\|\mathbf{d}'(\mathbf{D}) * \mathbf{d}'(\mathbf{D})\|}$$

where $H^{\mathrm{m,i}}(\mathbf{D})$ is either

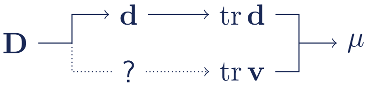
$$H^{\mathrm{m,0}}(\mathbf{D}) = 0, \text{ or} \tag{H0}$$

$$H^{\mathrm{m,1}}(\mathbf{D}) = 1.23 \cdot 10^{10} \cdot I_1(\mathbf{D})^4 \cdot I_2(\mathbf{D}') . \tag{H1}$$

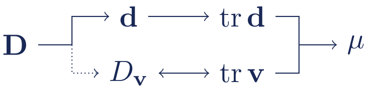
New question

How accurate is the modeling of the whole elasticity tensor?

State model – Shear modulus $\mu = \frac{1}{8}\left(2\,\mathrm{tr}\,\mathbf{v} - \mathrm{tr}\,\mathbf{d}\right)$



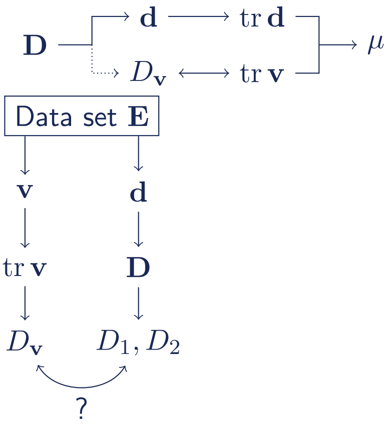
State model – Shear modulus $\mu = \frac{1}{8}\left(2\,\mathrm{tr}\,\mathbf{v} - \mathrm{tr}\,\mathbf{d}\right)$



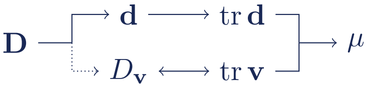
$D_{\mathbf{v}}$ such that $\mathrm{tr}\,\mathbf{v} = \mathrm{tr}\,\mathbf{v}_0(1 - D_{\mathbf{v}})$.

State model – Shear modulus $\mu = \frac{1}{8}\left(2\,\mathrm{tr}\,\mathbf{v} - \mathrm{tr}\,\mathbf{d}\right)$

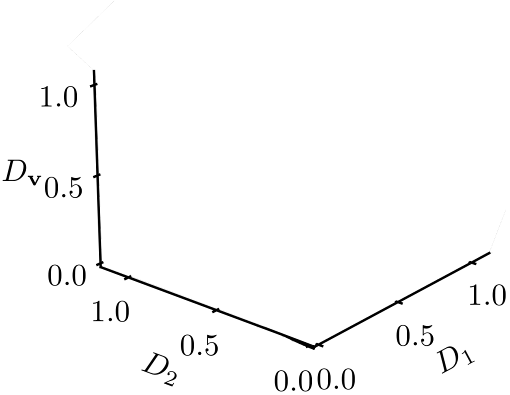
$D_{\mathbf{v}}$ such that $\mathrm{tr}\,\mathbf{v} = \mathrm{tr}\,\mathbf{v}_0(1 - D_{\mathbf{v}})$.



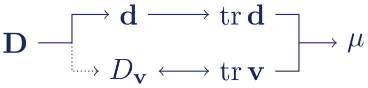
State model – Shear modulus $\mu = \frac{1}{8}\left(2\,\mathrm{tr}\,\mathbf{v} - \mathrm{tr}\,\mathbf{d}\right)$



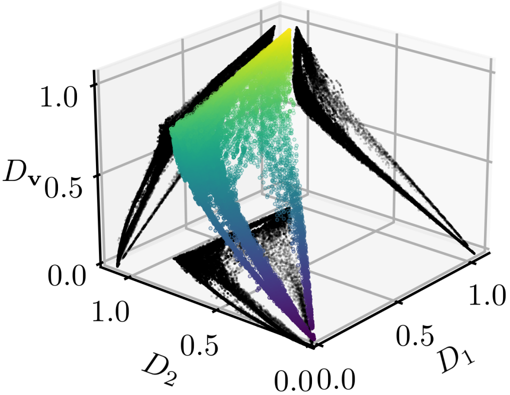
$D_{\mathbf{v}}$ such that $\mathrm{tr}\,\mathbf{v} = \mathrm{tr}\,\mathbf{v}_0(1 - D_{\mathbf{v}})$.



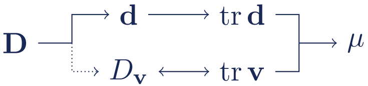
State model – Shear modulus $\mu = \frac{1}{8}\left(2\,\mathrm{tr}\,\mathbf{v} - \mathrm{tr}\,\mathbf{d}\right)$



$D_{\mathbf{v}}$ such that $\mathrm{tr}\,\mathbf{v} = \mathrm{tr}\,\mathbf{v}_0(1 - D_{\mathbf{v}})$.

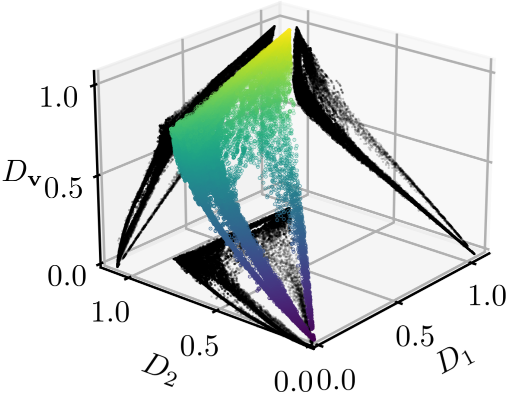


State model – Shear modulus $\mu = \frac{1}{8}\left(2\,\mathrm{tr}\,\mathbf{v} - \mathrm{tr}\,\mathbf{d}\right)$

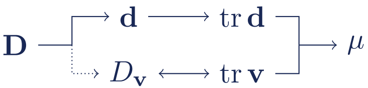


$D_{\mathbf{v}}$ such that $\mathrm{tr}\,\mathbf{v} = \mathrm{tr}\,\mathbf{v}_0(1 - D_{\mathbf{v}})$.
Let us model $D_{\mathbf{v}}$ as a function of

$$I_n(\mathbf{D}) = \mathrm{tr}(\mathbf{D}^n) = D_1^n + D_2^n$$



State model – Shear modulus $\mu = \frac{1}{8}\left(2\,\mathrm{tr}\,\mathbf{v} - \mathrm{tr}\,\mathbf{d}\right)$



$D_{\mathbf{v}}$ such that $\mathrm{tr}\,\mathbf{v} = \mathrm{tr}\,\mathbf{v}_0(1 - D_{\mathbf{v}})$.
Let us model $D_{\mathbf{v}}$ as a function of

