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Bases d'intégrité pour l'hyperélasticité plane à gradient

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Problematique (2D)

La modélisation de comportements hyperélastiques nécessite la détermination d'une densité d'énergie $\rho\psi$ reliant :

- tenseurs d'état de mesures de déformations
- aux tenseurs d'état de contraintes

Elasticité standard 2D : $\psi(\mathbf{F}^T \cdot \mathbf{F})$ est fonction de 2 invariants

Objectif :

Dans le cas de milieux micromorphes hyperélastiques, obtenir un jeu fini d'invariants des mesures de déformations dont est fonction toute densité d'énergie isotrope

Generalized degrees of freedom

$$DOF = \{\boldsymbol{u}, \underbrace{\boldsymbol{\chi}}_{\text{micromorphic order 1}}, \underbrace{{}^3\boldsymbol{\chi}}_{\text{micromorphic order 2}}, \underbrace{{}^4\boldsymbol{\chi}}_{\text{micromorphic order 3}}, \ldots\}$$

${}^n\boldsymbol{\chi} = (\chi_{iJ_1 \cdots J_{n-1}})$: tensor of order n,

$$DOF = \{u_i, \quad \chi_{iJ}, \quad \chi_{iJK}, \quad \chi_{iJKL}, \ldots\}$$

- Germain (1973) : $\chi_{i(J_1 \cdots J_{n-1})}$
- Forest,Sab (2020) : $\chi_{iJ_1 \cdots J_{n-1}}$

"*Germain's vision of this hierarchy of additional degrees of freedom is related to a Taylor expansion of the description around the material point. The general microdeformations are symmetric with respect to all the indices except the first one.*

In contrast to Germain's general micromorphic medium, the microdeformation tensor is viewed as the relaxed counterpart of $\mathbf{F}$ *meaning that it is a generally incompatible deformation field, in contrast to* $\mathbf{F}$*.* "

Interpretation of microdeformation tensors

- Forest,Sab (2020) : $\chi_{iJ_1 \cdots J_{n-1}}$

"The interpretation of the first microdeformation tensor as the linear transformation of a triad of directors attached to the microstructure is particularly illustrative.

Extensions to higher order microdeformation is possible based on higher order tensor products of directors.

The physical meaning and physical dimension of the higher order micromorphic variables will depend on the specific application."

Constitutive variables of micromorphic media

$$GRAD = \{\boldsymbol{u}, \quad \nabla\boldsymbol{u}, \quad \underbrace{\boldsymbol{\chi}, \quad \nabla\boldsymbol{\chi}}_{\substack{\text{micromorphic} \\ \text{order 1 and grade 1}}}, \quad \underbrace{{}^3\boldsymbol{\chi}, \quad \nabla{}^3\boldsymbol{\chi}}_{\substack{\text{micromorphic} \\ \text{order 2 and grade 1}}}, \ldots\}$$

$$GRAD = \{u_i, \quad u_{i,J}, \quad \chi_{iJ}, \quad \chi_{iJ,K}, \quad \chi_{iJK}, \quad \chi_{iJK,L}, \ldots\}$$

$$GRAD = \{\boldsymbol{u}, \quad \boldsymbol{u}\otimes\nabla^0, \quad \underbrace{\boldsymbol{\chi}, \quad \boldsymbol{\chi}\otimes\nabla^0}_{\substack{\text{micromorphic} \\ \text{order 1 and grade 1}}}, \quad \underbrace{{}^3\boldsymbol{\chi}, \quad {}^3\boldsymbol{\chi}\otimes\nabla^0}_{\substack{\text{micromorphic} \\ \text{order 2 and grade 1}}}, \ldots\}$$

Strain measures

Invariance properties of specific free energy lead to (on the reference configuration)

$$\psi_{\mathbf{F}}(\mathbf{F}, \boldsymbol{\chi}, \nabla\boldsymbol{\chi}, {}^3\boldsymbol{\chi}, \nabla{}^3\boldsymbol{\chi}, \cdots) = \psi_{\mathbf{C}}(\mathbf{C}, \boldsymbol{\Upsilon}, {}^3\mathbf{K}, {}^3\boldsymbol{\Upsilon}, {}^4\mathbf{K}, \cdots)$$

Set of strain measures

$$\mathbf{C} = \mathbf{F}^T \cdot \mathbf{F}$$

$$\left.\begin{aligned} \boldsymbol{\Upsilon} &= \boldsymbol{\chi}^{-1} \cdot \mathbf{F} \\ {}^3\mathbf{K} &= \boldsymbol{\chi}^{-1} \cdot (\nabla\boldsymbol{\chi}) \end{aligned}\right\} \text{order 1, grade 1}$$

$$\left.\begin{aligned} {}^3\boldsymbol{\Upsilon} &= \boldsymbol{\chi}^{-1} \cdot {}^3\boldsymbol{\chi} \\ {}^4\mathbf{K} &= \boldsymbol{\chi}^{-1} \cdot (\nabla{}^3\boldsymbol{\chi}) \end{aligned}\right\} \text{order 2, grade 1}$$

Invariance properties of specific free energy lead to (on the reference configuration)

$$\psi_{\mathbf{F}}(\mathbf{F}, \boldsymbol{\chi}, \nabla\boldsymbol{\chi}, {}^3\boldsymbol{\chi}, \nabla{}^3\boldsymbol{\chi}, \cdots) = \psi_{\mathbf{C}}(\mathbf{C}, \boldsymbol{\Upsilon}, {}^3\mathbf{K}, {}^3\boldsymbol{\Upsilon}, {}^4\mathbf{K}, \cdots)$$

