A generic passive-guaranteed structure for elastoplatic friction models

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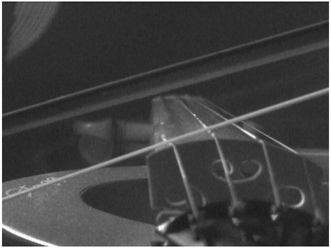
GDR Géométrie Différentielle et Mécanique

Context: Sound synthesis

Bowed string dynamics: Nonlinear dynamical system with self-oscillations due to an alternance between stick and slip phases.



Source: https://www.youtube.com/watch?v=6JeyiM0YNo4 (ViolinB0W)

Problem Statement

Aims

- Sound synthesis of a resonator with nonlinear interaction[a]
- Ensure power balance and therefore computational stability

[a] Silvin Willemsen, Stefan Bilbao, and Stefania Serafin. "Real-time implementation of an elasto-plastic friction model applied to stiff strings using finite difference schemes". In: *22nd International Conference on Digital Audio Effects*. 2019.

Framework: Port Hamiltonian Systems (PHS)

- PHS formulation[a]
- Modularity: interconnection of several PHS is a PHS
- **Passive guaranteed simulation** (stability)

[a] Bernhard M Maschke, Arjan J Van Der Schaft, and Pieter Cornelis Breedveld. "An intrinsic Hamiltonian formulation of network dynamics: Non-standard Poisson structures and gyrators". In: *Journal of the Franklin institute* 329.5 (1992), pp. 923–966.

Outline

Outline

Port-Hamiltonian framework: Generalities

Motivations

- Every physical systems are *passive*, *i.e.* they fullfill a power-balance $\frac{dE(t)}{dt} + D(t) + P(t) = 0$, with energy $E(t) \in \mathbb{R}^+$, dissipated power $D(t) \in \mathbb{R}^+$ and external power *sink* $P(t) \in \mathbb{R}$.
- Preserving this property ensures the stability of numerical schemes and control laws.

Origins

Port-based modeling Bond-graphs, network of components

Theoretical mechanics Symplectic, Poisson, Dirac structures

Non-equilibrium thermodynamics GENERIC formalism

Control theory (structured) input-state-output system

Port-Hamiltonian framework: Mathematical background

The pH framework relies on the definition of *Dirac structures*[1]. Consider

- a finite dimensional vector space $\mathcal{F}$ (*space of flux*),
- its dual space $\mathcal{E} = \mathcal{F}^*$ (*space of efforts*),
- the product space $\mathcal{P} = \mathcal{F} \times \mathcal{E}$,
- the duality bracket $\langle \cdot | \cdot \rangle : \begin{cases} \mathcal{P} & \to & \mathbb{R}, \\ (f, e) & \mapsto & \langle e | f \rangle, \end{cases}$

and define the (canonical) symmetric pairing

$$\langle\langle \cdot | \cdot \rangle\rangle : \begin{cases} \mathcal{P} \times \mathcal{P} & \to & \mathbb{R}, \\ ((f_1, e_1), (f_2, e_2)) & \mapsto & \langle e_1 | f_2 \rangle + \langle e_2 | f_1 \rangle. \end{cases}$$

A Dirac structure is a linear subspace $\mathcal{D} \subset \mathcal{P}$ s.t. $\mathcal{D} = \mathcal{D}^\perp$, with

$$\mathcal{D}^\perp = \left\{ (f, e) \in \mathcal{P},\ \langle\langle (f, e) | (\tilde{f}, \tilde{e}) \rangle\rangle = 0\ \forall (\tilde{f}, \tilde{e}) \in \mathcal{D} \right\}.$$

[1] Vincent Duindam et al. *Modeling and control of complex physical systems: the port-Hamiltonian approach*. Springer Science & Business Media, 2009.