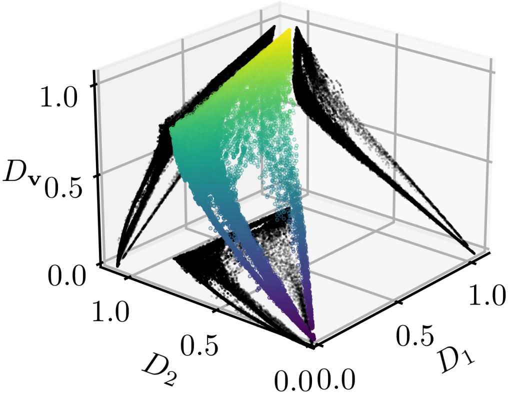
$$I_n\left(\mathbf{D}\right) = \mathrm{tr}(\mathbf{D}^n) = D_1^n + D_2^n$$

with assumptions

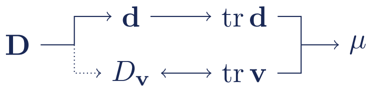
$$D_{\mathbf{v}}(\mathbf{D} = \mathbf{0}) = 0, \quad D_{\mathbf{v}}(\mathbf{D} = \mathbf{1}_2) = 1$$

$$\left.\frac{\partial D_{\mathbf{v}}}{\partial \mathbf{D}}\right|_{\mathbf{D}=\mathbf{0}} = \frac{\kappa_0}{2\mu_0 + \kappa_0}\mathbf{1}_2$$

Kachanov, 1992



State model – Shear modulus $\mu = \frac{1}{8}\left(2\,\mathrm{tr}\,\mathbf{v} - \mathrm{tr}\,\mathbf{d}\right)$



$D_{\mathbf{v}}$ such that $\mathrm{tr}\,\mathbf{v} = \mathrm{tr}\,\mathbf{v}_0(1 - D_{\mathbf{v}})$.
Let us model $D_{\mathbf{v}}$ as a function of

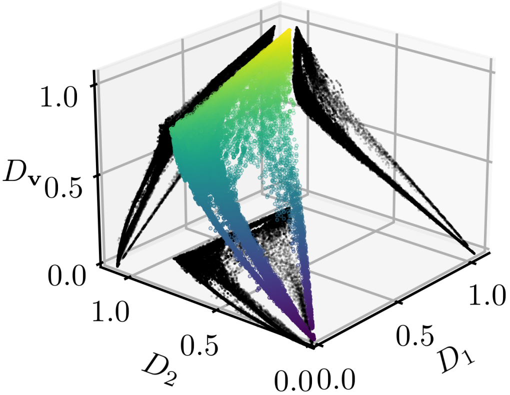
$$I_n(\mathbf{D}) = \mathrm{tr}(\mathbf{D}^n) = D_1^n + D_2^n$$

with assumptions

$$D_{\mathbf{v}}(\mathbf{D} = \mathbf{0}) = 0, \quad D_{\mathbf{v}}(\mathbf{D} = \mathbf{1}_2) = 1$$

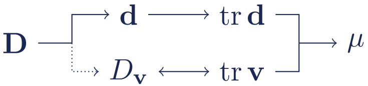
$$\left.\frac{\partial D_{\mathbf{v}}}{\partial \mathbf{D}}\right|_{\mathbf{D}=\mathbf{0}} = \frac{\kappa_0}{2\mu_0 + \kappa_0}\mathbf{1}_2$$

Kachanov, 1992



With 2 invariants $\quad D_{\mathbf{v}}^{\mathrm{m}} = c_1 I_1(\mathbf{D}) + c_2 I_2(\mathbf{D})$

State model – Shear modulus $\mu = \frac{1}{8}\left(2\,\mathrm{tr}\,\mathbf{v} - \mathrm{tr}\,\mathbf{d}\right)$



$D_{\mathbf{v}}$ such that $\mathrm{tr}\,\mathbf{v} = \mathrm{tr}\,\mathbf{v}_0(1 - D_{\mathbf{v}})$.
Let us model $D_{\mathbf{v}}$ as a function of

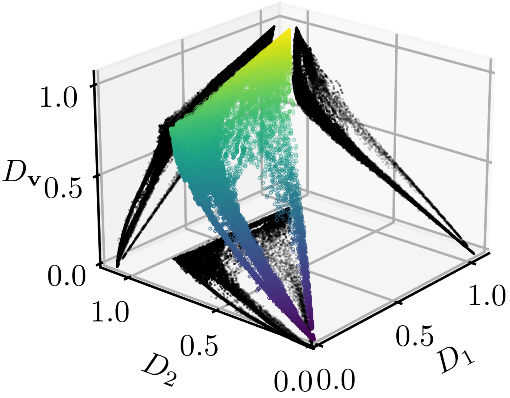
$$I_n\left(\mathbf{D}\right) = \mathrm{tr}(\mathbf{D}^n) = D_1^n + D_2^n$$

with assumptions

$$D_{\mathbf{v}}(\mathbf{D} = \mathbf{0}) = 0, \quad D_{\mathbf{v}}(\mathbf{D} = \mathbf{1}_2) = 1$$

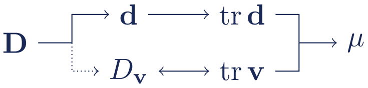
$$\left.\frac{\partial D_{\mathbf{v}}}{\partial \mathbf{D}}\right|_{\mathbf{D}=\mathbf{0}} = \frac{\kappa_0}{2\mu_0 + \kappa_0}\mathbf{1}_2$$

Kachanov, 1992



With 2 invariants $D_{\mathbf{v}}^{\mathrm{m}} = c_1 I_1\left(\mathbf{D}\right) + c_2 I_2\left(\mathbf{D}\right) \implies c_1 = \frac{\kappa_0}{2\mu_0 + \kappa_0},\ c_2 = \frac{1}{2} - c_1.$

State model – Shear modulus $\mu = \frac{1}{8}\left(2\,\mathrm{tr}\,\mathbf{v} - \mathrm{tr}\,\mathbf{d}\right)$



$D_{\mathbf{v}}$ such that $\mathrm{tr}\,\mathbf{v} = \mathrm{tr}\,\mathbf{v}_0(1 - D_{\mathbf{v}})$.
Let us model $D_{\mathbf{v}}$ as a function of

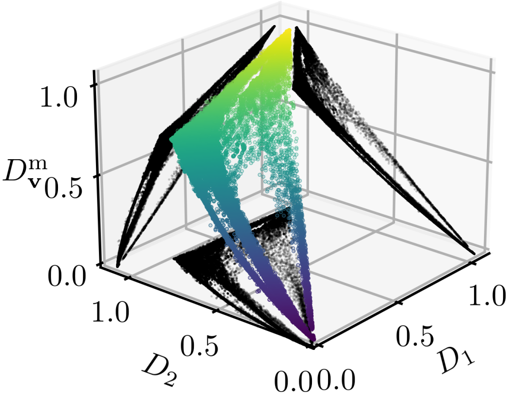
$$I_n\left(\mathbf{D}\right) = \mathrm{tr}(\mathbf{D}^n) = D_1^n + D_2^n$$

with assumptions

$$D_{\mathbf{v}}(\mathbf{D} = \mathbf{0}) = 0, \quad D_{\mathbf{v}}(\mathbf{D} = \mathbf{1}_2) = 1$$

$$\left.\frac{\partial D_{\mathbf{v}}}{\partial \mathbf{D}}\right|_{\mathbf{D}=\mathbf{0}} = \frac{\kappa_0}{2\mu_0 + \kappa_0}\mathbf{1}_2$$

