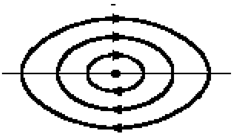
Formulation géométrique de la mécanique: histoire, philosophie, réligion...

Vladimir Salnikov

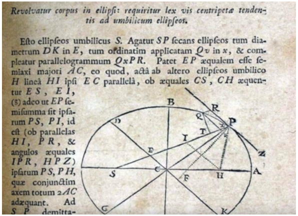
GDR Groupement de recherche
GDR-GDM n°2043
Géométrie Différentielle et Mécanique

La Rochelle, 8 Juillet 2021

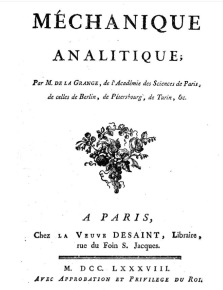
Very classical story

Canonical case: given $H\colon T^*Q \to \mathbb{R}$ $\dot{\mathbf{q}} = \dfrac{\partial H}{\partial \mathbf{p}}, \quad \dot{\mathbf{p}} = -\dfrac{\partial H}{\partial \mathbf{q}}$	Symplectic geometry $\omega = \sum_i dp_i \wedge dq^i$ $\iota_{X_H}\omega = \mathrm{d}H$	
More general case: given $H\colon M \to \mathbb{R}$ and an antisymmetric $J(\mathbf{x})$ $\dot{\mathbf{x}} = J(\mathbf{x})\dfrac{\partial H}{\partial \mathbf{x}}$	Poisson geometry $\{\cdot,\cdot\}$ on : $C^\infty(M)$ $X_H = \{H,\cdot\}$ $\dot{\mathbf{x}} = \{H,\mathbf{x}\}$	

History


Revolvatur corpus in ellipsi: requiritur lex vis centripetæ tendentis ad umbilicum ellipseos.

Esto ellipseos umbilicus S. Agatur SP secans ellipseos tum diametrum DK in E, tum ordinatim applicatam Qv in x, & compleatur parallelogrammum QxPR. Patet EP æqualem esse semiaxi majori AC, eo quod, actâ ab altero ellipseos umbilico H lineâ HI ipsi EC parallelâ, ob æquales CS, CH æquentur ES, EI, adeo ut EP semisumma sit ipsarum PS, PI, id est (ob parallelas HI, PR, & angulos æquales IPR, HPZ) ipsarum PS, PH, quæ conjunctim axem totum 2AC adæquant. Ad SP demitta-


MÉCHANIQUE

ANALITIQUE;

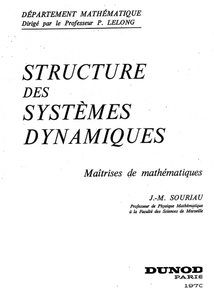
Par M. DE LA GRANGE, de l'Académie des Sciences de Paris, de celles de Berlin, de Pétersbourg, de Turin, &c.

A PARIS,

Chez LA VEUVE DESAINT, Libraire, rue du Foin S. Jacques.

M. DCC. LXXXVIII.

AVEC APPROBATION ET PRIVILEGE DU ROI.

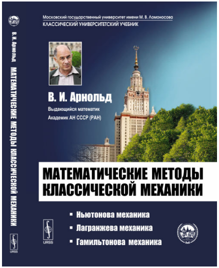

DÉPARTEMENT MATHÉMATIQUE
Dirigé par le Professeur P. LELONG

STRUCTURE DES SYSTÈMES DYNAMIQUES

Maîtrises de mathématiques

J.-M. SOURIAU
Professeur de Physique Mathématique à la Faculté des Sciences de Marseille