Port-Hamiltonian framework: Definition

To recover the power balance, the flows and efforts spaces split as

$$\begin{aligned} \mathcal{P} = \mathcal{F} \times \mathcal{E} &= \left(\mathcal{F}_s \times \mathcal{F}_d \times \mathcal{F}_{\mathrm{ext}}\right) \times \left(\mathcal{E}_s \times \mathcal{E}_d \times \mathcal{E}_{\mathrm{ext}}\right), \\ (f, e) &= \left(f_s, f_d, f_{\mathrm{ext}}, e_s, e_d, e_{\mathrm{ext}}\right), \end{aligned}$$

with

Storage state $\boldsymbol{x} \in \mathbb{R}^{n_x}$, Hamiltonian $H : \mathbb{R}^{n_x} \to \mathbb{R}^+$,
$f_s \triangleq \frac{d\boldsymbol{x}}{dt}$ and $e_s \triangleq \nabla H(\boldsymbol{x})$, s.t. $\frac{dH(\boldsymbol{x})}{dt} = \langle e_s | f_s \rangle$,

Dissipation variable $\boldsymbol{w} \in \mathbb{R}^{n_w}$, function $\boldsymbol{z} : \mathbb{R}^{n_w} \to \mathbb{R}^{n_w}$,
$f_d \triangleq \boldsymbol{w}$ and $e_d \triangleq \boldsymbol{z}(\boldsymbol{w})$ s.t. $D = \langle e_d | f_d \rangle \geq 0$,

Exterior control $\boldsymbol{u} \in \mathbb{R}^{n_y}$, observation $\boldsymbol{y} \in \mathbb{R}^{n_y}$,
$f_{\mathrm{ext}} \triangleq \boldsymbol{y}$ and $e_{\mathrm{ext}} \triangleq \boldsymbol{u}$ s.t. $P = \langle e_{\mathrm{ext}} | f_{\mathrm{ext}} \rangle$.

Now, if $\mathcal{P}$ is endowed with a Dirac structure $\mathcal{D}$, we have $\forall (f, e) \in \mathcal{D}$:

$$\langle\langle (f, e) | (f, e) \rangle\rangle = 2 \langle e | f \rangle = 0,$$

so that: $\langle e_s | f_s \rangle + \langle e_d | f_d \rangle + \langle e_{\mathrm{ext}} | f_{\mathrm{ext}} \rangle = \frac{dH(\boldsymbol{x})}{dt} + D + P = 0.$

Port-Hamiltonian framework: Example 1

The damped oscillator with external force:

$$m\frac{\mathrm{d}^2 q}{\mathrm{d}t^2} = -kq - a\frac{\mathrm{d}q}{\mathrm{d}t} - F_{\mathrm{m}\to\mathrm{ext}}.$$

Storage state $\boldsymbol{x} = (q, p = m\frac{\mathrm{d}q}{\mathrm{d}t})$, Hamiltonian $H(x) = \frac{kq^2}{2} + \frac{p^2}{2m}$,
$f_s = \frac{d\boldsymbol{x}}{dt} = (\frac{\mathrm{d}q}{\mathrm{d}t}, \frac{\mathrm{d}p}{\mathrm{d}t})$ and $e_s \triangleq \nabla H(\boldsymbol{x}) = (kq, \frac{\mathrm{d}q}{\mathrm{d}t})$,

Dissipation variable $w = \frac{\mathrm{d}q}{\mathrm{d}t}$, function $z(w) = aw$,
$f_d \triangleq w$ and $e_d \triangleq z(w)$ s.t. $D = aw^2 \geq 0$,

Exterior control $e_{\mathrm{ext}} = u = F_{\mathrm{m}\to\mathrm{ext}}$, observation $f_{\mathrm{ext}} = y = \frac{\mathrm{d}q}{\mathrm{d}t}$.