Kachanov, 1992



With 2 invariants $D_{\mathbf{v}}^{\mathrm{m}} = c_1 I_1\left(\mathbf{D}\right) + c_2 I_2\left(\mathbf{D}\right) \implies c_1 = \dfrac{\kappa_0}{2\mu_0 + \kappa_0},\ c_2 = \dfrac{1}{2} - c_1.$

State model – Partial formulation

Knowing isotropic $\mathbf{E}_0 \cong (\mu_0, \kappa_0, \mathbf{0}_2, \mathbf{0}_4)$ and $\mathbf{D}$, elasticity tensor $\mathbf{E}$ can be modelled as

$$\mathbf{E}(\mathbf{D}) = 2\mu(\mathbf{D})\mathbf{J} + \kappa(\mathbf{D})\mathbf{1}_2 \otimes \mathbf{1}_2 + \frac{1}{2}\left(\mathbf{d}'(\mathbf{D}) \otimes \mathbf{1}_2 + \mathbf{1}_2 \otimes \mathbf{d}'(\mathbf{D})\right) + \mathbf{H}(\mathbf{D})$$

where the invariants and covariants models are

$$\mu(\mathbf{D}) = \mu_0 - \frac{1}{4}\kappa_0(\operatorname{tr}\mathbf{D}) + \frac{1}{4}(\kappa_0 - 2\mu_0)(\mathbf{D}:\mathbf{D}) \qquad \mathbf{d}'(\mathbf{D}) = -2\kappa_0\mathbf{D}'$$

$$\kappa(\mathbf{D}) = \kappa_0\left(1 - \frac{1}{2}\operatorname{tr}\mathbf{D}\right) \qquad \mathbf{H}(\mathbf{D}) = \mathbf{E} - \mathbf{Iso} - \mathbf{Dil}$$

Questions How to model ✓ shear modulus $\mu(\mathbf{D})$? ○ harmonic part $\mathbf{H}(\mathbf{D})$?

State model – Harmonic part – Parametrization

How to parametrize the harmonic part?
(Vannucci, 2005) (Desmorat & Desmorat, 2015)

$$\text{Orthotropy} \implies \mathbf{H} = \|\mathbf{H}\| \left(\pm \frac{\mathbf{d}' * \mathbf{d}'}{\|\mathbf{d}' * \mathbf{d}'\|} \right)$$

where $\mathbf{d}' * \mathbf{d}' = \mathbf{d}' \otimes \mathbf{d}' - \frac{1}{2}(\mathbf{d}':\mathbf{d}')\mathbf{J}$.

State model – Harmonic part – Parametrization

How to parametrize the harmonic part?
(Vannucci, 2005) (Desmorat & Desmorat, 2015)

$$\text{Orthotropy} \implies \mathbf{H} = \|\mathbf{H}\| \left(\pm \frac{\mathbf{d}' * \mathbf{d}'}{\|\mathbf{d}' * \mathbf{d}'\|} \right)$$

where $\mathbf{d}' * \mathbf{d}' = \mathbf{d}' \otimes \mathbf{d}' - \frac{1}{2}(\mathbf{d}':\mathbf{d}')\mathbf{J}$.

Questions

- Model orientation ($\pm$)?
- Model norm $H(\mathbf{D}) = \|\mathbf{H}\|$?

State model – Harmonic part – Parametrization

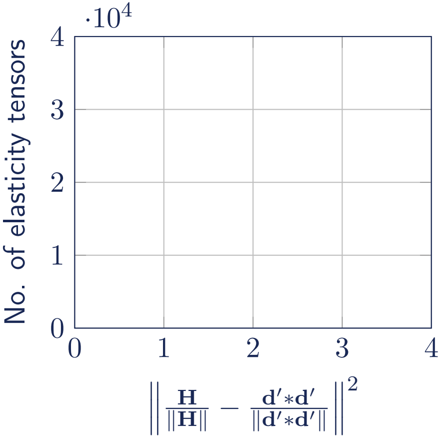
How to parametrize the harmonic part?
(Vannucci, 2005) (Desmorat & Desmorat, 2015)

Orthotropy $\implies \mathbf{H} = \|\mathbf{H}\| \left(\pm \frac{\mathbf{d}' * \mathbf{d}'}{\|\mathbf{d}' * \mathbf{d}'\|} \right)$

where $\mathbf{d}' * \mathbf{d}' = \mathbf{d}' \otimes \mathbf{d}' - \frac{1}{2}(\mathbf{d}' : \mathbf{d}')\mathbf{J}$.

Questions

- Model orientation $(\pm)$?
- Model norm $H(\mathbf{D}) = \|\mathbf{H}\|$?



State model – Harmonic part – Parametrization

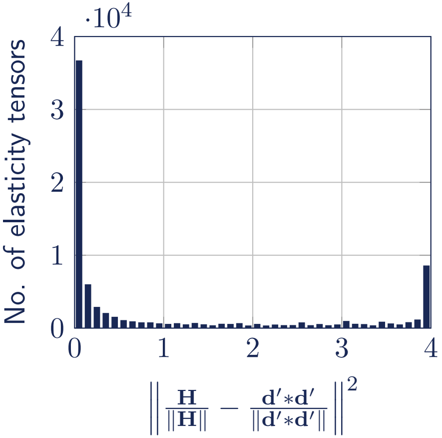
How to parametrize the harmonic part?
(Vannucci, 2005) (Desmorat & Desmorat, 2015)

$$\text{Orthotropy} \implies \mathbf{H} = \|\mathbf{H}\| \left(\pm \frac{\mathbf{d}' * \mathbf{d}'}{\|\mathbf{d}' * \mathbf{d}'\|} \right)$$

where $\mathbf{d}' * \mathbf{d}' = \mathbf{d}' \otimes \mathbf{d}' - \frac{1}{2}(\mathbf{d}' : \mathbf{d}')\mathbf{J}$.

Questions

- Model orientation ($\pm$)?
- Model norm $H(\mathbf{D}) = \|\mathbf{H}\|$?



State model – Harmonic part – Parametrization

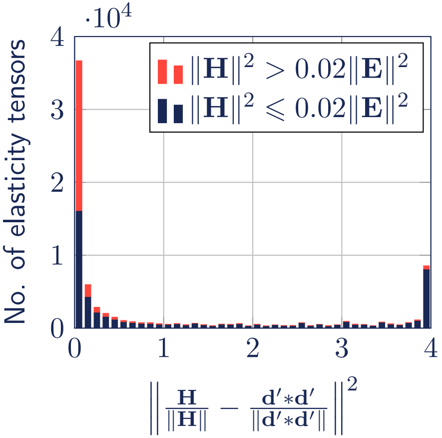
How to parametrize the harmonic part?
(Vannucci, 2005) (Desmorat & Desmorat, 2015)

$$\text{Orthotropy} \implies \mathbf{H} = \|\mathbf{H}\| \left(\pm \frac{\mathbf{d}' * \mathbf{d}'}{\|\mathbf{d}' * \mathbf{d}'\|} \right)$$

where $\mathbf{d}' * \mathbf{d}' = \mathbf{d}' \otimes \mathbf{d}' - \frac{1}{2}(\mathbf{d}':\mathbf{d}')\mathbf{J}$.

