On tangent geometry and microcontinuum with defects: A scaled material modeling

Nguyen Van Hoi
Joint work with Prof. G. Casale, Prof. L. Le Marrec

Department of Mathematics, IRMAR
University of Rennes 1.

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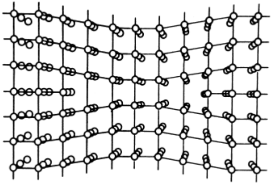
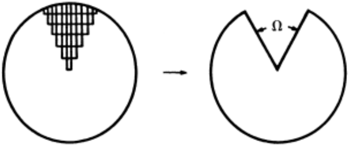
Some words about defects

In reality, any microstructure medium has a great number of defects.

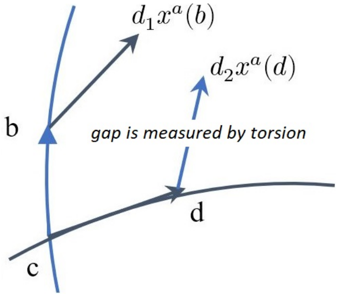


For instance, dislocation-line is performed by removing or adding a section of atom out of the media.

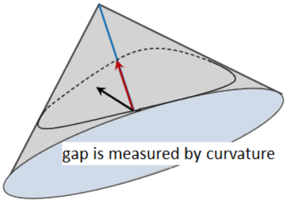
If many discs of missing or excess atoms come to lie close together, the defects is now disclination.



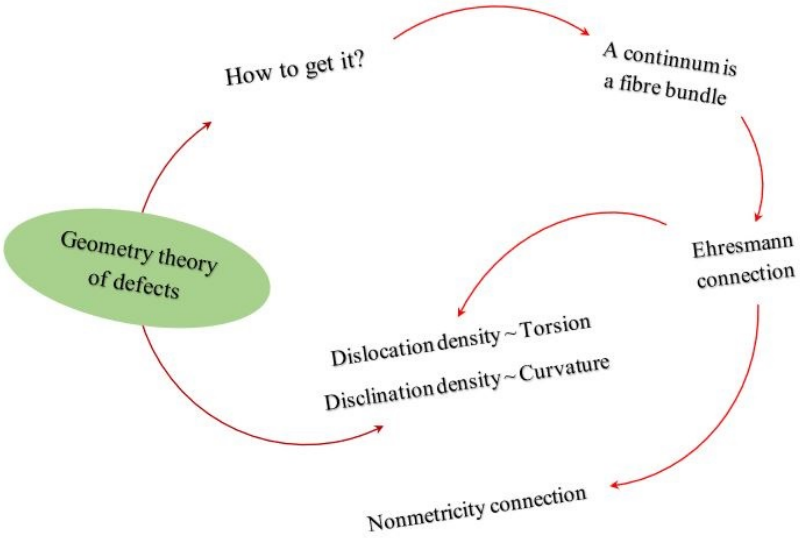
Since the 1950s, it has been appreciated that continuum with a distribution of defects has a close connection with Riemann-Cartan geometry (RC geometry).



Dislocation density being interpreted to torsion tensor of a material connection.



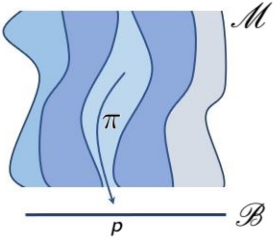
Disclinations density being identified with curvature tensor of a material connection.


How to get it?
A continnum is
a fibre bundle
Geometry theory
of defects
Ehresmann
connection
Dislocation density ~ Torsion
Disclination density ~ Curvature
Nonmetricity connection

Geometrical background

A microstructured material is modelised by a fibre bundle $\mathscr{M} \xrightarrow{\pi} \mathscr{B}$

- $\mathscr{B}$ is the material differentiable manifold.
- Fibre of the bundle at $p \in \mathscr{B}$ is denoted by $\mathscr{M}_p$. It is the set of microelements of the body at the point.



After the general definitions, one will consider $\mathscr{M} = T\mathscr{B}$ meaning that microelements are first order infinitesimal neighborhoods of geometrical points of the locus.