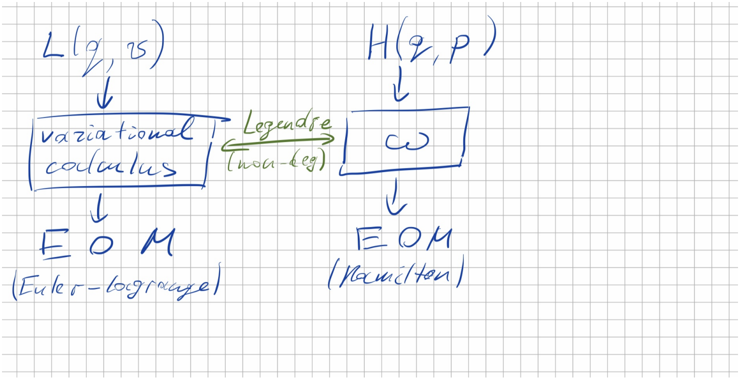
DUNOD
PARIS
1970



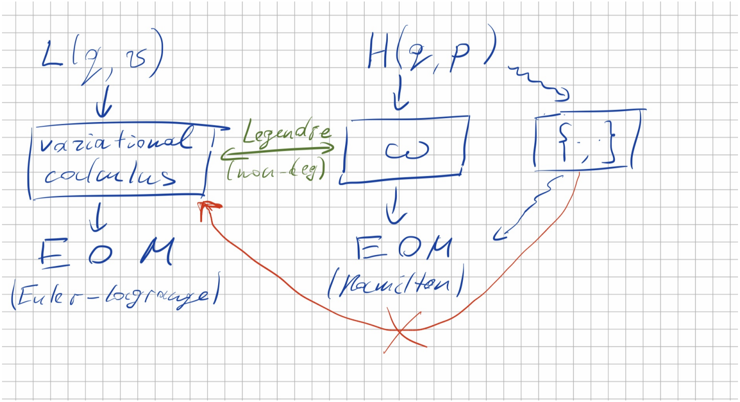
conférence de Jean-Pierre Bourguignon

https://www.youtube.com/watch?v=93hFolIBo0Q

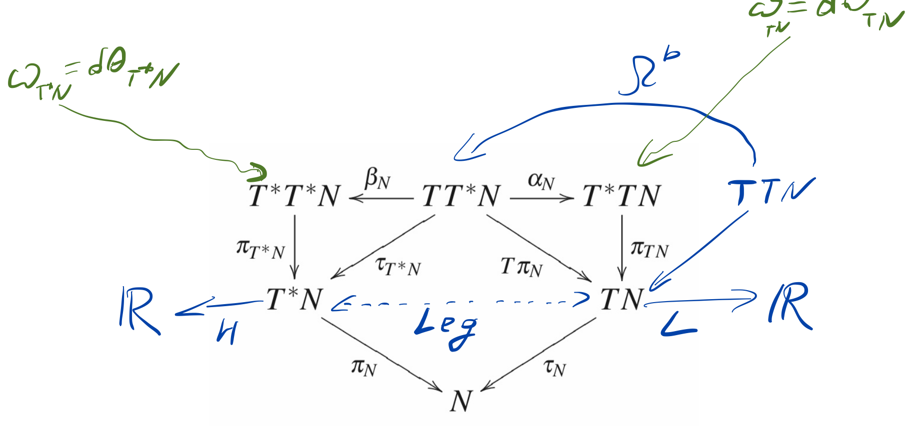
Classical story in modern language



Classical story in modern language



Classical story revisited (Tulczyjew)



Rappel: T^*Q, $\omega = \sum_i dp_i\, dq^i = d\big(\underbrace{\textstyle\sum p_i\, dq^i}_{\theta}\big)$

$p_i \quad q^i$

Classical story revisited (Tulczyjew)