Set of strain measures

$$C_{IJ} = F_{Ip}\, F_{pJ}$$

$$\left.\begin{aligned} \Upsilon_{IJ} &= \chi^{-1}_{Ip}\, F_{pJ} \\ K_{IJK} &= \chi^{-1}_{Ip}\, \chi_{pJ,K} \end{aligned}\right\} \text{order 1, grade 1}$$

$$\left.\begin{aligned} \Upsilon_{IJK} &= \chi^{-1}_{Ip}\, \chi_{pJK} \\ K_{IJKL} &= \chi^{-1}_{Ip}\, \chi_{pJK,L} \end{aligned}\right\} \text{order 2, grade 1}$$

Vector space of strain measures
of a micromorphic continuum of order 1 and of grade 1

Specific free energy (on the reference configuration) :

$$\psi_{\mathbf{F}}(\mathbf{F}, \boldsymbol{\chi}, \nabla\boldsymbol{\chi}) = \psi_{\mathbf{C}}(\mathbf{C}, \boldsymbol{\Upsilon}, {}^3\mathbf{K})$$

$$\mathbf{C} = \mathbf{F}^T \cdot \mathbf{F} \qquad \boldsymbol{\Upsilon} = \boldsymbol{\chi}^{-1} \cdot \mathbf{F} \qquad {}^3\mathbf{K} = \boldsymbol{\chi}^{-1} \cdot (\nabla\boldsymbol{\chi})$$

$$\mathbf{C} \in \mathbb{T}_{(IJ)} \qquad \boldsymbol{\Upsilon} \in \mathbb{T}_{IJ} \qquad {}^3\mathbf{K} \in \mathbb{T}_{IJK}$$

Vector space of strain measures (Micromorphic continuum)

$$\mathbb{V} = \mathbb{T}_{(IJ)} \oplus \mathbb{T}_{IJ} \oplus \mathbb{T}_{IJK}$$

Micromorphic medium of order 1 and grade 1 with $\chi = \mathbf{F}$

$$\psi_{\mathbf{F}}(\mathbf{F}, \chi, \nabla\chi) = \psi_{\mathbf{F}}(\mathbf{F}, \nabla\mathbf{F}) = \psi_{\mathbf{C}}(\mathbf{C}, {}^{3}\mathbf{K})$$

$$\mathbf{C} = \mathbf{F}^{T} \cdot \mathbf{F} \qquad {}^{3}\mathbf{K} = \mathbf{F}^{-1} \cdot \nabla\mathbf{F}$$

$$K_{IJK} = F^{-1}_{Ip} F_{pJ,K} = F^{-1}_{Ip} u_{p,JK}$$

$$\mathbf{C} \in \mathbb{T}_{(IJ)} \qquad {}^{3}\mathbf{K} \in \mathbb{T}_{I(JK)}$$

Vector space of strain measures (2nd gradient continuum)

$$\mathbb{V} = \mathbb{T}_{(IJ)} \oplus \mathbb{T}_{I(JK)}$$

The linear action $\star$ of the orthogonal group $\mathrm{O}(2)$ on a vector space $\mathbb{V}$ is a linear mapping

$$\mathrm{O}(2) \times \mathbb{V} \to \mathbb{V}, \qquad (g, \boldsymbol{v}) \mapsto g \star \boldsymbol{v},$$

such that $(g_1 g_2) \star \boldsymbol{v} = g_1 \star (g_2 \star \boldsymbol{v})$ and $\mathbf{1} \star \boldsymbol{v} = \boldsymbol{v}$

Particular case of $\mathbb{V} = \mathbb{T}_{(IJ)} \oplus \mathbb{T}_{IJ} \oplus \mathbb{T}_{IJK}$

$$g \star (\mathbf{C}, \boldsymbol{\Upsilon}, {}^3\mathbf{K}) = (g \star \mathbf{C}, g \star \boldsymbol{\Upsilon}, g \star {}^3\mathbf{K})$$

$$(g \star \boldsymbol{\Upsilon})_{IJ} = g_{IP} g_{JQ} \Upsilon_{PQ} \qquad (g \star {}^3\mathbf{K})_{IJK} = g_{IP} g_{JQ} g_{KR} K_{PQR}$$

$\mathbb{R}[\mathbb{V}]$: Algebra of polynomial functions on $\mathbb{V}$

Sub-algebra of $\mathrm{O}(2)-$*invariants* polynomial functions on $\mathbb{V}$

$$\mathbb{R}[\mathbb{V}]^{\mathrm{O}(2)} := \{\mathrm{p} \in \mathbb{R}[\mathbb{V}]; \quad \mathrm{p}(g \star \boldsymbol{v}) = \mathrm{p}(\boldsymbol{v}) \quad \forall g \in \mathrm{O}(2),\ \forall \boldsymbol{v} \in \mathbb{V}\}$$

Integrity basis

Any *finite* set of generators $\{I_1, \dots, I_N\}$ of $\mathbb{R}[\mathbb{V}]^{\mathrm{O}(2)}$ is called an *integrity basis*.