Questions

- Model orientation $(\pm)$?
- Model norm $H(\mathbf{D}) = \|\mathbf{H}\|$?



State model – Harmonic part – Parametrization

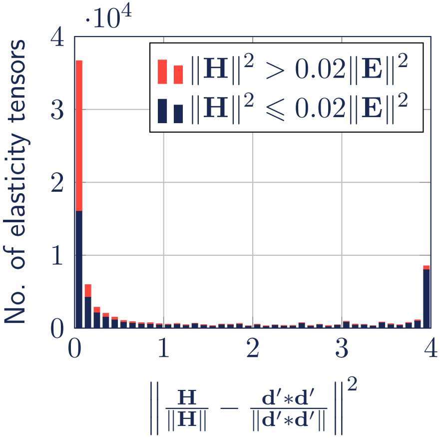
How to parametrize the harmonic part?
(Vannucci, 2005) (Desmorat & Desmorat, 2015)

$$\text{Orthotropy} \implies \mathbf{H} = \|\mathbf{H}\| \left(+ \frac{\mathbf{d}' * \mathbf{d}'}{\|\mathbf{d}' * \mathbf{d}'\|} \right)$$

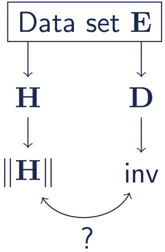
where $\mathbf{d}' * \mathbf{d}' = \mathbf{d}' \otimes \mathbf{d}' - \frac{1}{2}(\mathbf{d}':\mathbf{d}')\mathbf{J}$.

Questions

- ✓ Model orientation ($\pm$)?
- ○ Model norm $H(\mathbf{D}) = \|\mathbf{H}\|$?



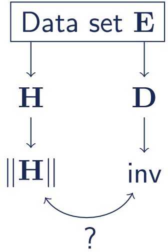
State model – Harmonic part – Norm modelling $H^{\mathrm{m}} : \mathbf{D} \mapsto H^{\mathrm{m}}(\mathbf{D}) \approx \|\mathbf{H}\|$?



State model – Harmonic part – Norm modelling $H^{\mathrm{m}} : \mathbf{D} \mapsto H^{\mathrm{m}}(\mathbf{D}) \approx \|\mathbf{H}\|$?

Invariants

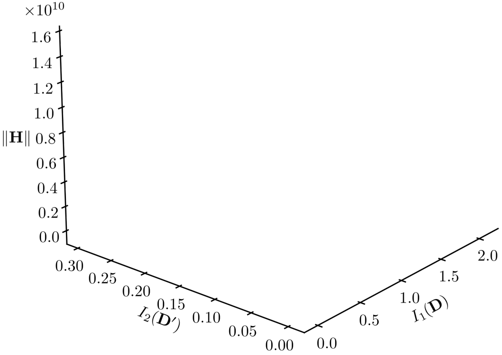
$$I_1\left(\mathbf{D}\right) = \mathrm{tr}(\mathbf{D}) \quad I_2\left(\mathbf{D}'\right) = \mathbf{D}':\mathbf{D}'$$



State model – Harmonic part – Norm modelling $H^{\mathrm{m}} : \mathbf{D} \mapsto H^{\mathrm{m}}(\mathbf{D}) \approx \|\mathbf{H}\|$?

Invariants

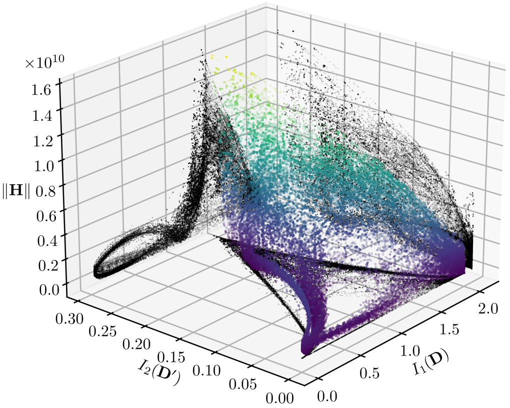
$$I_1\left(\mathbf{D}\right) = \mathrm{tr}(\mathbf{D}) \quad I_2\left(\mathbf{D}'\right) = \mathbf{D}':\mathbf{D}'$$



State model – Harmonic part – Norm modelling $H^{\mathrm{m}} : \mathbf{D} \mapsto H^{\mathrm{m}}(\mathbf{D}) \approx \|\mathbf{H}\|$?

Invariants

$$I_1\left(\mathbf{D}\right) = \mathrm{tr}(\mathbf{D}) \quad I_2\left(\mathbf{D}'\right) = \mathbf{D}'{:}\mathbf{D}'$$



State model – Harmonic part – Norm modelling $H^{\mathrm{m}} : \mathbf{D} \mapsto H^{\mathrm{m}}(\mathbf{D}) \approx \|\mathbf{H}\|$?

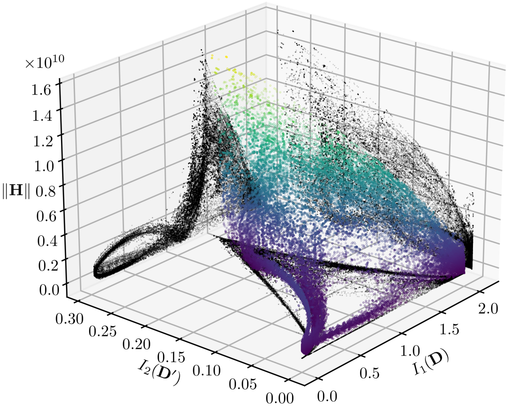
Invariants

$$I_1(\mathbf{D}) = \mathrm{tr}(\mathbf{D}) \quad I_2(\mathbf{D}') = \mathbf{D}':\mathbf{D}'$$

Assumptions

$$H^{\mathrm{m}}(\mathbf{D} = \mathbf{0}_2) = \mathbf{0}_4 \qquad \text{(Initial isotropy)}$$

$$H^{\mathrm{m}}(\mathbf{D} = \mathbf{1}_2) = \mathbf{0}_4 \qquad \text{(Fully damaged)}$$



State model – Harmonic part – Norm modelling $H^{\mathrm{m}} : \mathbf{D} \mapsto H^{\mathrm{m}}(\mathbf{D}) \approx \|\mathbf{H}\|$?