The solder form

Denote $V(\mathscr{M}) = \ker d\pi = \{v \in T\mathscr{M} \mid \pi v = 0\}$ is the vertical tangent bundle of $\mathscr{M} \xrightarrow{\pi} \mathscr{B}$. The fibre of the bundle at $p \in \mathscr{B}$ is the tangent bundle to the micro space $\mathscr{M}_p$, namely $T(\mathscr{M}_p)$.
One considers that the tangent at the microelement $m \in \mathscr{M}_p$, namely $T_m(\mathscr{M}_p) \equiv V_m\mathscr{M}$, should be $T_p\mathscr{B}$. It is formalized as

Definition

A solder form on $\mathscr{M}$ is an isomorphism $T\mathscr{B} \underset{\mathscr{B}}{\times} \mathscr{M} \xrightarrow{\vartheta} V(\mathscr{M})$.

This definition implies that the dimension of a micro space $\mathscr{M}_p$ equals the dimension of $\mathscr{B}$.
The main example of solder form is the canonical form on the tangent bundle: let (x^a, y^i) be the standard coordinate on $T\mathscr{B}$, the canonical form is

$$\vartheta_{can} = \delta^j_b \frac{\partial}{\partial y^j} \otimes dx^b.$$

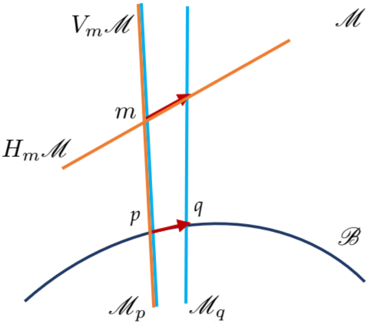
Connections

One must identify infinitesimally closed microelements. This task is performed by Ehresmann connections.

Definition

Formally, Ehresmann connection on $\mathscr{M}$ is a morphism $N : T\mathscr{B} \times_{\mathscr{B}} \mathscr{M} \to T\mathscr{M}$ such that $d\pi \circ N(v, m) = v$. In local coordinates

$$N = \left(\frac{\partial}{\partial x^a} - N_a^i(x, y) \frac{\partial}{\partial y^i} \right) \otimes dx^a$$



More geometrically, Ehresmann connection consists of a smooth assignment to each point $m \in \mathscr{M}$ of a decomposition $T_m\mathscr{M} = H_m\mathscr{M} \oplus V_m\mathscr{M}$ with

$$H_m\mathscr{M} = \text{span}\,(\frac{\delta}{\delta x^a} = \frac{\partial}{\partial x^a} - N^i_a(m)\frac{\partial}{\partial y^i}) \qquad \textit{and} \qquad V_m\mathscr{M} = \text{span}(\frac{\partial}{\partial y^i}).$$

The dual of the horizontal and vertical tangent spaces are given by

$$H^*_m\mathscr{M} = \text{span}(dx^a) \qquad \textit{and} \qquad V^*_m\mathscr{M} = \text{span}(\delta y^i = dy^i + N^i_a(m)dx^a).$$

Linear connection When $\mathscr{M} = T\mathscr{B}$, one may interest in linear connections N (i.e $N^i_a(x,y) = \Gamma^i_{aj}(x)y^j$). If the case, Γ is an affine connection on $\mathscr{B}$.

Definition

From a connection N and a solder form ϑ one can define another connection $N - \vartheta$ whose expression in local coordinate is

$$N - \vartheta = \left(\frac{\partial}{\partial x^a} - (N^i_a(m) + \vartheta^i_a(m))\frac{\partial}{\partial y^i}\right) \otimes dx^a.$$

Parallel and rolling transport

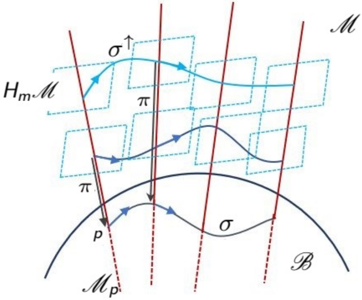
Let $\sigma\colon [0,1] \to \mathscr{B}$, $t \mapsto \sigma(t) = (x^a(t))$ be a curve on $\mathscr{B}$.

Parallel transport Thanks N, it possibles to define (at least locally) a unique lift $\sigma^{\uparrow}(t) = (y^i(t))$ of σ by solving

$$\dot{y}^i = -N^i_a(\sigma(t), y)\dot{x}^a.$$

Rolling transport For $N - \vartheta$, the lift of σ is obtained by solving

$$\dot{y}^i = -(N^i_a(\sigma(t), y) + \vartheta^i_a(\sigma(t), y))\dot{x}^a.$$



In this situation, *parallel* refers to N and *rolling* to $N - \vartheta$.
Particularly, $\mathscr{M} = T\mathscr{B}$, it can be read as a transport of vectors.