Port-Hamiltonian system

$$\underbrace{\begin{pmatrix} \frac{\mathrm{d}q}{\mathrm{d}t} \\ \frac{\mathrm{d}p}{\mathrm{d}t} \\ \hline w \\ \hline y \end{pmatrix}}_{f} = \underbrace{\left(\begin{array}{cc|c|c} 0 & +1 & 0 & 0 \\ -1 & 0 & -1 & -1 \\ \hline 0 & +1 & 0 & 0 \\ \hline 0 & +1 & 0 & 0 \end{array}\right)}_{J} \underbrace{\begin{pmatrix} \frac{\partial H(q,p)}{\partial q} \\ \frac{\partial H(q,p)}{\partial p} \\ \hline z(w) \\ \hline u \end{pmatrix}}_{e}.$$

Port-Hamiltonian framework: Example 1 (cont.)

Port-Hamiltonian system

$$\underbrace{\begin{pmatrix} \frac{dq}{dt} \\ \frac{dp}{dt} \\ \hline w \\ \hline y \end{pmatrix}}_{f} = \underbrace{\left(\begin{array}{cc|c|c} 0 & +1 & 0 & 0 \\ -1 & 0 & -1 & -1 \\ \hline 0 & +1 & 0 & 0 \\ \hline 0 & +1 & 0 & 0 \end{array}\right)}_{J} \underbrace{\begin{pmatrix} \frac{\partial H(q,p)}{\partial q} \\ \frac{\partial H(q,p)}{\partial p} \\ \hline z(w) \\ \hline u \end{pmatrix}}_{e}.$$

Constitutive relations

$$e^{\intercal} f = \underbrace{\nabla H(\mathbf{x})^{\intercal} \frac{d\mathbf{x}}{dt}}_{\frac{dH(\mathbf{x})}{dt}} + \underbrace{z(w)\, w}_{D \geq 0} + \underbrace{u\, y}_{P}.$$

Dirac structure $\mathcal{D} = \left\{(f, e) \in \mathbb{R}^4 \times \mathbb{R}^4,\ f = Je\right\}$

$$\langle e | f \rangle = e^{\intercal} f = e^{\intercal} J e = 0, \quad \forall (f, e) \in \mathcal{D}.$$

Port-Hamiltonian framework: Example 2

Linear (clamped) string model

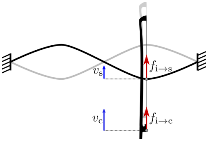
$$\mu \frac{\partial^2 u}{\partial t^2}(\xi,t) + a \frac{\partial u}{\partial t}(\xi,t) = T_0 \frac{\partial^2 u}{\partial \xi^2}(\xi,t) + f_{s\rightarrow ext}(\xi,t),$$

$$\begin{pmatrix} \frac{\partial q}{\partial t}(\xi,t) \\ \frac{\partial v}{\partial t}(\xi,t) \\ \hline w_s(\xi,t) \\ \hline y_s(\xi,t) \end{pmatrix} = \left(\begin{array}{cc|c|c} 0 & \frac{1}{\mu}\frac{\partial}{\partial \xi} & 0 & 0 \\ \frac{1}{\mu}\frac{\partial}{\partial \xi} & 0 & \frac{-1}{\mu} & \frac{-1}{\mu} \\ \hline 0 & \frac{1}{\mu} & 0 & 0 \\ \hline 0 & \frac{1}{\mu} & 0 & 0 \end{array}\right) \begin{pmatrix} \frac{\partial H_s}{\partial q}\left(q(\xi,t), v(\xi,t)\right) \\ \frac{\partial H_s}{\partial v}\left(q(\xi,t), v(\xi,t)\right) \\ \hline z_s\left(w_s(\xi,t)\right) \\ \hline u_s(\xi,t) \end{pmatrix}$$

with

- For the storing components: $\frac{\partial H_s}{\partial q} = T_0 q$ and $\frac{\partial H_s}{\partial v} = \mu v$
- For the dissipating components: $w_s = v$ and $z_s\left(w_s(\xi,t)\right) = av$
- For the power source: $u_s(\xi,t) = -f(\xi,t)$ and $y_s(\xi,t) = v(\xi,t)$

The bowed string



Friction model with two external ports

Control port with imposed velocity v_c and $f_{i\to c}$ the force exerted by the interaction.