Minimal integrity basis

No proper subset of it is an integrity basis

Each isotropic polynomial $J \in \mathbb{R}[\mathbb{V}]^{O(2)}$ is a polynomial function in $I_1, I_2, \ldots, I_n$:

$$J(\boldsymbol{v}) = \mathrm{p}(I_1(\boldsymbol{v}), \ldots, I_N(\boldsymbol{v})), \quad \boldsymbol{v} \in \mathbb{V}$$

in which $\boldsymbol{v} \in \mathbb{V}$ reads $(\mathbf{C}, \boldsymbol{\Upsilon}, {}^3\mathbf{K}) \in \mathbb{T}_{(IJ)} \oplus \mathbb{T}_{IJ} \oplus \mathbb{T}_{IJK}$

O(2)−invariant (i.e. isotropic) function

$\mathcal{F}(\mathbb{V})$: vector space of real valued functions on $\mathbb{V}$.

Algebra of O(2)−*invariant* functions on $\mathbb{V}$

$$\mathcal{F}(\mathbb{V})^{\mathrm{O}(2)} := \{\mathrm{f} \in \mathcal{F}(\mathbb{V}); \quad \mathrm{f}(g \star \boldsymbol{v}) = \mathrm{f}(\boldsymbol{v}) \quad \forall g \in \mathrm{O}(2),\ \forall \boldsymbol{v} \in \mathbb{V}\}$$

[Wineman Pipkin 1963]

Each isotropic function $\mathrm{f} \in \mathcal{F}(\mathbb{V})^{\mathrm{O}(2)}$ is a function in $I_1, \dots, I_n$:

$$\mathrm{f}(\boldsymbol{v}) = \mathrm{F}(I_1(\boldsymbol{v}), \dots, I_N(\boldsymbol{v})), \quad \boldsymbol{v} \in \mathbb{V}$$

where $\{I_1, \dots, I_N\}$ is a (finite minimal) *integrity basis* of $\mathbb{R}[\mathbb{V}]^{\mathrm{O}(2)}$.

Remark

Since the I_k are continuous functions on $\mathbb{V}$,
if f is continuous, one can choose F continuous.

$\mathcal{F}(\mathbb{V})$: vector space of real valued functions on $\mathbb{V}$.

Algebra of O(2)−*invariant* functions on $\mathbb{V}$

$$\mathcal{F}(\mathbb{V})^{\mathrm{O}(2)} := \{\mathrm{f} \in \mathcal{F}(\mathbb{V}); \quad \mathrm{f}(g \star \boldsymbol{v}) = \mathrm{f}(\boldsymbol{v}) \quad \forall g \in \mathrm{O}(2),\ \forall \boldsymbol{v} \in \mathbb{V}\}$$

[Wineman Pipkin 1963]

Each isotropic function $\psi \in \mathcal{F}(\mathbb{V})^{\mathrm{O}(2)}$ is a function in $I_1, \dots, I_n$:

$$\psi(\boldsymbol{v}) = \Psi(I_1(\boldsymbol{v}), \dots, I_N(\boldsymbol{v})), \quad \boldsymbol{v} \in \mathbb{V}$$

where $\{I_1, \dots, I_N\}$ is a (finite minimal) *integrity basis* of $\mathbb{R}[\mathbb{V}]^{\mathrm{O}(2)}$.

Remark

Since the I_k are continuous functions on $\mathbb{V}$,
if ψ is continuous, one can choose Ψ continuous.

Three steps procedure

[B. Desmorat et al. 2020]

1. Define a (minimal) integrity basis of $\mathbb{R}[\mathbb{V}]^{\mathrm{SO}(2)}$
2. Deduce a non-minimal integrity basis of $\mathbb{R}[\mathbb{V}]^{\mathrm{O}(2)}$
3. Obtain a **minimal** integrity basis of $\mathbb{R}[\mathbb{V}]^{\mathrm{O}(2)}$ by a cleaning procedure

Vector space of harmonic tensors of order $n \geq 1$:

$$\mathbb{H}^n = \left\{ T_{(i_1 \cdots i_n)} \in \mathbb{R} \quad | \quad \mathrm{tr}\left({}^n\mathbf{T}\right) = 0 \right\} \qquad \dim \mathbb{H}^n = 2 \text{ in 2D}$$

Vector space of isotropic scalars

$$\mathbb{H}^0 = \mathbb{R} \qquad \text{with } g \star \alpha := \alpha \qquad \dim \mathbb{H}^0 = 1$$

Vector space of hemitropic scalars (pseudo-scalars)

$$\mathbb{H}^{-1} = \mathbb{R} \qquad \text{with } g \star \xi := (\det g)\xi \qquad \dim \mathbb{H}^{-1} = 1$$

Harmonic Structure of strain measures vector space $\mathbb{V}$

$$\mathbf{C} \in \mathbb{T}_{(IJ)} \qquad \mathbb{T}_{(IJ)} \simeq \mathbb{H}^2 \oplus \mathbb{H}^0$$
$$\boldsymbol{\Upsilon} \in \mathbb{T}_{IJ} \qquad \mathbb{T}_{IJ} \simeq \mathbb{H}^2 \oplus \mathbb{H}^0 \oplus \mathbb{H}^{-1}$$
$$^3\mathbf{K} \in \mathbb{T}_{IJK} \qquad \mathbb{T}_{IJK} \simeq \mathbb{H}^3 \oplus 3\mathbb{H}^1$$

Harmonic decomposition of vector space of strain measures (Micromorphic medium of order 1 and grade 1)

$$\mathbb{V} = \mathbb{T}_{(IJ)} \oplus \mathbb{T}_{IJ} \oplus \mathbb{T}_{IJK} \simeq \mathbb{H}^3 \oplus 2\mathbb{H}^2 \oplus 3\mathbb{H}^1 \oplus 2\mathbb{H}^0 \oplus \mathbb{H}^{-1}$$

$$(\mathbf{C}, \boldsymbol{\Upsilon}, {}^3\mathbf{K}) = ({}^3\mathbf{H}, \mathbf{h}_1, \mathbf{h}_2, \boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \alpha_1, \alpha_2, \xi)$$