Curvature and torsion

Total curvature of our material is measured by the compatibility of $N - \vartheta$ with Lie brackets of vector fields on $\mathscr{B}$:

$$\mathcal{R}(V,U) = (N-\vartheta)[V,U] - [(N-\vartheta)V,(N-\vartheta)U] = \mathfrak{R}(V,U) + \mathfrak{T}(V,U).$$

is a vertical vector over $\mathscr{M}$.

Ehresmann curvature $\mathfrak{R}(V,U) = N[V,U] - [NV,NU]$.

Torsion $\mathfrak{T}(V,U) = [NV,\vartheta U] + [\vartheta V, NU] - \vartheta[V,U] - [\vartheta V,\vartheta U]$.

Lemma If N is linear (*ie* $N_a^i(x,y) = \Gamma^i_{aj}(x)y^j$) and $\vartheta = \vartheta_{can}$, we can write

$$\mathfrak{R} = -\mathrm{R}^i_{jab}y^j\frac{\partial}{\partial y^i}\otimes dx^a\otimes dx^b \quad and \quad \mathfrak{T} = \mathrm{T}^i_{ab}\frac{\partial}{\partial y^i}\otimes dx^a\otimes dx^b$$

with R and T are respectively usual curvature and torsion of the connection Γ

$$\mathrm{R}^a_{bcd} = \partial_c\Gamma^a{}_{db} - \partial_d\Gamma^a{}_{cb} + \Gamma^a{}_{ce}\Gamma^e{}_{db} - \Gamma^a{}_{de}\Gamma^e{}_{cb} \quad and \quad \mathrm{T}^a{}_{bc} = \Gamma^a{}_{bc} - \Gamma^a{}_{cb}.$$

Sasaki metric and Objective

Compatible with the split structure of the tangent space $T\mathscr{M}$ defined by the connection N, on $\mathscr{M}$ one has a special metric tensor g-so-called Sasaki metric:

$$\mathrm{g}(x,y) = \mathrm{g}^h{}_{ab}(x,y)dx^a \otimes dx^b + \mathrm{g}^v{}_{ij}(x,y)\delta y^i \otimes \delta y^j .$$

It states that there exist two independent mechanisms respect to these component.
As we mentioned before, the microstructured continuum is the fibre bundle $\mathscr{M}$ over $\mathscr{B}$ and the evolution of the continuum in Euclidean space is presented by a bundle map

$$\Upsilon : \mathscr{M} \to \mathbb{E}^n$$

where $\mathbb{E}^n$ is the Euclidean space. The configuration of the body is the induced geometry structure pull-backed by Υ.
Our objective *is to restrain our construction to obtain linear Ehresmann connections. The solder form is always assumed to be the canonical form.*

Non-scale material modeling

Formally, the material transformation can be represented by

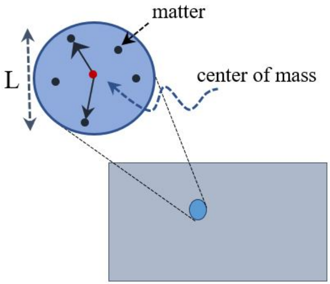
$$\Upsilon : T\mathscr{B} \to T\mathbb{E}^3$$
$$(X, V) \mapsto (\phi(X), \Psi(X, V))$$

is smooth, has a smooth inverses *st* $\Psi(X, V) = 0$ iff $V = 0$.

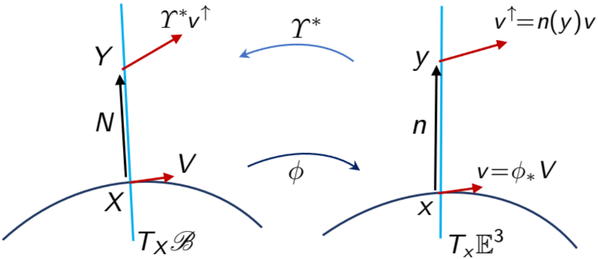
Let $\mathfrak{g}$ be a Sasaki metric on $T\mathbb{E}^3$:

$$\mathfrak{g} = \mathrm{g}_{ab}(x) dx^a \otimes dx^b + \mathrm{g}_{ij}(x) \delta y^i \otimes \delta y^j,$$

and $n^i_a = \gamma^i_{aj} y^j$ and γ is Levi-Civita connection according to the metric g. If (x, y) is Cartesian chart, $\gamma^i_{aj} = 0$.