Interaction port with string velocity v_s and $f_{i\to s}$ the force exerted on the string.

Power emitted through the ext. ports : $\boldsymbol{u}^T\boldsymbol{y} = f_{i\to s}v_s + f_{i\to c}v_c$.

Outline

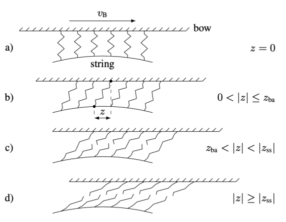
1. Port Hamiltonian Systems
2. Friction models
3. Application to a bowed string

Dupont friction model

Elastoplastic friction model

The interaction forces depend on the *relative* velocity $v_{\mathrm{rel}} = v_{\mathrm{c}} - v_{\mathrm{i}}$ which is decomposed as $v_{\mathrm{rel}} = v_{\mathrm{e}} + v_{\mathrm{p}}$.

- Sticking phase with potential elastic energy
- Presliding phase mixing elastic and plastic behaviour
- Slipping phase with purely plastic behaviour



Microscopic displacement of the bristles between the bow and the string[a]

[a] Willemsen, Bilbao, and Serafin, "Real-time implementation of an elasto-plastic friction model applied to stiff strings using finite difference schemes".

Dupont friction model [2]

- Includes a compliance damping σ_c and a fluid damping σ_f.
- Is given as an implicit relation

$$\begin{cases} f_{i\to s} &= k_e x_e + \sigma_c \frac{\mathrm{d}x_e}{\mathrm{d}t} + \sigma_f v_{rel} \\ \frac{\mathrm{d}x_e}{\mathrm{d}t} &= v_{rel}\left(1 - \frac{|v_{rel}|\alpha(x_e, w_{rel})k_e}{v_{rel} f_{ss}(v_{rel})} x_e\right) \end{cases}$$

where the steady state friction force (Stribeck curve) is defined by:

$$f_{\mathrm{ss}}(v_{\mathrm{rel}}) = \frac{1}{k_{\mathrm{e}}}\left(f_C + (f_{\mathrm{S}} - f_C)\exp\left(-\left(\frac{v_{\mathrm{rel}}}{v_{\mathrm{S}}}\right)^2\right)\right),$$

and $\alpha(x_{\mathrm{e}}, v_{\mathrm{rel}}) \in (0,1)$ is an adhesion map.

[2] Dupont, P., Hayward, V., Armstrong, B., Altpeter, F. (2002). Single state elastoplastic friction models. IEEE Transactions on automatic control, 47(5), 787-792.

Dupont friction model

Elastoplastic friction model

Elastic behavior:

$$x_{\mathrm{e}}(t) = x_{\mathrm{e}}(0) + \int_0^t v_{\mathrm{e}}(\tau)\mathrm{d}\tau$$

with potential energy $h_{\mathrm{e}}(x_{\mathrm{e}}) = \frac{k_{\mathrm{e}} x_{\mathrm{e}}^2}{2}$ and associated elastic force

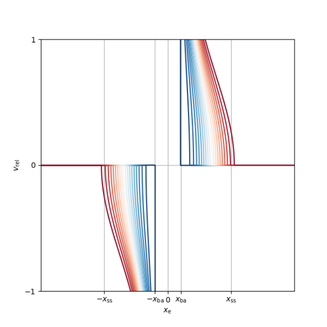
$$f_{\mathrm{e}\to\mathrm{s}} = \frac{dh_{\mathrm{e}}}{dx_{\mathrm{e}}} = k_{\mathrm{e}} x_{\mathrm{e}}$$