4.3.5 Theorem (W.M. Tulczyjew). *With the notations specified above (4.3.4), let* $X_H : T^*N \to TT^*N$ *be the Hamiltonian vector field on the symplectic manifold* $(T^*N, \mathrm{d}\theta_N)$ *associated to the Hamiltonian* $H : T^*N \to \mathbb{R}$*, defined by* $\mathrm{i}(X_H)\mathrm{d}\theta_N = -\mathrm{d}H$*. Then*

$$X_H(T^*N) = \beta_N^{-1}\big(\mathrm{d}H(T^*N)\big) .$$

Moreover, the equality

$$\alpha_N^{-1}\big(\mathrm{d}L(TN)\big) = \beta_N^{-1}\big(\mathrm{d}H(T^*N)\big)$$

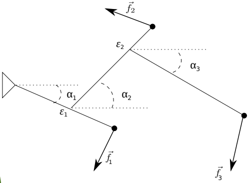
holds if and only if the Lagrangian L is hyper-regular and such that

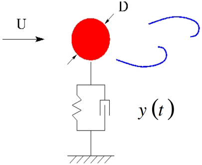
$$\mathrm{d}H = \mathrm{d}\big(E_L \circ \mathcal{L}_L^{-1}\big) ,$$

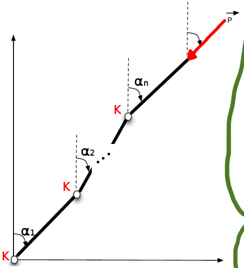
where $\mathcal{L}_L : TN \to T^*N$ *is the Legendre map and* $E_L : TN \to \mathbb{R}$ *the energy associated to the Lagrangian L.*

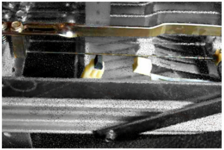
Beyond: port-Hamiltonian systems; constraints

A lot of examples
(ask Antoine Falaize, David Roze) :









Conjecture (VS): Everything is port-Hamiltonian.

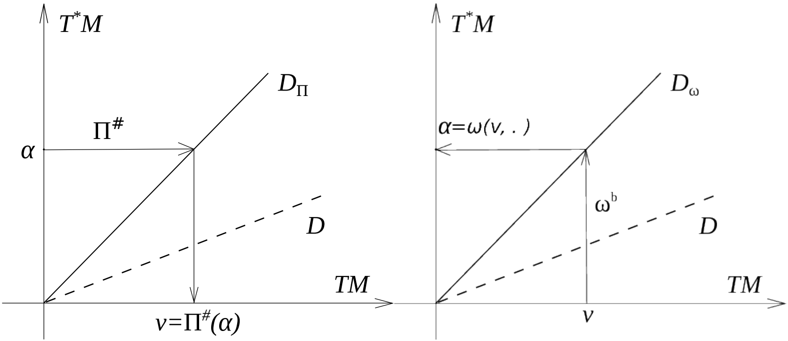
almost

Dirac structures

Geometry behind: Courant algebroids, Dirac structures

On $E = TM \oplus T^*M$ (or more generally $F \oplus F^*$)
Symmetric pairing: $< v \oplus \eta, v' \oplus \eta' >= \eta(v') + \eta'(v)$,
Dorfman bracket: $[v \oplus \eta, v' \oplus \eta']_D = [v, v']_{\mathsf{Lie}} \oplus (\mathcal{L}_v \eta' - \mathrm{d}\eta(v'))$.

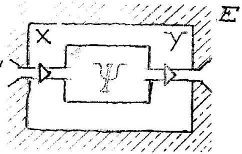
A *Dirac structure* $\mathcal{D}$ is a maximally isotropic (Lagrangian) subbundle of E closed w.r.t. $[\cdot, \cdot]_D$



$\mathcal{D}_\Pi = graph(\Pi^\sharp) = \{(\Pi^\sharp \alpha, \alpha)\}$ $\qquad$ $\mathcal{D}_\omega = graph(\omega^\flat) = \{(v, \iota_v \omega)\}$

SYSTEMS ENGINEERING SEMINAR

"PORTS, ENERGY, SYSTEMS"



1959

Professor Henry M. Paynter will present a seminar on the subject, "Ports, Energy and Thermodynamic Systems" on Friday, April 24 at 3:15 p.m. in Room B 103 of the Mechanical Engineering Building.

