Blowups of differential equations at singular points

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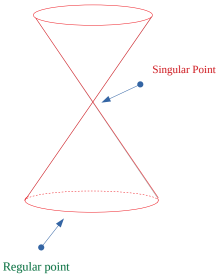
Blowup at a point

Blow-up of a singular point of a foliation

Looking for a good blowup !

Resolution of singularities

Let $X \subseteq \mathbb{R}^d$ or $X \subseteq \mathbb{C}^d$ be a variety with a few singularities :

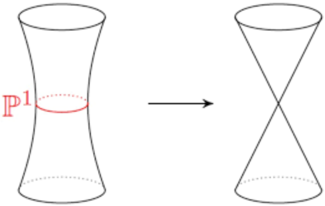


Formal definition

A variety $X \subseteq \mathbb{R}^d$ or $\mathbb{C}^d$ is the set of solutions of equations, $\varphi_1 = 0, \ldots, \varphi_k = 0$, with $\varphi_1, \ldots, \varphi_k$ are polynomial functions.

Resolution of singularities

The aim is to replace X with a good « approximation » with the same dimension.



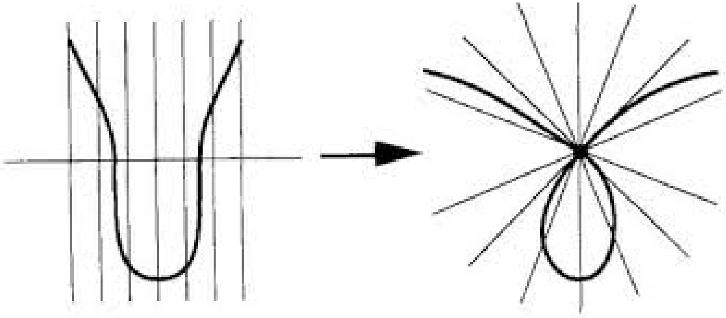
Formal definition

Let X, Y be varieties. A morphism $\pi\colon Y \to X$ is a resolution of singularities if Y has no singular points and π is proper and birational, that is there exist two open set $U \subset Y, V \subset X$ with $V = \pi(U)$ such that $\pi\colon U \to V$ is an isomorphism.

The simplest transformation which could resolve a singularity is called blow-up !

Blowups on varieties !

Roughly speaking "blow up of a point" means that we replace the point by all the lines that pass through that point, equivalently by a sphere where we identify antipodal points !

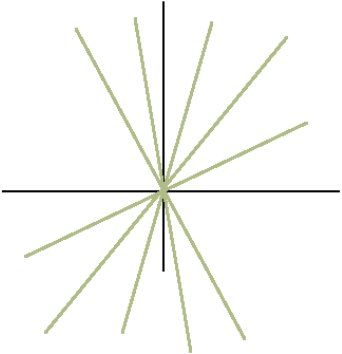


Review of Projective Space

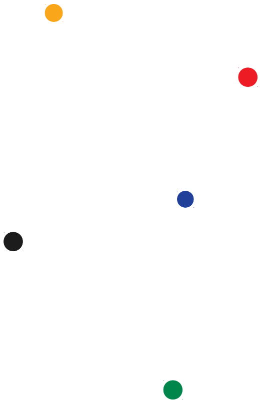
Some ways to realize projective space $\mathbb{P}^d_{\mathbb{C}}$

1. As $\mathbb{C}^{d+1} \setminus \{0\}/\sim$, under the identification $(a_0, \ldots, a_d) \sim (\lambda a_0, \ldots, \lambda a_d)$, for $\lambda \in \mathbb{C}^{\times}$, « straight lines through the origin ».
2. As $S^d/\sim$, that is $\{x \in \mathbb{C}^{d+1} \mid \|x\| = 1\}$/"antipodal points".

We usually denote a « point = straight line » of $\mathbb{P}^d_{\mathbb{C}}$ by $(a_0 : a_1 : \cdots : a_d)$.



Blowup $\mathbb{C}^d$ at the origin



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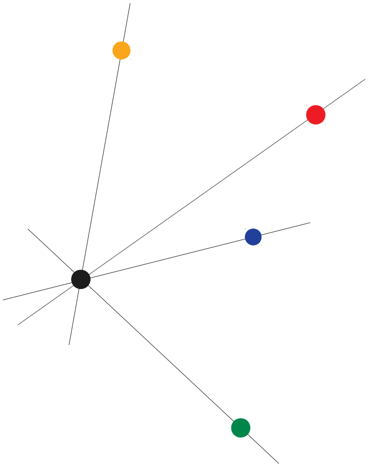

Blowup $\mathbb{C}^d$ at the origin

The Blowup space :

Consider the closed subvariety,

$$\begin{aligned} B_0(\mathbb{C}^d) &= \left\{(x,\ell) \in \mathbb{C}^d \times \mathbb{P}^{d-1}_{\mathbb{C}} \mid x \in \mathbb{C}^d, \ell \in \mathbb{P}^d_{\mathbb{C}}\right\} \subset \mathbb{C}^d \times \mathbb{P}^{d-1}_{\mathbb{C}} \\ &= \left\{((x_1,\dots,x_d),(y_1:\dots:y_d)) \in \mathbb{C}^d \times \mathbb{P}^{d-1}_{\mathbb{C}} \mid x_i y_j - x_j y_i = 0\right\} \end{aligned}$$

together with the projection

$$\pi\colon B_0(\mathbb{C}^d) \longrightarrow \mathbb{C}^d, (x,\ell) \mapsto x.$$

So, π restricts to an isomorphism between $B_0(\mathbb{C}^d) \setminus \pi^{-1}(0)$ and $\mathbb{C}^d \setminus \{0\}$.