Dissipation relative to the plastic behaviour:

$$\alpha(x_e, v_{rel}) = \begin{cases} 0 & \text{if} \quad \text{sign}(x_e) \neq \text{sign}(v_{rel}), \\ 0 & \text{if} \quad \text{sign}(x_e) = \text{sign}(v_{rel}), \quad |x_e| \leq x_{ba}, \\ \widehat{\alpha}(x_e, v_{rel}) & \text{if} \quad \text{sign}(x_e) = \text{sign}(v_{rel}), \quad x_{ba} < |x_e| < x_{ss}(v_{rel}), \\ 1 & \text{if} \quad \text{sign}(x_e) = \text{sign}(v_{rel}), \quad x_{ss}(w_{rel}) \leq |x_e|, \end{cases}$$

where for $x_{ba} < |x_e| < x_{ss}(w_{rel})$:

$$\widehat{\alpha}(x_e, v_{rel}) = \frac{1}{2}\left(1 + \sin\left(\pi \frac{|x_e| - \left(\frac{x_{ss}(v_{rel}) + x_{ba}}{2}\right)}{x_{ss}(v_{rel}) - x_{ba}}\right)\right).$$



PHS Formulation

Dissipation variables

$$\begin{aligned} w_{\mathrm{rel}} &= \quad v_{\mathrm{rel}} &= \dot{x}_{\mathrm{e}} + v_{\mathrm{p}}, \\ w_{\mathrm{p}} &= \quad k_{\mathrm{e}} x_{\mathrm{e}} &= h'_{\mathrm{e}}(x_{\mathrm{e}}), \end{aligned}$$

the Dupont model of friction reads:

$$\begin{cases} f_{i\to s} &= k_{\mathrm{e}} x_{\mathrm{e}} + (\sigma_{\mathrm{c}} + \sigma_{\mathrm{f}}) w_{\mathrm{rel}} - \sigma_{\mathrm{c}}\, r_{\mathrm{Lu}}(w_{\mathrm{rel}})\, w_{\mathrm{p}}, \\ \frac{\mathrm{d}x_e}{\mathrm{d}t} &= v_{\mathrm{rel}} - r_{Du}(x_e, w_{rel}) w_{\mathrm{p}}, \end{cases}$$

where $r_{Du}(x_e, w_{rel}) \triangleq \alpha(x_e, w_{rel}) \frac{|w_{rel}|}{f_{ss}(w_{rel})} > 0$.

PHS formulation

The PHS formulation of the Dupont model of friction is then

$$\begin{pmatrix} \frac{\mathrm{d}x_\mathrm{e}}{\mathrm{d}t} \\ \hline w_\mathrm{rel} \\ w_\mathrm{p} \\ \hline f_{\mathrm{i}\to\mathrm{s}} \\ f_{\mathrm{i}\to\mathrm{c}} \end{pmatrix} = \left(\begin{array}{c|cc|cc} 0 & 0 & -1 & -1 & +1 \\ \hline 0 & 0 & 0 & -1 & +1 \\ +1 & 0 & 0 & 0 & 0 \\ \hline +1 & +1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 \end{array}\right) \begin{pmatrix} h'_\mathrm{e}(x_\mathrm{e}) \\ \hline z_\mathrm{rel}\big(w_\mathrm{rel}, w_\mathrm{p}; x_\mathrm{e}\big) \\ z_\mathrm{p}\big(w_\mathrm{rel}, w_\mathrm{p}; x_\mathrm{e}\big) \\ \hline v_\mathrm{i} \\ v_\mathrm{c} \end{pmatrix},$$

with dissipation variables $\boldsymbol{w} = (w_\mathrm{rel}, w_\mathrm{p})^\intercal$, dissipation function $\boldsymbol{z}(\boldsymbol{w}; x_\mathrm{e}) = R(\boldsymbol{w}; x_\mathrm{e})\,\boldsymbol{w}$ where

$$R(\boldsymbol{w}; x_\mathrm{e}) = \begin{pmatrix} \sigma_c + \sigma_f & -\sigma_c\, r_{Du}(x_e, w_{rel}) \\ 0 & r_{Du}(x_e, w_{rel}) \end{pmatrix} \succeq 0.$$