Invariants

$$I_1(\mathbf{D}) = \mathrm{tr}(\mathbf{D}) \quad I_2(\mathbf{D}') = \mathbf{D}':\mathbf{D}'$$

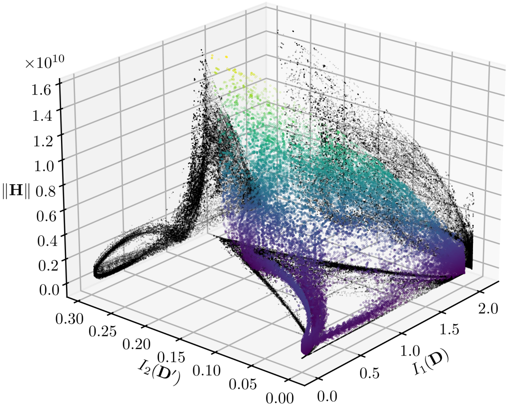
Assumptions

$$H^{\mathrm{m}}(\mathbf{D} = \mathbf{0}_2) = \mathbf{0}_4 \qquad \text{(Initial isotropy)}$$

$$H^{\mathrm{m}}(\mathbf{D} = \mathbf{1}_2) = \mathbf{0}_4 \qquad \text{(Fully damaged)}$$

Proposition – Polynomial of invariants

$$H^{\mathrm{m}}(\mathbf{D}) = \sum_{n,m} c_{n,m} \, I_1(\mathbf{D})^n \cdot I_2(\mathbf{D}')^m$$



State model – Harmonic part – Norm modelling $H^{\mathrm{m}} : \mathbf{D} \mapsto H^{\mathrm{m}}(\mathbf{D}) \approx \|\mathbf{H}\|$?

Invariants

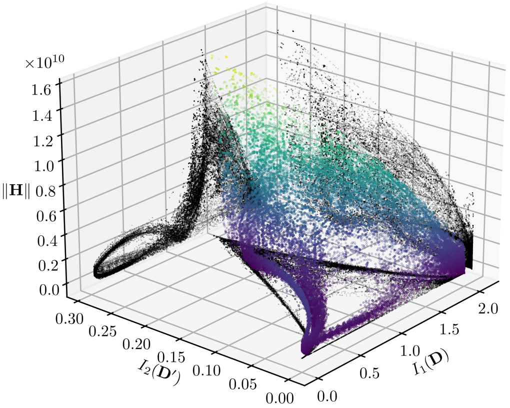
$$I_1(\mathbf{D}) = \mathrm{tr}(\mathbf{D}) \quad I_2(\mathbf{D}') = \mathbf{D}':\mathbf{D}'$$

Assumptions

$$H^{\mathrm{m}}(\mathbf{D} = \mathbf{0}_2) = \mathbf{0}_4 \qquad \text{(Initial isotropy)}$$
$$H^{\mathrm{m}}(\mathbf{D} = \mathbf{1}_2) = \mathbf{0}_4 \qquad \text{(Fully damaged)}$$

Proposition – Polynomial of invariants

$$H^{\mathrm{m}}(\mathbf{D}) = \sum_{n,m} c_{n,m}\, I_1(\mathbf{D})^n \cdot I_2(\mathbf{D}')^m$$



Sparse regression $\implies H^{\mathrm{m}}(\mathbf{D}) = 12 \cdot 10^9 \cdot I_1(\mathbf{D})^4 \cdot I_2(\mathbf{D}')$

Harmonic part – Harmonic part – Parametrization

Based on (Vannucci, 2005) and (Desmorat & Desmorat, 2015), intuition from (Kachanov, 1992).

How to parametrize the harmonic part?

Harmonic part – Harmonic part – Parametrization

Based on (Vannucci, 2005) and (Desmorat & Desmorat, 2015), intuition from (Kachanov, 1992).

How to parametrize the harmonic part?

Anisotropy $$\mathbf{H} = \|\mathbf{H}\|(\pm \underbrace{\mathbf{e}'_1 * \mathbf{e}'_1}_{\mathbf{e}'_1 \otimes \mathbf{e}'_1 - \frac{1}{2}(\mathbf{e}'_1 : \mathbf{e}'_1)\mathbf{J}})$$

Harmonic part – Harmonic part – Parametrization

Based on (Vannucci, 2005) and (Desmorat & Desmorat, 2015), intuition from (Kachanov, 1992).

How to parametrize the harmonic part?

Anisotropy $\mathbf{H} = \|\mathbf{H}\|(\pm \underbrace{\mathbf{e}'_1 * \mathbf{e}'_1}_{\mathbf{e}'_1 \otimes \mathbf{e}'_1 - \frac{1}{2}(\mathbf{e}'_1 : \mathbf{e}'_1)\mathbf{J}})$

Orthotropy $\mathbf{H} = \|\mathbf{H}\| \left(\pm \dfrac{\mathbf{d}' * \mathbf{d}'}{\|\mathbf{d}' * \mathbf{d}'\|} \right)$

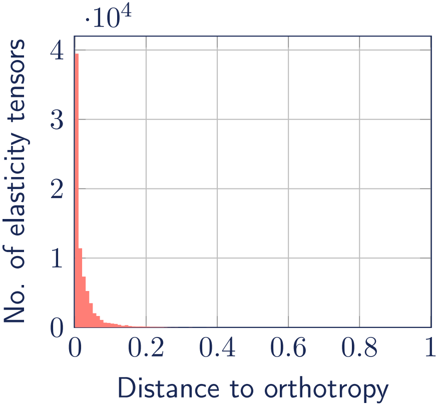
Harmonic part – Harmonic part – Parametrization

Based on (Vannucci, 2005) and (Desmorat & Desmorat, 2015), intuition from (Kachanov, 1992).

How to parametrize the harmonic part?

Anisotropy $$\mathbf{H} = \|\mathbf{H}\|(\pm \underbrace{\mathbf{e}'_1 * \mathbf{e}'_1}_{\mathbf{e}'_1 \otimes \mathbf{e}'_1 - \frac{1}{2}(\mathbf{e}'_1 : \mathbf{e}'_1)\mathbf{J}})$$

Orthotropy $$\mathbf{H} = \|\mathbf{H}\| \left(\pm \frac{\mathbf{d}' * \mathbf{d}'}{\|\mathbf{d}' * \mathbf{d}'\|} \right)$$



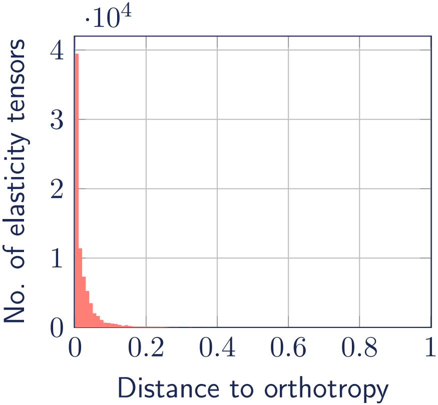
Harmonic part – Harmonic part – Parametrization

Based on (Vannucci, 2005) and (Desmorat & Desmorat, 2015), intuition from (Kachanov, 1992).

How to parametrize the harmonic part?