$r-contraction$ of two tensors

$$^{n_1}\mathbf{T}_1 \overset{(r)}{\cdot} {}^{n_2}\mathbf{T}_2$$

$\Rightarrow$ tensor of order $n_1 + n_2 - 2r$

Skew-symmetric contraction between two symmetric tensors

$$({}^{n_1}\mathbf{S}_1 \times {}^{n_2}\mathbf{S}_2) := -({}^{n_1}\mathbf{S}_1 \cdot \boldsymbol{\varepsilon} \cdot {}^{n_2}\mathbf{S}_2)^{SYM}$$

$\Rightarrow$ in 2D, symmetric tensor of order $n_1 + n_2 - 2$

$\boldsymbol{\varepsilon}$: 2D second order Levi-Civita tensor

Harmonic product between two tensors

$$\left({}^{n_1}\mathbf{T}_1 * {}^{n_2}\mathbf{T}_2\right) := \left({}^{n_1}\mathbf{T}_1 \odot {}^{n_2}\mathbf{T}_2\right)_0$$

$\Rightarrow$ harmonic tensor of order $n_1 + n_2$

Particular case of two harmonic tensors (in 2D)

$$\mathbf{h}_1 * \mathbf{h}_2 = \mathbf{h}_1 \odot \mathbf{h}_2 - \frac{1}{4}(\mathbf{h}_1 : \mathbf{h}_2)\mathbf{1} \odot \mathbf{1}$$

$${}^3\mathbf{H}_1 * {}^3\mathbf{H}_2 = {}^3\mathbf{H}_1 \odot {}^3\mathbf{H}_2 - \frac{1}{8}({}^3\mathbf{H}_1 \vdots {}^3\mathbf{H}_2)\mathbf{1} \odot \mathbf{1} \odot \mathbf{1}$$

Degrees 1-2 : Isotropic invariants / Product of hemitropic invariants

#	degree	Formula
1	1	α_1
2	1	α_2

Degrees 1-2 : Isotropic invariants / Product of hemitropic invariants

#	degree	Formula
1	1	α_1
2	1	α_2

#	degree	Formula
3	2	$\boldsymbol{v}_1 \cdot \boldsymbol{v}_1$
4	2	$\boldsymbol{v}_2 \cdot \boldsymbol{v}_2$
5	2	$\boldsymbol{v}_3 \cdot \boldsymbol{v}_3$
6	2	$\mathbf{h}_1 : \mathbf{h}_1$
7	2	$\mathbf{h}_2 : \mathbf{h}_2$
8	2	${}^3\mathbf{H} \vdots {}^3\mathbf{H}$
9	2	$\boldsymbol{v}_1 \cdot \boldsymbol{v}_2$
10	2	$\boldsymbol{v}_1 \cdot \boldsymbol{v}_3$
11	2	$\boldsymbol{v}_2 \cdot \boldsymbol{v}_3$
12	2	$\mathbf{h}_1 : \mathbf{h}_2$

#	degree	Formula
13	2	ξ^2

#	degree	Formula
14	3	$\mathbf{h}_1 : (\boldsymbol{v}_1 \otimes \boldsymbol{v}_1)$
15	3	$\mathbf{h}_1 : (\boldsymbol{v}_2 \otimes \boldsymbol{v}_2)$
16	3	$\mathbf{h}_1 : (\boldsymbol{v}_3 \otimes \boldsymbol{v}_3)$
17	3	$\mathbf{h}_2 : (\boldsymbol{v}_1 \otimes \boldsymbol{v}_1)$
18	3	$\mathbf{h}_2 : (\boldsymbol{v}_2 \otimes \boldsymbol{v}_2)$
19	3	$\mathbf{h}_2 : (\boldsymbol{v}_3 \otimes \boldsymbol{v}_3)$
20	3	$\mathbf{h}_1 : (\boldsymbol{v}_1 \otimes \boldsymbol{v}_2)$
21	3	$\mathbf{h}_1 : (\boldsymbol{v}_1 \otimes \boldsymbol{v}_3)$
22	3	$\mathbf{h}_1 : (\boldsymbol{v}_2 \otimes \boldsymbol{v}_3)$
23	3	$\mathbf{h}_2 : (\boldsymbol{v}_1 \otimes \boldsymbol{v}_2)$
24	3	$\mathbf{h}_2 : (\boldsymbol{v}_1 \otimes \boldsymbol{v}_3)$
25	3	$\mathbf{h}_2 : (\boldsymbol{v}_2 \otimes \boldsymbol{v}_3)$
26	3	${}^3\mathbf{H} \vdots (\boldsymbol{v}_1 \otimes \mathbf{h}_1)$
27	3	${}^3\mathbf{H} \vdots (\boldsymbol{v}_1 \otimes \mathbf{h}_2)$
28	3	${}^3\mathbf{H} \vdots (\boldsymbol{v}_2 \otimes \mathbf{h}_1)$
29	3	${}^3\mathbf{H} \vdots (\boldsymbol{v}_2 \otimes \mathbf{h}_2)$
30	3	${}^3\mathbf{H} \vdots (\boldsymbol{v}_3 \otimes \mathbf{h}_1)$
31	3	${}^3\mathbf{H} \vdots (\boldsymbol{v}_3 \otimes \mathbf{h}_2)$