Remark: (i) the dissipated power is $\boldsymbol{w}^\intercal \boldsymbol{z}(\boldsymbol{w}; x_\mathrm{e}) \geq 0, \forall x_\mathrm{e}$. (ii) Generic model (Dalh for $\sigma_c = \sigma_f = 0$ and $f_{ss} = f_C$, Lugre for $\alpha(x_e, w_{rel}) = 1$).

Infinite dimensional PHS

Uncoupled concatenation of the string model with the friction model:

$$\begin{pmatrix} \frac{\partial \boldsymbol{x}_s}{\partial t}(\xi,t) \\ \hline w_s(\xi,t) \\ \hline y_s(\xi,t) \\ \hline \frac{d\boldsymbol{x}_i}{dt}(t) \\ \hline \boldsymbol{w}_i(t) \\ \hline y_i(t) \\ y_c(t) \end{pmatrix} = \left(\begin{array}{c|c|c|c|c|cc} \mathcal{J} & \mathcal{K} & \mathcal{G} & 0 & 0 & 0 & 0 \\ \hline -\mathcal{K}^\intercal & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline -\mathcal{G}^\intercal & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \mathrm{J}_i & \mathrm{K}_i & \mathrm{G}_i & \mathrm{G}_c \\ \hline 0 & 0 & 0 & -\mathrm{K}_i^\intercal & 0 & \mathrm{C}_i & \mathrm{C}_c \\ \hline 0 & 0 & 0 & -\mathrm{G}_i^\intercal & -\mathrm{C}_i^\intercal & 0 & 0 \\ 0 & 0 & 0 & -\mathrm{G}_c^\intercal & -\mathrm{C}_c^\intercal & 0 & 0 \end{array}\right) \begin{pmatrix} \frac{\partial h_s}{\partial \boldsymbol{x}_s}(\boldsymbol{x}_s(\xi,t)) \\ \hline z_s(w_s(\xi,t)) \\ \hline u_s(\xi,t) \\ \hline \frac{\partial \mathrm{H}_i}{\partial \boldsymbol{x}_i}(\boldsymbol{x}_i(t)) \\ \hline \boldsymbol{z}_i(\boldsymbol{w}_i(t)) \\ \hline u_i(t) \\ u_c(t) \end{pmatrix},$$

with the coupling:

$$\begin{cases} u_s(\xi,t) = -y_i(t)\,\gamma(\xi), \\ u_i(t) = \int_\Omega \gamma(\xi) y_s(\xi,t) d\xi, \end{cases}$$

where $\gamma : \Omega \to [0,1]$ is the spatial distribution of the interaction force along the string.

Finite dimensional uncoupled PHS (modal projection)

$$\begin{pmatrix} \widehat{M}_x \frac{d\widehat{\mathbf{x}}_s}{dt}(t) \\ \frac{d\mathbf{x}_i}{dt}(t) \\ \hline \widehat{M}_w \, \widehat{\mathbf{w}}_s(t) \\ \mathbf{w}_i(t) \\ \hline \widehat{y}_s(\xi_B, t) \\ y_i(t) \\ y_c(t) \end{pmatrix} = \left(\begin{array}{cc|cc|ccc} \widehat{J}_s & 0 & \widehat{K}_s & 0 & \widehat{G}_s & 0 & 0 \\ 0 & J_i & 0 & K_i & 0 & G_i & G_c \\ \hline -\widehat{K}_s^T & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -K_i^T & 0 & 0 & 0 & C_i & C_c \\ \hline -\widehat{G}_s^T & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -G_i^T & 0 & -C_i^T & 0 & 0 & 0 \\ 0 & -G_c^T & 0 & -C_c^T & 0 & 0 & 0 \end{array}\right) \begin{pmatrix} \frac{\partial \widehat{H}_s}{\partial \widehat{\mathbf{x}}_s}\left(\widehat{\mathbf{x}}_s(t)\right) \\ \frac{\partial \widehat{H}_i}{\partial \mathbf{x}_i}\left(\mathbf{x}_i(t)\right) \\ \hline \widehat{\mathbf{z}}_s\left(\widehat{\mathbf{w}}_s(t)\right) \\ \mathbf{z}_i\left(\mathbf{w}_i(t); \mathbf{x}_i(t)\right) \\ \hline \widehat{u}_s(\xi_B, t) \\ u_i(t) \\ u_c(t) \end{pmatrix}$$