Anisotropy $$\mathbf{H} = \|\mathbf{H}\|(\pm \underbrace{\mathbf{e}'_1 * \mathbf{e}'_1}_{\mathbf{e}'_1 \otimes \mathbf{e}'_1 - \frac{1}{2}(\mathbf{e}'_1 : \mathbf{e}'_1)\mathbf{J}})$$

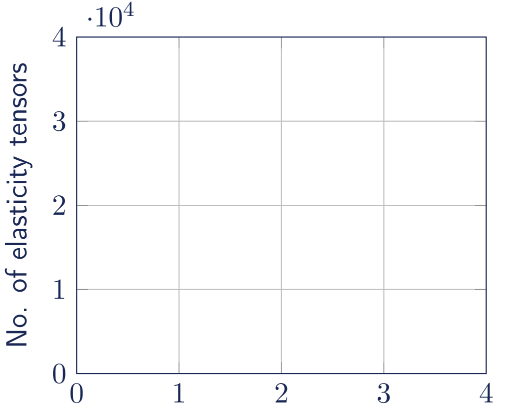
Orthotropy $$\mathbf{H} = \|\mathbf{H}\| \left(\pm \frac{\mathbf{d}' * \mathbf{d}'}{\|\mathbf{d}' * \mathbf{d}'\|} \right)$$



Questions

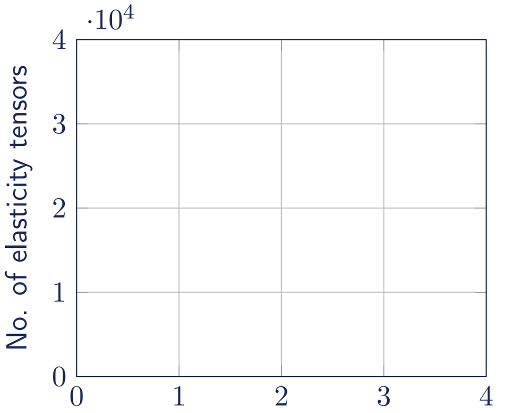
- Model orientation ($\pm$)?
- Model norm $H(\mathbf{D}) = \|\mathbf{H}\|$?

Harmonic part – Orientation of the harmonic part



Orientation $O_+(\mathbf{E}) = \left\| \frac{\mathbf{H}}{\|\mathbf{H}\|} - \frac{\mathbf{d}'*\mathbf{d}'}{\|\mathbf{d}'*\mathbf{d}'\|} \right\|^2$

Harmonic part – Orientation of the harmonic part

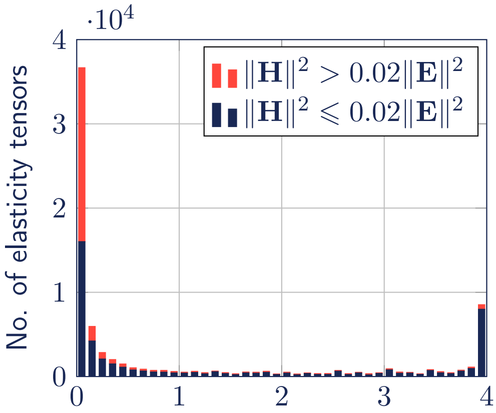


Orientation $O_+(\mathbf{E}) = \left\| \frac{\mathbf{H}}{\|\mathbf{H}\|} - \frac{\mathbf{d}' * \mathbf{d}'}{\|\mathbf{d}' * \mathbf{d}'\|} \right\|^2$

Indications

$$O_+(\mathbf{E}) = 0 \iff \mathbf{H} \propto +\mathbf{d}' * \mathbf{d}'$$
$$O_+(\mathbf{E}) = 4 \iff \mathbf{H} \propto -\mathbf{d}' * \mathbf{d}'$$

Harmonic part – Orientation of the harmonic part



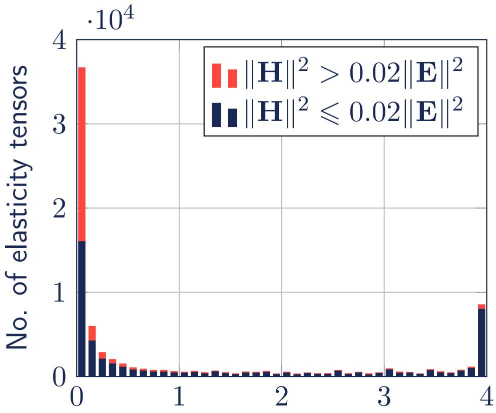
Orientation $O_+(\mathbf{E}) = \left\| \frac{\mathbf{H}}{\|\mathbf{H}\|} - \frac{\mathbf{d}' * \mathbf{d}'}{\|\mathbf{d}' * \mathbf{d}'\|} \right\|^2$

Indications

$$O_+(\mathbf{E}) = 0 \iff \mathbf{H} \propto +\mathbf{d}' * \mathbf{d}'$$

$$O_+(\mathbf{E}) = 4 \iff \mathbf{H} \propto -\mathbf{d}' * \mathbf{d}'$$

Harmonic part – Orientation of the harmonic part



Orientation $O_+(\mathbf{E}) = \left\| \frac{\mathbf{H}}{\|\mathbf{H}\|} - \frac{\mathbf{d}' * \mathbf{d}'}{\|\mathbf{d}' * \mathbf{d}'\|} \right\|^2$

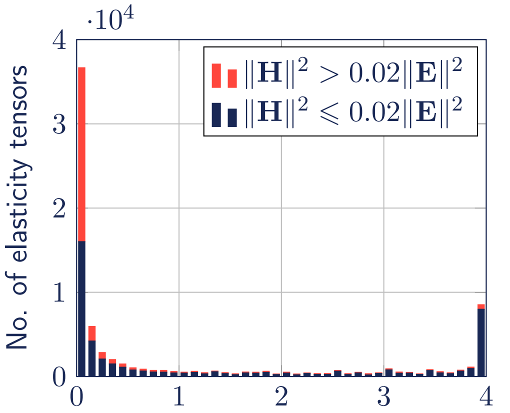
Indications

$$O_+(\mathbf{E}) = 0 \iff \mathbf{H} \propto +\mathbf{d}' * \mathbf{d}'$$
$$O_+(\mathbf{E}) = 4 \iff \mathbf{H} \propto -\mathbf{d}' * \mathbf{d}'$$

Conclusion

$$\mathbf{H}^{\mathrm{m}}(\mathbf{D}) = +H^{\mathrm{m}}(\mathbf{D}) \, \frac{\mathbf{d}'(\mathbf{D}) * \mathbf{d}'(\mathbf{D})}{\|\mathbf{d}'(\mathbf{D}) * \mathbf{d}'(\mathbf{D})\|}$$

Harmonic part – Orientation of the harmonic part



Orientation $O_+(\mathbf{E}) = \left\| \frac{\mathbf{H}}{\|\mathbf{H}\|} - \frac{\mathbf{d}' * \mathbf{d}'}{\|\mathbf{d}' * \mathbf{d}'\|} \right\|^2$

Indications

$$O_+(\mathbf{E}) = 0 \iff \mathbf{H} \propto +\mathbf{d}' * \mathbf{d}'$$
$$O_+(\mathbf{E}) = 4 \iff \mathbf{H} \propto -\mathbf{d}' * \mathbf{d}'$$