Matte's size is not considered. In this context, the continuum is driven by: ϕ controls the placement of element, its stretch is defined by Ψ.



Let V be an arbitrary tangent vector to $\mathscr{B}$ at a point $X \in \mathscr{B}$, the induced Ehresmann connection is naturally defined by considering its horizontal lift:

$$N(Y)V = \Upsilon^* n(\phi(X), \Psi(X,Y))\phi_* V$$

with coefficients $\quad N^I_A = \partial_j \Psi^I n^j_a F^a_A + \partial_i \Psi^I \partial_A \Psi^i.$

Lemma The connection N has zero curvature $\mathfrak{R}$.

Usually, an induced metric $\mathcal{G} = \Upsilon^* \mathfrak{g}$. Respect to the connection N, it splits into

$$\mathcal{G}(X,Y) = \mathtt{G}^h{}_{AB}(X) dX^A \otimes dX^B + \mathtt{G}^v{}_{IJ}(X,Y) \delta Y^I \otimes \delta Y^J$$

with $\quad \mathtt{G}^h{}_{AB} = F^a_A \mathrm{g}_{ab} F^b_B \quad$ *and* $\quad \mathtt{G}^v{}_{IJ} = \partial_I \Psi^i \mathrm{g}_{ij} \partial_J \Psi^j.$

Conclusion

Keeping in mind that our objective was to construct a linear Ehresmann connection with torsion and curvature, the present bundle maps fails for the following reasons: (1) the metric is generally dependent on fibre coordinate; (2) the connection is not linear; (3) the curvature of the connection is always zero. However, it could be another interesting theory. In the linear situation one has:

Theorem

If Υ is linear ie $\Psi^a(X,V) = \Psi^a_A(X)V^A$ for any $(X,V) \in T\mathscr{B}$,

$N^I_A(X,Y) = {\Gamma^I}_{AJ}(X)Y^J$; with ${\Gamma^C}_{AB} = \Psi^C_c \partial_A \Psi^c_B$ if Cartesian coordinate applied on the Euclidean space $\mathbb{E}^3$.

The metric G^v induces a metric on $\mathscr{B}$ by $\mathtt{G} = \vartheta^*_{can}\mathtt{G}^v$, locally $\mathtt{G}_{AB} = \Psi^a_A \mathtt{g}_{ab} \Psi^b_B$.

They construct a Weitzenböck manifold $(\mathscr{B}, \Gamma, \mathtt{G})$ with metric-compatible connection has torsion while vanishing curvature.

Scaled material modeling

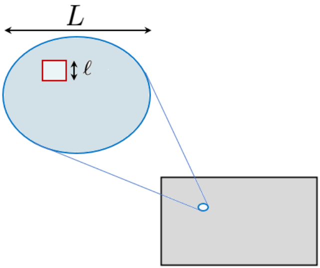
The preceding bundle map is not sufficient to our purpose since the pull-back is performed by $D\Upsilon$.

To go further, let consider the material transformation as the following

$$\begin{array}{rrcl} \Upsilon^v : & VT\mathscr{B} & \to & VT\mathbb{E}^3 \\ & X & \mapsto & \phi(X) \\ & V & \mapsto & F(X)V \\ & W & \mapsto & \Theta(X)W \end{array}$$

To construct an induced Ehresmann connection on $\mathscr{B}$. The idea is to define

$$\boxed{\begin{array}{rrcl} \Upsilon : & TT\mathscr{B} & \to & TT\mathbb{E}^3 \ st \\ & \Upsilon_{|VT\mathscr{B}} & = & \Upsilon^v \end{array}}$$



$F = D\phi$ and Θ define the stretch of macro/micro-element, respectively. Notice that scaling effect is no more redundant since the maps related to each scale are separated.