with

$$\begin{pmatrix} \widehat{u}_s(\xi_b, t) \\ u_i \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \sum_k \widehat{y}_s(t) e_k(\xi_B) \\ y_i \end{pmatrix}$$

when the interaction force is located in one point.

Finite dimensional coupled PHS

$$\left(\begin{array}{c} \widehat{M}_{x}\frac{d\widehat{\mathbf{x}}_s}{dt}(t) \\ \frac{d\mathbf{x}_i}{dt}(t) \\ \hline \widehat{M}_{w}\,\widehat{\mathbf{w}}_s(t) \\ \mathbf{w}_i(t) \\ \hline y_c(t) \end{array}\right) = \left(\begin{array}{cc|cc|c} \widehat{J}_s & +\widehat{G}_s G_i^T & +\widehat{K}_s & +\widehat{G}_s C_i^T & 0 \\ -G_i\widehat{G}_s^T & J_i & 0 & +K_i & +G_c \\ \hline -\widehat{K}_s^T & 0 & 0 & 0 & 0 \\ -C_i\widehat{G}_s^T & -K_i^T & 0 & 0 & +C_c \\ \hline 0 & -G_c^T & 0 & -C_c^T & 0 \end{array}\right) \left(\begin{array}{c} \frac{\partial \widehat{H}_s}{\partial \widehat{\mathbf{x}}_s}(\widehat{\mathbf{x}}_s(t)) \\ \frac{\partial H_i}{\partial \mathbf{x}_i}(\mathbf{x}_i(t)) \\ \hline \widehat{\mathbf{z}}_s(\widehat{\mathbf{w}}_s(t)) \\ \mathbf{z}_i(\mathbf{w}_i(t);\mathbf{x}_i(t)) \\ \hline u_c(t) \end{array}\right)$$

- Input $u_c(t)$: control velocity
- Output $y_c(t)$: control force

Outline

PHS of the bowed string

The finite dimensional model used for the simulations is obtained by

1. Projecting the PHS of the string on the modal basis $e_k(\xi) = \sqrt{\frac{2}{L}}\sin(\frac{\kappa\pi\xi}{L})$,
2. Connecting the ouptput of the string (velocity) to the input of the interaction model and the output of the interaction model (force) to the input of the string.

$$\begin{pmatrix} \frac{\mathrm{d}Q}{\mathrm{d}t}(t) \\ \frac{\mathrm{d}P}{\mathrm{d}t}(t) \\ \frac{\mathrm{d}x_\mathrm{e}}{\mathrm{d}t} \\ \hline W_\mathrm{s}(t) \\ w_\mathrm{rel} \\ w_\mathrm{p} \\ \hline f_{\mathrm{i}\to\mathrm{c}} \end{pmatrix} = \left(\begin{array}{ccc|cccc|c} 0 & \frac{1}{\mu}\mathrm{I}_K & 0 & 0 & 0 & 0 & 0 \\ \frac{-1}{\mu}\mathrm{I}_K & 0 & \frac{-1}{\mu}E(\xi_B) & \frac{-1}{\mu}\mathrm{I}_K & \frac{-1}{\mu}E(\xi_B) & 0 & 0 \\ 0 & \frac{1}{\mu}E^T(\xi_B) & 0 & 0 & 0 & -1 & -1 \\ \hline 0 & \frac{1}{\mu}\mathrm{I}_K & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\mu}E^T(\xi_B) & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & +1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & +1 & 0 & +1 & 0 & 0 \end{array}\right) \begin{pmatrix} T_0 D_k^2 Q(t) \\ \mu P(t) \\ h'_\mathrm{e}(x_\mathrm{e}) \\ \hline aP(t) \\ z_\mathrm{rel}(w_\mathrm{rel}, w_\mathrm{p}; x_\mathrm{e}) \\ z_\mathrm{p}(w_\mathrm{rel}, w_\mathrm{p}; x_\mathrm{e}) \\ \hline v_\mathrm{c} \end{pmatrix}$$