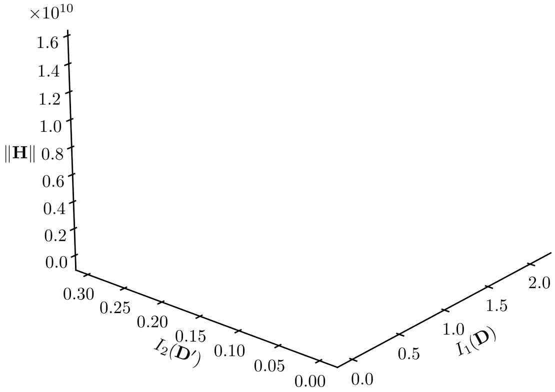
Conclusion

$$\mathbf{H}^{\mathrm{m}}(\mathbf{D}) = +H^{\mathrm{m}}(\mathbf{D}) \, \frac{\mathbf{d}'(\mathbf{D}) * \mathbf{d}'(\mathbf{D})}{\|\mathbf{d}'(\mathbf{D}) * \mathbf{d}'(\mathbf{D})\|}$$

Questions

- ✓ Model orientation $(\pm)$?
- ○ Model norm $H(\mathbf{D}) = \|\mathbf{H}\|$?

Harmonic part – Modeling the norm $H^{\mathrm{m}}(\mathbf{D}) \approx \|\mathbf{H}\|$?



Invariants

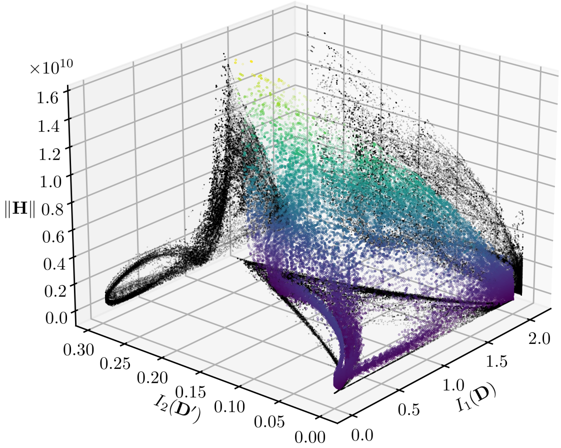
$$I_1(\mathbf{D}) = \mathrm{tr}(\mathbf{D})$$

$$I_2(\mathbf{D}') = \mathbf{D}':\mathbf{D}'$$

Remarks

- Initial isotropy
  $$\mathbf{D} = \mathbf{0}_2 \implies \|\mathbf{H}\| = \mathbf{0}_4$$
- $\mathbf{E} = \mathbf{0}_4$ when fully damaged
  $$\mathbf{D} = \mathbf{1}_2 \implies \|\mathbf{H}\| = \mathbf{0}_4$$

Harmonic part – Modeling the norm $H^{\mathrm{m}}(\mathbf{D}) \approx \|\mathbf{H}\|$?



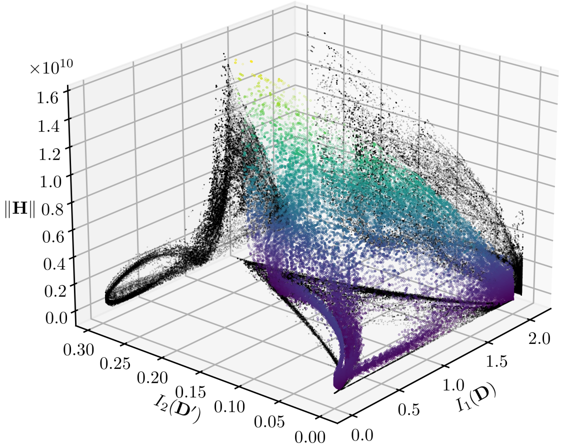
Invariants

$$I_1(\mathbf{D}) = \mathrm{tr}(\mathbf{D})$$
$$I_2(\mathbf{D}') = \mathbf{D}' : \mathbf{D}'$$

Remarks

- Initial isotropy
  $$\mathbf{D} = \mathbf{0}_2 \implies \|\mathbf{H}\| = \mathbf{0}_4$$
- $\mathbf{E} = \mathbf{0}_4$ when fully damaged
  $$\mathbf{D} = \mathbf{1}_2 \implies \|\mathbf{H}\| = \mathbf{0}_4$$

Harmonic part – Modeling the norm $H^{\mathrm{m}}(\mathbf{D}) \approx \|\mathbf{H}\|$?



Invariants

$$I_1(\mathbf{D}) = \mathrm{tr}(\mathbf{D})$$
$$I_2(\mathbf{D}') = \mathbf{D}':\mathbf{D}'$$

Remarks

- Initial isotropy
  $$\mathbf{D} = \mathbf{0}_2 \implies \|\mathbf{H}\| = \mathbf{0}_4$$
- $\mathbf{E} = \mathbf{0}_4$ when fully damaged
  $$\mathbf{D} = \mathbf{1}_2 \implies \|\mathbf{H}\| = \mathbf{0}_4$$

Conclusion

$H^{\mathrm{m}}(\mathbf{D}) \approx \|\mathbf{H}\|$ should be possible

Harmonic part – Modeling the norm $H^{\mathrm{m}}(\mathbf{D}) \approx \|\mathbf{H}\|$

Proposition

Polynomial of invariants

$$H^{\mathrm{m}}(\mathbf{D}) = \sum_{n,m} c_{n,m} \; I_1 \left(\mathbf{D}\right)^n \cdot I_2 \left(\mathbf{D}'\right)^m$$

Harmonic part – Modeling the norm $H^{\mathrm{m}}(\mathbf{D}) \approx \|\mathbf{H}\|$

Proposition

Polynomial of invariants

$$H^{\mathrm{m}}(\mathbf{D}) = \sum_{n,m} c_{n,m} \; I_1\left(\mathbf{D}\right)^n \cdot I_2\left(\mathbf{D}'\right)^m$$

⚠ Large number of parameters $c_{n,m}$

Harmonic part – Modeling the norm $H^{\mathrm{m}}(\mathbf{D}) \approx \|\mathbf{H}\|$

Proposition

Polynomial of invariants

$$H^{\mathrm{m}}(\mathbf{D}) = \sum_{n,m} c_{n,m} \; I_1\left(\mathbf{D}\right)^n \cdot I_2\left(\mathbf{D}'\right)^m$$

⚠ Large number of parameters $c_{n,m}$

Tool Sparse regression (LASSO)

$$\mathbf{c}^* = \underset{\mathbf{c}\,\in\mathbb{R}^N}{\operatorname{argmin}} \left\| [\|\mathbf{H}\|] - \mathbf{c} \cdot \mathbf{Inv} \right\|_2$$

subject to $\|\mathbf{c}\|_1 < \alpha$

- Constrained linear regression
- Minimize number of non zero coeff
- Introduces a hyper-parameter α

Harmonic part – Modeling the norm $H^{\mathrm{m}}(\mathbf{D}) \approx \|\mathbf{H}\|$