#	degree	Formula
32	3	$\xi\, \boldsymbol{v}_1 \times \boldsymbol{v}_2$
33	3	$\xi\, \boldsymbol{v}_1 \times \boldsymbol{v}_3$
34	3	$\xi\, \boldsymbol{v}_2 \times \boldsymbol{v}_3$
35	3	$\xi\, (\mathrm{I} \times \mathbf{h}_1) : \mathbf{h}_2$

Degree 4 : Isotropic invariants / Product of hemitropic invariants

#	degree	Formula
36	4	${}^3\mathbf{H} \vdots (\boldsymbol{v}_1 \otimes \boldsymbol{v}_1 \otimes \boldsymbol{v}_1)$
37	4	${}^3\mathbf{H} \vdots (\boldsymbol{v}_2 \otimes \boldsymbol{v}_2 \otimes \boldsymbol{v}_2)$
38	4	${}^3\mathbf{H} \vdots (\boldsymbol{v}_3 \otimes \boldsymbol{v}_3 \otimes \boldsymbol{v}_3)$
39	4	${}^3\mathbf{H} \vdots (\boldsymbol{v}_1 \otimes \boldsymbol{v}_2 \otimes \boldsymbol{v}_2)$
40	4	${}^3\mathbf{H} \vdots (\boldsymbol{v}_1 \otimes \boldsymbol{v}_3 \otimes \boldsymbol{v}_3)$
41	4	${}^3\mathbf{H} \vdots (\boldsymbol{v}_2 \otimes \boldsymbol{v}_1 \otimes \boldsymbol{v}_1)$
42	4	${}^3\mathbf{H} \vdots (\boldsymbol{v}_2 \otimes \boldsymbol{v}_3 \otimes \boldsymbol{v}_3)$
43	4	${}^3\mathbf{H} \vdots (\boldsymbol{v}_3 \otimes \boldsymbol{v}_1 \otimes \boldsymbol{v}_1)$
44	4	${}^3\mathbf{H} \vdots (\boldsymbol{v}_3 \otimes \boldsymbol{v}_2 \otimes \boldsymbol{v}_2)$
45	4	${}^3\mathbf{H} \vdots (\boldsymbol{v}_1 \otimes \boldsymbol{v}_2 \otimes \boldsymbol{v}_3)$
46	4	$(\mathbf{h}_1 * \mathbf{h}_1) :: (\boldsymbol{v}_1 \otimes {}^3\mathbf{H})$
47	4	$(\mathbf{h}_2 * \mathbf{h}_2) :: (\boldsymbol{v}_1 \otimes {}^3\mathbf{H})$
48	4	$(\mathbf{h}_1 * \mathbf{h}_1) :: (\boldsymbol{v}_2 \otimes {}^3\mathbf{H})$
49	4	$(\mathbf{h}_2 * \mathbf{h}_2) :: (\boldsymbol{v}_2 \otimes {}^3\mathbf{H})$
50	4	$(\mathbf{h}_1 * \mathbf{h}_1) :: (\boldsymbol{v}_3 \otimes {}^3\mathbf{H})$
51	4	$(\mathbf{h}_2 * \mathbf{h}_2) :: (\boldsymbol{v}_3 \otimes {}^3\mathbf{H})$
52	4	$(\mathbf{h}_1 * \mathbf{h}_2) :: (\boldsymbol{v}_1 \otimes {}^3\mathbf{H})$
53	4	$(\mathbf{h}_1 * \mathbf{h}_2) :: (\boldsymbol{v}_2 \otimes {}^3\mathbf{H})$
54	4	$(\mathbf{h}_1 * \mathbf{h}_2) :: (\boldsymbol{v}_3 \otimes {}^3\mathbf{H})$