Linear contribution

A typical example of the extension is

$$\begin{array}{rccc} \Upsilon : & TT\mathscr{B} & \rightarrow & TT\mathbb{E}^3 \\ & (X, Y, W) & \mapsto & (\phi(X), F(X)Y, \Omega(X,Y)W) \end{array}$$

with $\det F = D\phi > 0$; $\det \Psi > 0$, $\det \Theta > 0$, and Ω is in the following form

$$\Omega = F^a_A \partial_a \otimes dX^A + \Omega^i_A \partial_i \otimes dX^A + \Theta^i_J \partial_i \otimes dY^J.$$

Until now, Ω^i_A is still free. At the same spirit as before, an induced Ehresmann connection is naturally defined by

$$N(Y)V = \Omega^{-1} n(F(X)Y)(\phi_* V)$$

with coefficients $\quad N^J_A = \Theta^J_i \Omega^i_A + \Theta^J_i n^i_a F^a_A.$

Lemma

The connection is linear ie $N^I_A(X,Y) = \Gamma^I{}_{AJ}(X) Y^J$ *if and only if* $\Omega^i_A(X,Y)$ *is linear ie* $\Omega^i_A(X,Y) = \Omega^i_{AJ}(X) Y^J$*. At this context, one gets*

$$\Gamma^I{}_{AJ} = \Theta^I_i \Omega^i_{AJ} + \Theta^I_i \gamma^i_{aj} F^j_J F^a_A.$$

Induced structure

The letter Ω^i_{AI} is still free. In order to remove this indeterminacy, it is constructed by a linear balance between the stretching variation at each scale:

$$\boxed{\Omega^i_{AI} = ((1-\zeta)\partial_A F^i_I + \zeta\partial_A \Theta^i_I)}$$

where $0 < \zeta \leq 1$ is a free parameter controlling the scaling effect (for example, $\zeta = \ell/L$). At this stage, one gets, if Cartesian chart applied on $\mathbb{E}^3$,

$$\boxed{\Gamma^I{}_{AJ} = \Theta^I_i((1-\zeta)\partial_A F^i_I + \zeta\partial_A \Theta^i_I)}$$

Usually, an induced metric $\mathcal{G} = \Upsilon^* \mathfrak{g}$. Respect to the connection N, it splits into

$$\mathcal{G}(X,Y) = \mathtt{G}^h{}_{AB}(X) dX^A \otimes dX^B + \mathtt{G}^v{}_{IJ}(X)\delta Y^I \otimes \delta Y^J$$

$$\text{with} \qquad \mathtt{G}^h{}_{AB} = F^a_A \mathrm{g}_{ab} F^b_B \qquad \textit{and} \qquad \mathtt{G}^v{}_{IJ} = \Theta^i_I \mathrm{g}_{ij} \Theta^j_J.$$

They are functions of base coordinate alone. $\mathtt{G}^v$ is independent on N.

Conclusion

The split structure of the transformation and metrics allows to describe the current state as the superposition of a microscopic and macroscopic processes by considering a micro-manifold $(\mathscr{B}, \Gamma, \mathtt{G})$ with $\mathtt{G} = \vartheta^*_{can}\mathtt{G}^v$ and a macro-manifold $(\mathscr{B}, \mathsf{L}, \mathtt{G}^h)$ where L is Levi-Civita connection of the horizontal metric, it has no torsion and no curvature and is metric-compatible. The properties of $(\mathscr{B}, \Gamma, \mathtt{G})$ is more richer:

Theorem

$(\mathscr{B}, \Gamma, \mathtt{G})$ *is sufficient to state the shape change as well as the defected state of the material. It satisfies:*

- *If* $\Theta = F$, *it yields that* $\mathsf{T} = 0$; $\mathsf{R} = 0$ *and* $\nabla\mathtt{G} = 0$. *The manifold is Euclidean.*
- *If* $\zeta = 1$, *then* $\mathsf{T} \neq 0$, $\mathsf{R} = 0$ *and* $\nabla\mathtt{G} = 0$. *It behaves as a Weitzenböck manifold.*
- *If* $\Theta \neq F$ *and* $0 < \zeta < 1$, *then* $\mathsf{T} \neq 0$, $\mathsf{R} \neq 0$ *and* $\nabla\mathtt{G} \neq 0$. *The manifold behaves as a Weyl manifold.*