Definition

The *blowup of* $\mathbb{C}^d$ *at the origin* is the projection $\pi\colon B_0(\mathbb{C}^d) \longrightarrow \mathbb{C}^d$. The fibers of $\pi : B_0(\mathbb{C}^d) \longrightarrow \mathbb{C}^d$ are

$$\pi^{-1}(x) = \begin{cases} (x,[x]) & \text{if } x \neq 0, \ \text{(no other choice)} \\ \{0\} \times \mathbb{P}^{d-1} & \text{if } x = 0, \ \text{(lines through the origin)} \end{cases}$$

$E = \pi^{-1}(0) \subset B_0\mathbb{C}^n$ is called the exceptional divisor. Points on E are in bijection with lines through 0 of $\mathbb{C}^d$.

Expression in local coordinates

Let for instance $d = 3$. In the z-chart, π is the map
$(x, y, z) \mapsto (xz, yz, z)$.

The Blow-Up of $W \subset \mathbb{C}^d$ at $p \in W$

Define the *strict transform* of W as

$$B_pW = \overline{\pi^{-1}(W \setminus \{p\})} \subset B_p\mathbb{C}^d.$$

The blowup of W at p is the projection map $\pi|_{B_pW} \colon B_pW \longrightarrow W$.

When $p \in \mathbb{C}^d$ is not the origin we shift the origin to p by the transformation $x \longmapsto x - p$.

Resolution of Kleinian Singularities

Let X be the surface given by the equation $A_n : x^2 + y^2 + z^{n+1} = 0$.

Its blow up at the origin

In the z-direction, we obtain $z^2(x^2 + y^2 + z^{n-1}) = 0$. Notice that it is exactly a singularity of type A_{n-2}.

After several blow-ups, it reduces to the case of A_0 or A_{-1} which are smooth.

For $A_2 : x^2 + y^2 + z^3 = 0$

A single blow-up is enough to resolve its singularity. We obtain A_0 which is smooth !

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Blowup a singular vector field

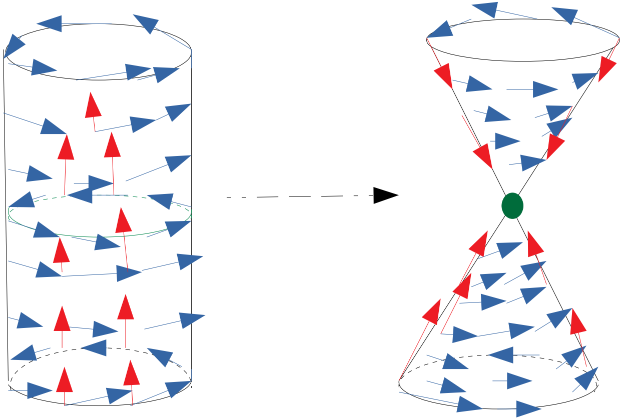
Definition

Let X be a vector field on $\mathbb{C}^d$ such that $X(0) = 0$. The blow-up of X at 0 is the pull-back vector field $\pi^*(X)$ of X by the blow up map $\pi \colon B_0W \longrightarrow W$.

Example

1. The blow-up of the Euler vector field $E = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}$ in the z-direction is the vector field $z\frac{\partial}{\partial z}$.
2. The blow up the vector field $x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$ in the z-direction is itself again.

Blowup a singular vector field



What is a good blow-up for a foliation ?

We need these properties

- Diffeomorphism on regular points
- Less singular
- The same number of leaves

Good blow up !

Blowup singular foliations

Definition

Given a foliation $\mathcal{F}$ of W and $p \in W$, the blow-up of $\mathcal{F}$ at p is the pull-back foliation $\pi^*(\mathcal{F})$ of $\mathcal{F}$ by the blow up map $\pi\colon B_pW \longrightarrow W$.

Example (Concentric spheres)

Let $\mathcal{F}$ be the singlar foliation generated by the vector fields, $x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$, $y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}$, $z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}$. $\mathcal{F}$ has one singular leaf which is $\{0\}$.

Its blow-up yields a regular foliation.

That is a good blow up !

The blowup in the sense above may be a bit lacking

It adds leaves

Consider the foliation $\mathcal{F} = Span_{C^\infty(\mathbb{R}^d)}\{x_i x_j \frac{\partial}{\partial x_k}, i, j, k = 1, \ldots, d\}$ made of vector fields vanishing to order 2 at 0 on $\mathbb{R}^d$!