#	degree	Formula
55	4	$(\boldsymbol{v}_1 \times \boldsymbol{v}_2)\,((\mathrm{I} \times \mathbf{h}_1) : \mathbf{h}_2)$
56	4	$(\boldsymbol{v}_1 \times \boldsymbol{v}_3)\,((\mathrm{I} \times \mathbf{h}_1) : \mathbf{h}_2)$
57	4	$(\boldsymbol{v}_2 \times \boldsymbol{v}_3)\,((\mathrm{I} \times \mathbf{h}_1) : \mathbf{h}_2)$
58	4	$\xi\,(\mathrm{I} \times \mathbf{h}_1) : (\boldsymbol{v}_1 \otimes \boldsymbol{v}_1)$
59	4	$\xi\,(\mathrm{I} \times \mathbf{h}_1) : (\boldsymbol{v}_2 \otimes \boldsymbol{v}_2)$
60	4	$\xi\,(\mathrm{I} \times \mathbf{h}_1) : (\boldsymbol{v}_3 \otimes \boldsymbol{v}_3)$
61	4	$\xi\,(\mathrm{I} \times \mathbf{h}_2) : (\boldsymbol{v}_1 \otimes \boldsymbol{v}_1)$
62	4	$\xi\,(\mathrm{I} \times \mathbf{h}_2) : (\boldsymbol{v}_2 \otimes \boldsymbol{v}_2)$
63	4	$\xi\,(\mathrm{I} \times \mathbf{h}_2) : (\boldsymbol{v}_3 \otimes \boldsymbol{v}_3)$
64	4	$\xi\,(\mathrm{I} \times \mathbf{h}_1) : (\boldsymbol{v}_1 \otimes \boldsymbol{v}_2)$
65	4	$\xi\,(\mathrm{I} \times \mathbf{h}_1) : (\boldsymbol{v}_1 \otimes \boldsymbol{v}_3)$
66	4	$\xi\,(\mathrm{I} \times \mathbf{h}_1) : (\boldsymbol{v}_2 \otimes \boldsymbol{v}_3)$
67	4	$\xi\,(\mathrm{I} \times \mathbf{h}_2) : (\boldsymbol{v}_1 \otimes \boldsymbol{v}_2)$
68	4	$\xi\,(\mathrm{I} \times \mathbf{h}_2) : (\boldsymbol{v}_1 \otimes \boldsymbol{v}_3)$
69	4	$\xi\,(\mathrm{I} \times \mathbf{h}_2) : (\boldsymbol{v}_2 \otimes \boldsymbol{v}_3)$
70	4	$\xi\,(\mathrm{I} \times {}^3\mathbf{H}) \vdots (\boldsymbol{v}_1 \otimes \mathbf{h}_1)$
71	4	$\xi\,(\mathrm{I} \times {}^3\mathbf{H}) \vdots (\boldsymbol{v}_1 \otimes \mathbf{h}_2)$
72	4	$\xi\,(\mathrm{I} \times {}^3\mathbf{H}) \vdots (\boldsymbol{v}_2 \otimes \mathbf{h}_1)$
73	4	$\xi\,(\mathrm{I} \times {}^3\mathbf{H}) \vdots (\boldsymbol{v}_2 \otimes \mathbf{h}_2)$
74	4	$\xi\,(\mathrm{I} \times {}^3\mathbf{H}) \vdots (\boldsymbol{v}_3 \otimes \mathbf{h}_1)$
75	4	$\xi\,(\mathrm{I} \times {}^3\mathbf{H}) \vdots (\boldsymbol{v}_3 \otimes \mathbf{h}_2)$

Degree 5 : Isotropic invariants / Product of hemitropic invariants

#	degree	Formula
76	5	$({}^3\mathbf{H} * {}^3\mathbf{H}) \vdots\vdots (\mathbf{h}_1 \otimes \mathbf{h}_1 \otimes \mathbf{h}_1)$
77	5	$({}^3\mathbf{H} * {}^3\mathbf{H}) \vdots\vdots (\mathbf{h}_2 \otimes \mathbf{h}_2 \otimes \mathbf{h}_2)$
78	5	$({}^3\mathbf{H} * {}^3\mathbf{H}) \vdots\vdots (\mathbf{h}_1 \otimes \mathbf{h}_2 \otimes \mathbf{h}_2)$
79	5	$({}^3\mathbf{H} * {}^3\mathbf{H}) \vdots\vdots (\mathbf{h}_2 \otimes \mathbf{h}_1 \otimes \mathbf{h}_1)$

#	degree	Formula
80	5	$\xi\,(\mathrm{I} \times {}^3\mathbf{H}) \vdots (\boldsymbol{v}_1 \otimes \boldsymbol{v}_1 \otimes \boldsymbol{v}_1)$
81	5	$\xi\,(\mathrm{I} \times {}^3\mathbf{H}) \vdots (\boldsymbol{v}_2 \otimes \boldsymbol{v}_2 \otimes \boldsymbol{v}_2)$
82	5	$\xi\,(\mathrm{I} \times {}^3\mathbf{H}) \vdots (\boldsymbol{v}_1 \otimes \boldsymbol{v}_2 \otimes \boldsymbol{v}_2)$
83	5	$\xi\,(\mathrm{I} \times {}^3\mathbf{H}) \vdots (\boldsymbol{v}_2 \otimes \boldsymbol{v}_1 \otimes \boldsymbol{v}_1)$
84	5	$\xi\,((\mathrm{I} \times \mathbf{h}_1) * \mathbf{h}_1) :: (\boldsymbol{v}_1 \otimes {}^3\mathbf{H})$
85	5	$\xi\,((\mathrm{I} \times \mathbf{h}_2) * \mathbf{h}_2) :: (\boldsymbol{v}_1 \otimes {}^3\mathbf{H})$
86	5	$\xi\,((\mathrm{I} \times \mathbf{h}_1) * \mathbf{h}_1) :: (\boldsymbol{v}_2 \otimes {}^3\mathbf{H})$
87	5	$\xi\,((\mathrm{I} \times \mathbf{h}_2) * \mathbf{h}_2) :: (\boldsymbol{v}_2 \otimes {}^3\mathbf{H})$
88	5	$\xi\,((\mathrm{I} \times \mathbf{h}_1) * \mathbf{h}_1) :: (\boldsymbol{v}_3 \otimes {}^3\mathbf{H})$
89	5	$\xi\,((\mathrm{I} \times \mathbf{h}_2) * \mathbf{h}_2) :: (\boldsymbol{v}_3 \otimes {}^3\mathbf{H})$
90	5	$\xi\,((\mathrm{I} \times \mathbf{h}_1) * \mathbf{h}_2) :: (\boldsymbol{v}_1 \otimes {}^3\mathbf{H})$
91	5	$\xi\,((\mathrm{I} \times \mathbf{h}_1) * \mathbf{h}_2) :: (\boldsymbol{v}_2 \otimes {}^3\mathbf{H})$
92	5	$\xi\,((\mathrm{I} \times \mathbf{h}_1) * \mathbf{h}_2) :: (\boldsymbol{v}_3 \otimes {}^3\mathbf{H})$