Dr. Paynter is Assistant Professor of Mechanical Engineering at M.I.T. and Director of the American Center for Analog Computing (a facility of Pi-Square Engineering Company). He is prominently recognized for his work in controls, dynamic systems, analog simulation and related fields. He is the author of very many authoritative papers covering a wide range of topics. He has also done extensive consulting work in industry and government.

Dr. Paynter is a very interesting and stimulating speaker. His viewpoints are novel and thought-provoking.

CASE INSTITUTE OF TECHNOLOGY

Cleveland, Ohio

– Henry M. Paynter, Analysis and Design of Engineering Systems, MIT Press, Cambridge, Massachusetts, 1961.

– Jean U. Thoma, Introduction to Bond Graphs and Their Applications, Pergamon Press, Oxford, 1975.

Philosophy

https://math.ucr.edu/home/baez/week292.html

	displacement q	flow q'	momentum p	effort p'
Mechanics (translation)	position	velocity	momentum	force
Mechanics (rotation)	angle	angular velocity	angular momentum	torque
Electronics	charge	current	flux linkage	voltage
Hydraulics	volume	flow	pressure momentum	pressure
Thermodynamics	entropy	entropy flow	temperature momentum	temperature
Chemistry	moles	molar flow	chemical momentum	chemical potential

(et Économie :))

"This Week's Finds in Mathematical Physics", by John Baez

Philosophy

https://math.ucr.edu/home/baez/week292.html

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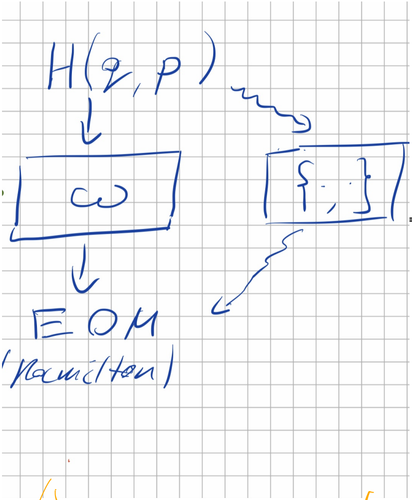
Chemistry	moles	molar flow	chemical momentum	chemical potential

J.-M. Souriau's thermodynamics,
cf. also C.-M. Marle `https://arxiv.org/abs/1608.00103`

What is the conceptual difference?



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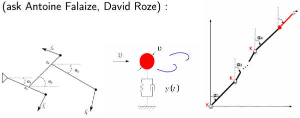

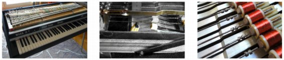

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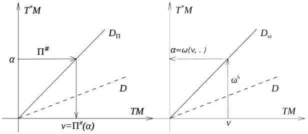

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⚠ Numerics (ask Anthony Gravouil)

Some directions

– Geometrical Mechanics on algebroids
K.Grabowska, J.Grabowski, P.Urbański,
https://arxiv.org/abs/math-ph/0509063

– Variational formulation of dynamics on Dirac structures,
j/w O. Cosserat, C. Laurent-Gengoux, A. Kotov, L. Ryvkin.

– . . .

a bit of religion

Quarks
$SU(3)$ symmetry

$\Rightarrow \Rightarrow$
$\Rightarrow \Rightarrow$

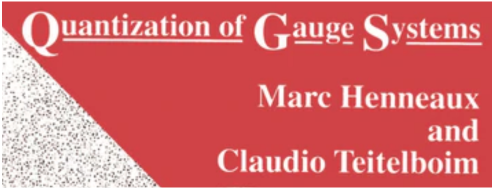
Gauging

$\Rightarrow \Rightarrow$
$\Rightarrow \Rightarrow$

8 connection 1-forms
Gluons

(j/w Thomas Strobl – Lyon, Alexei Kotov – Hradec Králové)

Methods from HEP



Important "tool": graded geometry, differential graded manifolds.