Let $n = 3$, and look at the blow up in the z-direction. All the the vector fields of $\widetilde{\widetilde{\mathcal{F}}}$ vanish on $\pi^{-1}(0)$. Which gives an infinity of leaves to $\widetilde{\widetilde{\mathcal{F}}}$.

Not a good blow up !

More generally, We can do blow up along a sub-variety

Let $X \subset \mathbb{R}^d, \mathbb{C}^d$ be variety and let $Z = \{\psi_1 = 0, \dots, \psi_k = 0\}$ be a subvariety of X. Consider the morphism

$$\gamma \colon X \setminus Z \longrightarrow \mathbb{P}^{d-1},\ x \mapsto (\psi_1(x) : \cdots : \psi_k(x))$$

Definition

The Zariski closure $\widetilde{X}$ of the graph Γ of γ inside $X \times \mathbb{P}^{d-1}$ together with the restriction $\pi \colon \widetilde{X} \longrightarrow X$ of the projection map $X \times \mathbb{P}^{d-1} \longrightarrow X$ is the blowup of X along Z.

Properties of foliations can be studied in a purely algebraic way

Given a singular foliation $\mathcal{F}$ over a manifold M we use the relations between its generators, relations of its relations... to bring an algebraic structure behind $\mathcal{F}$.

$$\cdots \to C^\infty(M)^r \to C^\infty(M)^p \xrightarrow{\begin{pmatrix}\text{minimal}\\ \text{system of}\\ \text{relations}\\ \text{on generators}\\ \text{of } \mathcal{F}\end{pmatrix}} C^\infty(M)^q \xrightarrow{\begin{pmatrix}\text{minimal}\\ \text{system of}\\ \text{generators}\\ \text{of } \mathcal{F}\end{pmatrix}} \mathcal{F}$$

We reproduce the Lie bracket of $\mathcal{F}$ on the module $C^\infty(M)^q$. Unfortunately, it does not satisfy "Jacobi identity" in general. i.e,

$$[[X,Y],Z] + \circlearrowleft (X,Y,Z) \neq 0.$$

This will give rise to an extra bracket "3-ary-bracket", $C(X,Y,Z)$....

Properties of foliations can be studied in a purely algebraic way

$$3 \text{ generators} \longrightarrow 1 \text{ relation} = C(X, Y, Z).$$

Look at an effect of blowup on on this algebraic structure

On the blow-up space of the hypersurface $W = \{\varphi = \sum_{i=1}^{d} x_i^3 + t^3 = 0\}$ at the origin

Consider the singular foliation

$$\mathcal{F} = \{\text{Vector fields on } \mathbb{R}^d \text{ such that } X[\varphi] = 0\}$$

$\mathcal{F}$ is generated by the vector fields $x_i^2 \frac{\partial}{\partial x_j} - x_j^2 \frac{\partial}{\partial x_i}$,.....

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We know that there is no way to get Jacobi in this case.

The blowup of W at zero in the t-direction is given by the equation,

$$\sum_{i=1}^{d} x_i^3 = -1,$$

with t left to be a free variable.

$\widetilde{\mathcal{F}}$ is generated by the vector fields $\widetilde{\Delta} ij = t\left(x_i^2 \frac{\partial}{\partial x_j} - x_j^2 \frac{\partial}{\partial x_i}\right)$, $\widetilde{\Delta}_{it} = t\left(t x_i^2 \frac{\partial}{\partial t} - \frac{\partial}{\partial x_i} - x_i^2 \sum_{j=1}^{d} x_j \frac{\partial}{\partial x_j}\right)$. We have

$$\widetilde{\mathcal{F}} = Span\{\widetilde{\Delta}_{it},\ i = 1, \dots N\}$$

The blowup foliation $\widetilde{\mathcal{F}}$ coincides with the Lie algebroid on the trivial bundle E_{-1} whose sections are the $\mathcal{O}_{\widetilde{W} \cap U}$-module generated by some set $(e_i,\ i = 1 \dots N)$, equipped with the Lie bracket :

$$[e_i, e_j] := 2t\left(x_i^2 e_j - x_j^2 e_i\right) \tag{1}$$

together with the anchor map ρ which assigns e_i to $\widetilde{\Delta}_{it}$, for $i = 1 \dots N$.

Proposition

After blow-up, we have Jacobi. But we have « too much » leaves.

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A blow-up through bi-submersions

A mecanic process, but faillable, that gives a good resolution.

Theorem.(RL)

Let $\Gamma \rightrightarrows M$ a Lie groupoid (bissubmersion) and $C \subset \overline{S}$ a crossing ($S \subset M$ a leaf). Take B to be the normalization of C that is, the vector bundle $B \to C \cap S$ defined over $C \cap S$ by

$$B_x = \rho^{-1}(T_x C), \; x \in C \cap S.$$

Then, $B \backslash \Gamma_C \longrightarrow \overline{S}$ is a good resolution. Where $\Gamma_C := s^{-1}(C)$.

I thank you all !

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