Degree 6 : Product of hemitropic invariants

#	degree	Formula
93	6	$\xi\left(\left(\mathrm{I}\times{}^3\mathbf{H}\right)*{}^3\mathbf{H}\right)\vdots\left(\mathbf{h}_1\otimes\mathbf{h}_1\otimes\mathbf{h}_1\right)$
94	6	$\xi\left(\left(\mathrm{I}\times{}^3\mathbf{H}\right)*{}^3\mathbf{H}\right)\vdots\left(\mathbf{h}_2\otimes\mathbf{h}_2\otimes\mathbf{h}_2\right)$
95	6	$\xi\left(\left(\mathrm{I}\times{}^3\mathbf{H}\right)*{}^3\mathbf{H}\right)\vdots\left(\mathbf{h}_1\otimes\mathbf{h}_2\otimes\mathbf{h}_2\right)$
96	6	$\xi\left(\left(\mathrm{I}\times{}^3\mathbf{H}\right)*{}^3\mathbf{H}\right)\vdots\left(\mathbf{h}_2\otimes\mathbf{h}_1\otimes\mathbf{h}_1\right)$

Name / Degree	1	2	3	4	5	6	tot
Micromorphic	2	11	22	40	17	4	96

2D Finite case ([Forest Sievert 2006] in 3D)

$$\chi = \mathbf{R} \cdot \mathbf{U} \qquad \mathbf{R} \in SO(2)$$

Name	#	DOF	Assumption
Cauchy	2	$\boldsymbol{u}$	$\chi = \mathbf{1}$
Microdilatation	3	$\boldsymbol{u}, \alpha$	$\mathbf{R} = \mathbf{1}$ $\mathbf{U} = \alpha\mathbf{1}$
Cosserat	3	$\boldsymbol{u}, \xi$	$\mathbf{U} = \mathbf{1}$
Microstretch	4	$\boldsymbol{u}, \alpha, \xi$	$\mathbf{U} = \alpha\mathbf{1}$
Incompressible microstrain	4	$\boldsymbol{u}, \mathbf{U}'$	$\mathbf{R} = \mathbf{1}$ $\det(\mathbf{U}) = 1$
Microstrain	5	$\boldsymbol{u}, \mathbf{U}$	$\mathbf{R} = \mathbf{1}$
Incompressible micromorphic	5	$\boldsymbol{u}, \mathbf{U}', \xi$	$\det(\mathbf{U}) = 1$
Micromorphic	6	$\boldsymbol{u}, \chi$	

Harmonic structures of strain measures

Name	dim	$\mathbf{C}$	$\boldsymbol{\Upsilon}$	${}^3\mathbf{K}$
Cauchy	3	$\mathbb{H}^2 \oplus \mathbb{H}^0$		
Microdilatation	6	$\mathbb{H}^2 \oplus \mathbb{H}^0$	$\mathbb{H}^0$	$\mathbb{H}^1$
Cosserat	6	$\mathbb{H}^2 \oplus \mathbb{H}^0$	$\mathbb{H}^{-1}$	$\mathbb{H}^1$
Microstretch	9	$\mathbb{H}^2 \oplus \mathbb{H}^0$	$\mathbb{H}^0 \oplus \mathbb{H}^{-1}$	$2\mathbb{H}^1$
Incompressible microstrain	9	$\mathbb{H}^2 \oplus \mathbb{H}^0$	$\mathbb{H}^2$	$\mathbb{H}^3 \oplus \mathbb{H}^1$
Microstrain	10	$\mathbb{H}^2 \oplus \mathbb{H}^0$	$\mathbb{H}^2 \oplus \mathbb{H}^0$	$\mathbb{H}^3 \oplus 2\mathbb{H}^1$
Incompressible micromorphic	10	$\mathbb{H}^2 \oplus \mathbb{H}^0$	$\mathbb{H}^2 \oplus \mathbb{H}^{-1}$	$\mathbb{H}^3 \oplus 2\mathbb{H}^1$
Micromorphic	15	$\mathbb{H}^2 \oplus \mathbb{H}^0$	$\mathbb{H}^2 \oplus \mathbb{H}^0 \oplus \mathbb{H}^{-1}$	$\mathbb{H}^3 \oplus 3\mathbb{H}^1$

$$\mathbf{C} = \mathbf{F}^T \cdot \mathbf{F} \qquad \boldsymbol{\Upsilon} = \chi^{-1} \cdot \mathbf{F} \qquad {}^3\mathbf{K} = \chi^{-1} \cdot (\nabla \chi)$$

Integrity basis

Name / Degree	1	2	3	4	5	6	tot
Cauchy	1	1	0	0	0	0	2
Microdilatation	2	2	1	0	0	0	5
Cosserat	1	3	1	1	0	0	6
Microstretch	2	5	3	3	0	0	13
Incompressible microstrain	1	5	4	4	4	0	18
Microstrain	2	7	10	8	4	0	31
Incompressible micromorphic	1	8	12	20	14	4	59
Micromorphic	2	11	22	40	17	4	96