On comparison with nonholonomic principle

The nonholonomic principle		The scaled material modeling
$\mathscr{B} \to \mathbb{E}^3$, $X \mapsto x$ is multivalued $dx^a = \mathrm{e}^a_A dX^A$ Base on RC point of view	Setting	$\Upsilon^v : VT\mathscr{B} \to VT\mathbb{E}^3$ smooth, single $(X, V, W) \mapsto (\phi(X), F(X)V, \Theta(X)W)$ Base on Ehresmann point of view
$\hat{\Gamma}^C_{AB} = \mathrm{e}^C_c \partial_A \mathrm{e}^c_B$ *and* $\hat{\mathrm{G}}_{AB} = \mathrm{e}^a_A \mathrm{g}_{ab} \mathrm{e}^b_B$ Require ∇ *and* G single-valued If $X \mapsto x$ is singlevalued, defect-free If e is single-valued, we have only T If e is multivalued, $\mathrm{T} \neq 0$ *and* $\mathrm{R} \neq 0$	Result	$\Gamma^C_{AB} = \Theta^C_c((1-\zeta)\partial_A F^c_B + \zeta \partial_A \Theta^c_B)$ *and* $\mathrm{G}_{AB} = \Theta^a_A \mathrm{g}_{ab} \Theta^b_B$. NO If $\Theta = F$, defect-free If $\zeta = 1$, we have only T Otherwise, $\mathrm{T} \neq 0$, $\mathrm{R} \neq 0$ and $\nabla \mathrm{G} \neq 0$

Other comments

Spin connection: For the single-valued tetrads, connection, having both torsion and curvature, cannot be obtained only in terms tetrads. A more rigorous method would be introducing a new additional fields to give a connection

$$\tilde{\Gamma}^A_{BC} = \mathrm{e}^A_c \partial_B \mathrm{e}^c_C + \mathrm{e}^A_c \overline{\Gamma}^c{}_{bC} \mathrm{e}^b_B = \text{Weitzenböck} + \text{spin connection}.$$

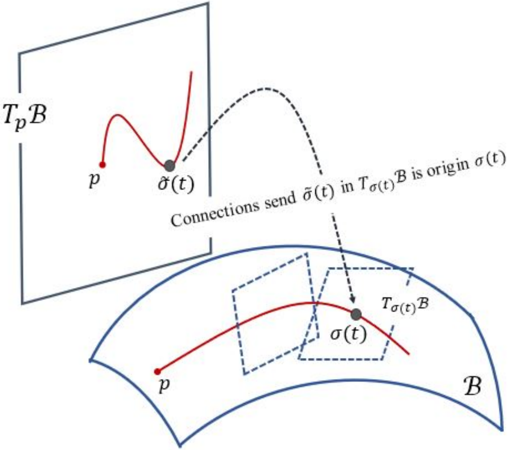
This connection has in common with ours,

$$\Gamma^A_{BC} = \zeta \Theta^A_a \partial_A \Theta^a_C + (1-\zeta) \Theta^A_a \partial_B F^a_C.$$

But, the sum of the two distribution is controlled by the factor ζ.

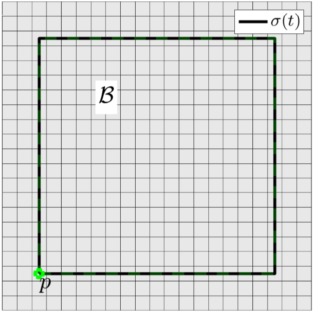
Kröner-Lee-decomposition: Our model seems to be an alternative to $\mathbb{F} = \mathbb{F}_e \mathbb{F}_p$ generally used for elastoplastic transformation of a material. Clearly that the total gradient is Θ which control stretch of microelement as smallest pieces from the initial to final state. Elastic part is then the gradient F and hence $\mathbb{F}_p = F^{-1}\Theta$. It implies the intermediate configuration is not needed.

Explicit transformations producing curvature, torsion and metricity tensor



In order to expose the application of scaled material modeling, illustrations are restricted to in-plane motion in the Euclidean ambient space endowed with $n = 0$ and $\mathrm{g} = \delta$.

Each colored cell is related to a micro-element (considered as $VT\mathscr{B}$). Note that all transformation as below are finite, and loop is infinitesimal.