Proposition

Polynomial of invariants

$$H^{\mathrm{m}}(\mathbf{D}) = \sum_{n,m} c_{n,m}\ I_1\left(\mathbf{D}\right)^n \cdot I_2\left(\mathbf{D}'\right)^m$$

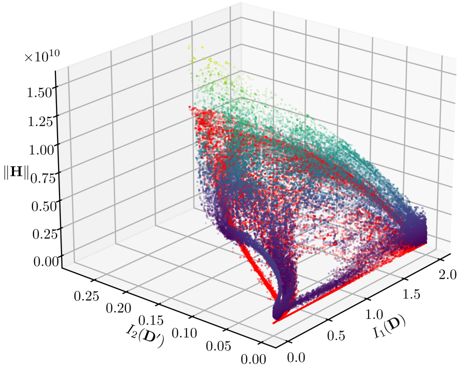
⚠ Large number of parameters $c_{n,m}$

Tool Sparse regression (LASSO)

$$\mathbf{c}^* = \underset{\mathbf{c}\ \in \mathbb{R}^N}{\operatorname{argmin}} \left\| [\|\mathbf{H}\|] - \mathbf{c} \cdot \mathbf{Inv} \right\|_2$$

subject to $\|\mathbf{c}\|_1 < \alpha$

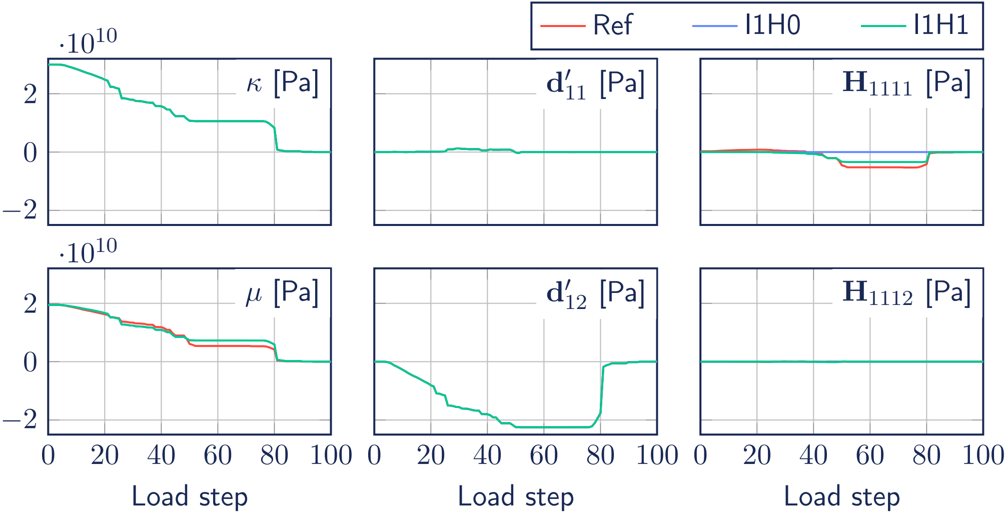
- Constrained linear regression
- Minimize number of non zero coeff
- Introduces a hyper-parameter α



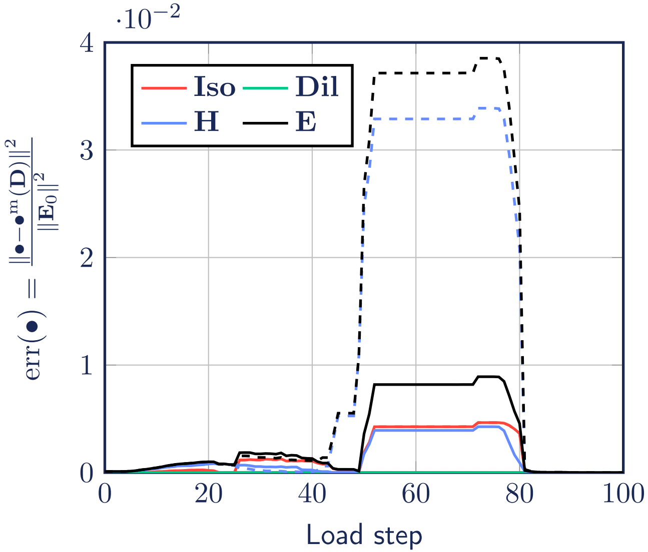
Resulting model

$$H^{\mathrm{m}}(\mathbf{D}) = 12 \cdot 10^9 \cdot I_1\left(\mathbf{D}\right)^4 \cdot I_2\left(\mathbf{D}'\right)$$

Verifications – Application to a shear → tension loading



Verifications – Application to a shear → tension loading



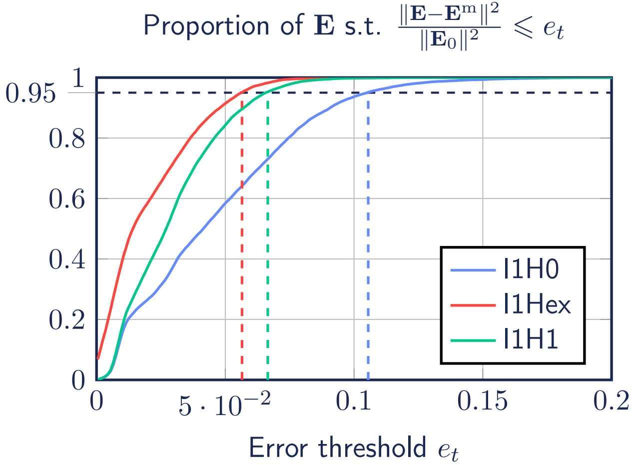
Legend

- I1H0 – Dashed
- I1H1 – Continuous

Partial conclusions

- H1 increase accuracy
- $\text{err}(\mathbf{Iso}) \approx \text{err}(\mathbf{H})$

Verifications – Over the whole dataset



Conclusion

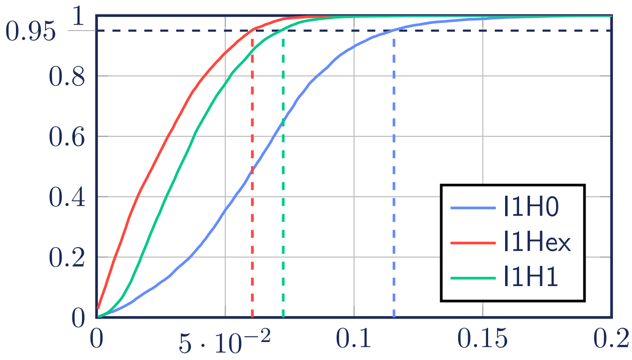
- H1 increases accuracy

Question

Accuracy at high damage ?

Verifications – For highly damaged tensors ($D_1 > 0.8$ or $D_2 > 0.8$)

Proportion of $\mathbf{E}$ s.t. $\frac{\|\mathbf{E}-\mathbf{E}^{\mathrm{m}}\|^2}{\|\mathbf{E}_0\|^2} \leqslant e_t$



Error threshold e_t

Conclusion

- Keep same levels of error