Micromorphic medium of order 1 and grade 1 with

$$\boldsymbol{\chi} = \mathbf{F}$$

$$\psi_{\mathbf{F}}(\mathbf{F}, \boldsymbol{\chi}, \nabla\boldsymbol{\chi}) = \psi_{\mathbf{F}}(\mathbf{F}, \nabla\mathbf{F}) = \psi_{\mathbf{C}}(\mathbf{C}, {}^{3}\mathbf{K})$$

with

$$\mathbf{C} = \mathbf{F}^{T} \cdot \mathbf{F} \qquad {}^{3}\mathbf{K} = \mathbf{F}^{-1} \cdot \nabla\mathbf{F}$$

$$C_{IJ} = F_{Ip} F_{pJ} \qquad K_{IJK} = F^{-1}_{Ip} F_{pJ,K} = F^{-1}_{Ip} u_{p,JK}$$

$$\mathbf{C} \in \mathbb{T}_{(IJ)} \simeq \mathbb{H}^{2} \oplus \mathbb{H}^{0} \qquad {}^{3}\mathbf{K} \in \mathbb{T}_{I(JK)} \simeq \mathbb{H}^{3} \oplus 2\mathbb{H}^{1}$$

Name	dim	$\mathbf{C}$	$\boldsymbol{\Upsilon}$	${}^3\mathbf{K}$
Cauchy	3	$\mathbb{H}^2 \oplus \mathbb{H}^0$		
Microdilatation	6	$\mathbb{H}^2 \oplus \mathbb{H}^0$	$\mathbb{H}^0$	$\mathbb{H}^1$
Cosserat	6	$\mathbb{H}^2 \oplus \mathbb{H}^0$	$\mathbb{H}^{-1}$	$\mathbb{H}^1$
Microstretch	9	$\mathbb{H}^2 \oplus \mathbb{H}^0$	$\mathbb{H}^0 \oplus \mathbb{H}^{-1}$	$2\mathbb{H}^1$
Incompressible microstrain	9	$\mathbb{H}^2 \oplus \mathbb{H}^0$	$\mathbb{H}^2$	$\mathbb{H}^3 \oplus \mathbb{H}^1$
Strain gradient	9	$\mathbb{H}^2 \oplus \mathbb{H}^0$		$\mathbb{H}^3 \oplus 2\mathbb{H}^1$
Microstrain	10	$\mathbb{H}^2 \oplus \mathbb{H}^0$	$\mathbb{H}^2 \oplus \mathbb{H}^0$	$\mathbb{H}^3 \oplus 2\mathbb{H}^1$
Incompressible micromorphic	10	$\mathbb{H}^2 \oplus \mathbb{H}^0$	$\mathbb{H}^2 \oplus \mathbb{H}^{-1}$	$\mathbb{H}^3 \oplus 2\mathbb{H}^1$
Micromorphic	15	$\mathbb{H}^2 \oplus \mathbb{H}^0$	$\mathbb{H}^2 \oplus \mathbb{H}^0 \oplus \mathbb{H}^{-1}$	$\mathbb{H}^3 \oplus 3\mathbb{H}^1$

Integrity basis

Name / Degree	1	2	3	4	5	6	tot
Cauchy	1	1	0	0	0	0	2
Microdilatation	2	2	1	0	0	0	5
Cosserat	1	3	1	1	0	0	6
Microstretch	2	5	3	3	0	0	13
Incompressible microstrain	1	5	4	4	4	0	18
Strain gradient	1	5	5	7	1	0	19
Microstrain	2	7	10	8	4	0	31
Incompressible micromorphic	1	8	12	20	14	4	59
Micromorphic	2	11	22	40	17	4	96

Conclusion

- we provide the minimal integrity basis for the strain measures vector space relative to micromorphic medium of order 1 and grade 1, and all related simplified models
- we provide the minimal integrity basis for the strain measures vector space relative to strain gradient model as the particular case of micromorphic medium of order 1 and grade with an internal constraint
- any isotropic specific energy density (i.e. not only polynomial) related to hyperelastic constitutive laws can be written in terms of the integrity basis (i.e. the polynomial invariants generators of the algebra of polynomial invariants)

Perspective

Minimal integrity basis are available (with the use of a numerical procedure) for the action of $\mathrm{O}(2)$ on any 2D tensor vector space :

- micromorphic model of order 3, 4, ... and grade 1,
- micromorphic of order 1 and grade 2, 3, 4, ...
- micromoprhic model with any type of internal constraint
- $n-th$ gradient model
- ...

Les outils utilisés sont le fruit d'un travail collaboratif avec Rodrigue Desmorat, Boris Kolev, Nicolas Auffray, Marc Olive

La discussion est ouverte !

$\mathcal{F}(\mathbb{V})$: vector space of real valued functions on $\mathbb{V}$.

Algebra of $\mathrm{O}(2)-$*invariant* functions on $\mathbb{V}$

$$\mathcal{F}(\mathbb{V})^{\mathrm{O}(2)} := \{\mathrm{f} \in \mathcal{F}(\mathbb{V}); \quad \mathrm{f}(g \star \boldsymbol{v}) = \mathrm{f}(\boldsymbol{v}) \quad \forall g \in \mathrm{O}(2),\ \forall \boldsymbol{v} \in \mathbb{V}\}$$

Functional basis

A finite set $\{\phi_1, \cdots, \phi_s\} \in {\mathcal{F}(\mathbb{V})^{\mathrm{O}(2)}}^s$ is a functional basis of $\mathcal{F}(\mathbb{V})^{\mathrm{O}(2)}$ if for any $\mathrm{f} \in \mathcal{F}(\mathbb{V})^{\mathrm{O}(2)}$, there exists a function $\mathrm{F} : \mathbb{R}^s \to \mathbb{R}$ such that

$$\mathrm{f}(\boldsymbol{v}) = F(\phi_1(\boldsymbol{v}), \cdots, \phi_s(\boldsymbol{v})), \forall \boldsymbol{v} \in \mathbb{V}$$

Minimal functional basis

No proper subset of it is a functional basis