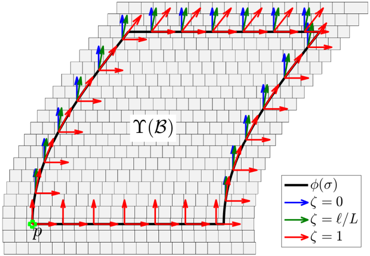
Pure-non-metric transformation

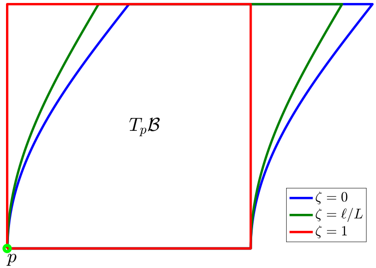
Let consider Θ is identity and

$$\phi \quad \begin{array}{ccc} X^1 & \rightarrow & x^1 = X^1 + h(X^2) \\ X^2 & \rightarrow & x^2 = X^2 \end{array}$$

This situation is illustrated with $h' = \pi/4 \sin\left(X^2\pi/8L\right)$

- $\mathtt{G} = \delta$.
- $\mathrm{T} = \mathrm{R} = 0$.
- $\nabla_2 \mathtt{G}_{12} = -(1-\zeta)h''$.





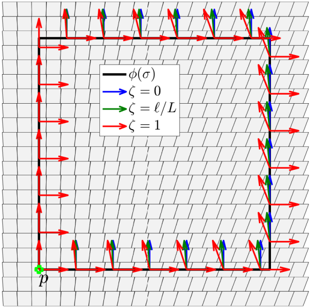
Torsion with no curvature

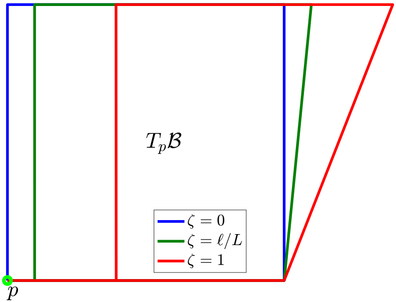
Here, ϕ identity, but

$$\Theta = \begin{pmatrix} 1 & \theta(X^1) \\ 0 & 1 \end{pmatrix}$$

- $\nabla_1 \mathtt{G}_{22} = 2(1-\zeta)\theta\theta'$ and $\nabla_1 \mathtt{G}_{12} = (1-\zeta)\theta'$.
- $\mathrm{R} = 0$ while $\mathrm{T}^1_{12} = \zeta\theta'$.

Illustrations are given with $\theta(X^1) = \dfrac{\pi}{4}\cos\left(\frac{X^1}{4L}\pi\right)$.





Curvature with no torsion

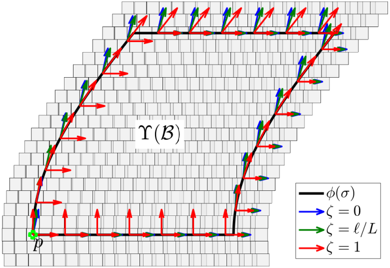
Let consider:

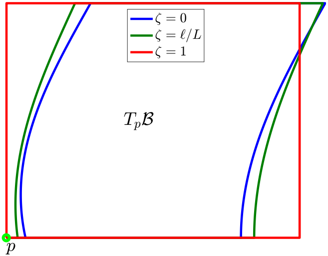
$$F = \begin{pmatrix} 1 & f(X^2) \\ 0 & 1 \end{pmatrix}, \ \Theta = \begin{pmatrix} 1+\theta(X^1) & 0 \\ 0 & 1 \end{pmatrix}$$

$$ex \ \theta(X^1) = \frac{1}{2L}\cos\left(\frac{X^1}{4L}\pi\right), \ f(X^2) = \frac{\pi}{4}\sin\left(\frac{X^2}{8L}\pi\right)$$

- Torsion is null.
- $\mathrm{R}^1{}_{212} = -(1-\zeta)^2/(1+\theta)^2 f'\theta'$
- $\nabla_1 \mathtt{G}_{11} = 2(1-\zeta)(1+\theta)\theta'$

  $\nabla_2 \mathtt{G}_{12} = -(1-\zeta)(1+\theta)f'$.





Conclusion remark

The kinematic (finite) models contain many advantage features and open several interesting issues:

- The new theory is easy to handle for both theoretical and numerical analysis rather focusing on multivalued fields. It brings microscopic defects into macroscopic observations.
- The existence of the extension bundle map is not unique. In this work, it is proposed by introducing just a scalar zeta as a weight of macro and micro effect in a linear way. It is sufficient to illustrate the presence of various defect types and avoids adding new unknown fields in the theory.
- Other possible extensions as well as the ratio parameter-family may be explored some others interesting physical phenomena.
- Study of the solder form can be another interesting subject in order to take into account specific micro-structured materials and nano-material.
- Last but not at least, another possible issue consists in introducing time, in the Galilean or in the Lorentzian framework

THANK FOR YOUR ATTENTION :)