Graded manifolds – details

"...graded manifolds are just manifolds with a few bells and whistles..." (D. Roytenberg)

Graded geometry: definitions (do not read)

- Graded vector space V is a collection of vector spaces $V = \oplus V_i$ ($i \in \mathbb{Z}$ or $i \in \mathbb{Z}_{\geq 0}$); if $v \in V_i$, $deg(v) = i$.
- Homomorphism shifting the grading by p: $(V[p])_i = V_{i-p}$.
- Assume the base to be of degree 0, the dual vector space $(V_i)^*$ is defined as $(V^*)_{-i}$.
- Graded algebra structure $\cdot$: $V \otimes V \to V$, s.t. $V_p \otimes V_q \to V_{p+q}$.
- Graded commutator $[a, b] = ab - (-1)^{deg(a)deg(b)} ba$.
- Graded symmetric algebra over V: $S(V) = \text{Tensor}(V)/[\cdot, \cdot]$

Definition. Graded manifold M is a couple $(M_0, \mathcal{O}_M)$, where M_0 is a smooth manifold and the sheaf of functions $\mathcal{O}_M$ is locally isomorphic to $C^\infty(U_0) \otimes S(V)$, where U_0 is an open subset of M_0.

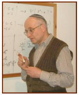
- Top degree of the generators of $\mathcal{O}_M$ – is called degree of M.

Standard abuse of notations: V_k-vector bundle or sheaf of sections.

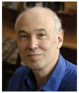
Graded manifolds, super manifolds

History

Felix Berezin



Joseph Bernstein



Pierre Deligne

Dimitry Leites

Philosophy

Dmitry Roytenberg

"...graded manifolds are just manifolds with a few bells and whistles..."

Graded manifolds

D. Roytenberg: "...bells and whistles..."

Prop. (D.Roytenberg) Given a non-negatively graded manifold $(M, \mathcal{O}_M)$ there is a tower of fibrations

$$M = M_n \to M_{n-1} \to \cdots \to M_1 \to M_0,$$

where any M_k is a graded manifold of degree at most k, for $k > 0$ $M_{k+1} \to M_k$ is an affine bundle.

Remark. Gradings can be encoded in the Euler vector field $\epsilon = deg(q^\alpha) q^\alpha \frac{\partial}{\partial q^\alpha}$; V_i corresponds to the i-eigenspace of ϵ.

Remark. Gradings can be encoded in the homogeneity structure $h\colon \mathbb{R}_+ \times \mathcal{M} \to \mathcal{M}$ such that $(q^1, \ldots, q^N) \mapsto h_t(q^1, \ldots, q^N) \equiv (t^{deg(q^1)} q^1, \ldots, t^{deg(q^N)} q^N)$.

For graded good

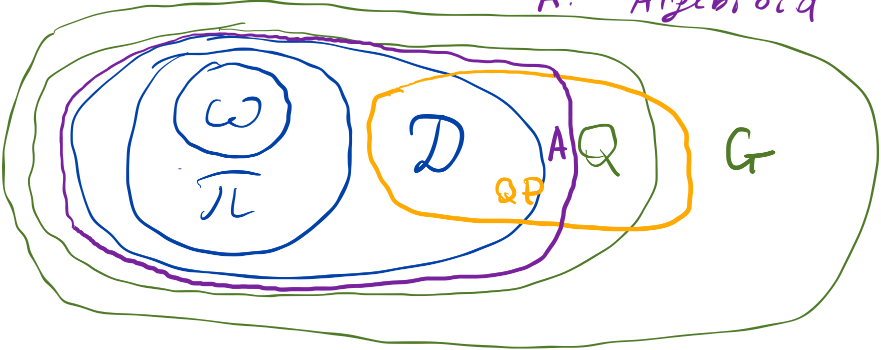
G := Graded

Q := Differential Graded

QP := DG symplectic

A := Algebroid

– "Folkloric" applications in geometry



– Other applications (in mechanics?):
A. Bruce, K. Grabowska, J. Grabowski, Higher order mechanics on graded bundles https://arxiv.org/abs/1412.2719