where $E^T(\xi) = (e_1(\xi)e_2(\xi)\dots e_K(\xi))$ and ξ_B denotes the position of the bow along the string.

Time discretisation

(i) Time derivative

$$\frac{\mathrm{d}x}{\mathrm{d}t}(t) \approx \frac{x(k+1)-x(k)}{T} = \frac{\delta x}{T},$$

where T is the discrete time step.

(ii) Gradient $\nabla H(x)$

The discrete gradient[a]

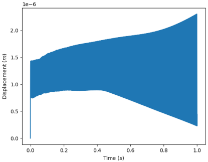
$$\nabla_d H(x, x+\delta x) = \frac{H(x+\delta x)-H(x)}{\delta x}\frac{\delta x}{T}.$$

In the case of a linear system (quadratic hamiltonian: $H = \frac{1}{2}x^T Qx$), this is:

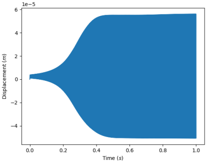
$$\nabla_d H(x, x+\delta x) = \mathrm{Q}\left(x(k) + \frac{\delta x(k)}{2}\right).$$

[a] Toshiaki Itoh and Kanji Abe. "Hamiltonian-conserving discrete canonical equations based on variational difference quotients". In: *Journal of Computational Physics* 76.1 (1988), pp. 85–102.

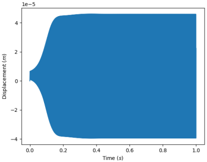
Sound synthesis results



(a) $\xi_B = L/10$



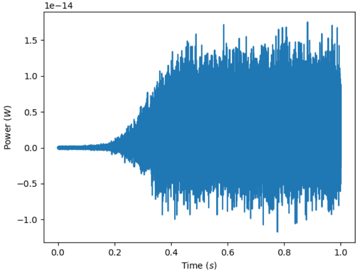
(b) $\xi_B = L/4$



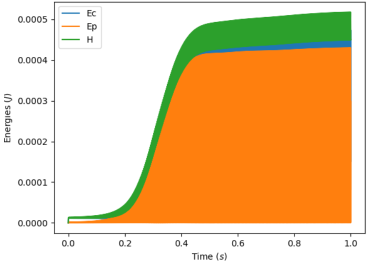
(c) $\xi_B = L/2$

Figure: Displacement of the string at the interaction point for differents bowing points.

Sound synthesis results



(a) Power balance
$\frac{dH(x)}{dt} + D + P$



(b) Energies

Power balance is preserved as it can be seen in the first column which present the difference signal between the variation of the Hamiltonian and the power damped minus the power of the sources.

Conclusion

Results

- Generic interaction model (collisions, friction) in the PHS formalism
- Passive-guaranteed numerical simulation
- Self-oscillations emergence depends on several physical and interaction parameters (bowing point, velocity and force of the bow)

Perspectives

- Application to nonlinear resonators (e.g. string or plates in large deformations)
- Optimisation of numerical schemes and code with an aim for real-time synthesis
- Identification of Dupont model parameters