Derived bracket construction

Let $(\mathcal{M}, Q)$ be a Q-manifold (i.e. graded manifold with a degree 1 homological vector field Q), and
$\mathcal{G}$ be degree -1 vector fields ε on $\mathcal{M}$.
Define the Q-derived bracket: $[\varepsilon, \varepsilon']_Q := [\varepsilon, [Q, \varepsilon']]$.

Remark. Good for equivariant Q-cohomology.
V.S. "Graded geometry in gauge theories and beyond", JGP, 2015.

Example 1. $(T^*[1]M, Q_\pi)$
$\varepsilon = \varepsilon_i(x)\frac{\partial}{\partial p_i} \leftrightarrow \varepsilon_i(x)dx^i \in \Omega^1(M)$.
If ε is exact, i.e. $\varepsilon_i dx^i = \epsilon_{,i} dx^i$, then $[\varepsilon, \varepsilon']_Q = \{\epsilon, \epsilon'\}_{,i}\frac{\partial}{\partial p_i}$

Example 2. Dirac structures.

Equivariant cohomology and gaugin in a nutshell

"equations of motion"	$\leftrightarrow$	Q-morphisms
"symmetries"	$\leftrightarrow$	Q-homotopies
"gauge invariant"	$\leftrightarrow$	"equivariantly Q-closed"

Example: Dirac sigma model – functional on vector bundle morphisms from $T\Sigma$ to $\mathcal{D}$.

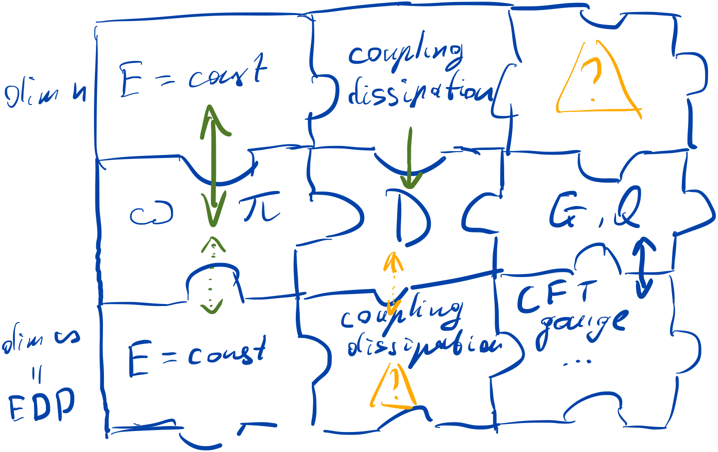
$$S_{DSM} = \int_\Sigma g(\mathrm{d}X,^\wedge (1+\mathcal{O})A) + g(A,^\wedge \mathcal{O}A) + \int_{\Sigma_3} H.$$

Theorem. The algebra of infinitesimal gauge transformations is given by smooth maps $\Sigma \to \Gamma(\mathcal{D})$, subject to $\mathrm{d}\eta - \iota_v H = 0$. $(*)$
Conversely, DSM is obtained by gauging of an extension of (*).

Theorem. DSM is the most general theory obtained by gauging the Wess-Zumino term in space-time dimension 2.

– V.S., T.Strobl, "Dirac Sigma Models from Gauging", JHEP, 2013.
– A.Kotov, V.S., T.Strobl, "2d Gauge theories and generalized geometry", JHEP, 08, 2014.

Instead of conclusion – big puzzle and questions



And what precisely about mechanics? What phenomena?

→ Thermo de Souriau (resp. Hamdaoui de Saxcé)

→ Calcul de Vinogradov (resp. Roubtsov)