Behaviour of a Kirshhoff rod loaded by a pure moment

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Joint work with Loïc Le Marrec and Jean Lerbet

Rencontre du GDR GDM à La Rochelle
7-9 Juillet 2021

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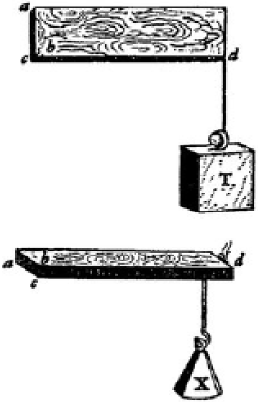
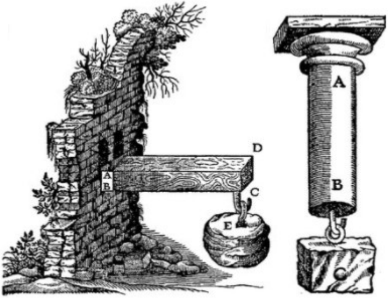
Beam:structure element that resists loads



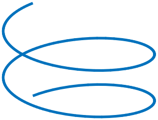
L. Da Vinci 1493



G. Galilei 1638

Rod/Beam theories

- Kirshhoff Rod

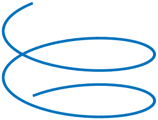


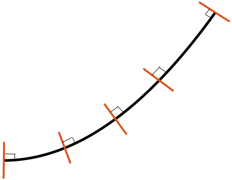
Kinematical constraint
Constrained load
Moments

G. Kirchhoff 1859

Rod/Beam theories

- Kirshhoff Rod
- Euler-Bernoulli Beam





Kinematical constraint
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Kinematical constaint
Any type of load
Moments

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L. Euler and D. Bernoulli 1750

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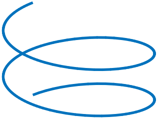

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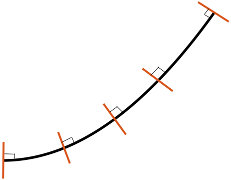
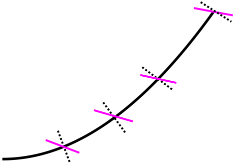


Kinematical constaint
Any type of load
Moments

L. Euler and D. Bernoulli 1750

- Timoshenko beam



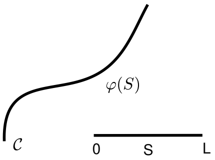
Shear effect
Any type of load
Moments

S.P. Timoshenko 1921

Kirshhoff rod model

Cosserat beam model

- Material curve $\mathcal{C}$.
- S curvilinear coordinate of $\mathcal{C}$.
- $\varphi(S)$ placement of $\mathcal{C}$.



Kirshhoff rod model

Cosserat beam model

- Material curve $\mathcal{C}$.
- S curvilinear coordinate of $\mathcal{C}$.
- $\varphi(S)$ placement of $\mathcal{C}$.
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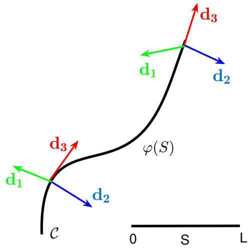
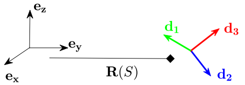
  $\{\boldsymbol{d_i}\} := (\boldsymbol{d}_1, \boldsymbol{d}_2, \boldsymbol{d}_3)$
  $\boldsymbol{d}_3$ tangent to the centerline.





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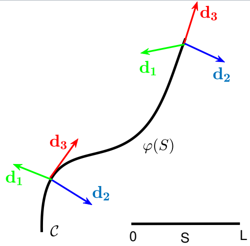

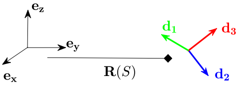

With

$$\frac{d\boldsymbol{d_i}}{dS} = \boldsymbol{\kappa} \times \boldsymbol{d_i}$$

Generalised curvatures

$$\boldsymbol{\kappa} = \kappa_1 \boldsymbol{d}_1 + \kappa_2 \boldsymbol{d}_2 + \kappa_3 \boldsymbol{d}_3$$

Basic assumption

Internal forces and moment

$$\begin{aligned} \boldsymbol{N} &= N_1 \boldsymbol{d}_1 + N_2 \boldsymbol{d}_2 + N_3 \boldsymbol{d}_3 \\ \boldsymbol{M} &= M_1 \boldsymbol{d}_1 + M_2 \boldsymbol{d}_2 + M_3 \boldsymbol{d}_3 \end{aligned}$$

With

N_1, N_2 shear forces N_3 normal force

M_1, M_2 bending moments M_3 torsion

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Static equilibrium

$$\begin{aligned} \frac{d\boldsymbol{N}}{dS} &= 0 \\ \frac{d\boldsymbol{M}}{dS} + \boldsymbol{d}_3 \times \boldsymbol{N} &= 0 \end{aligned}$$

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No external forces

$$\boldsymbol{N} = 0$$

Basic assumption

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$$\frac{d\mathbf{N}}{dS} = 0 \Longrightarrow \mathbf{N}(S) = \mathbf{N}$$

No external forces

$$\mathbf{N} = 0$$

$$\frac{d\mathbf{M}}{dS} = 0$$

Projecting along $\mathbf{d}_i$

$$\begin{aligned} \frac{dM_1}{dS} + M_3\kappa_2 - M_2\kappa_3 &= 0 \\ \frac{dM_2}{dS} + M_1\kappa_3 - M_3\kappa_1 &= 0 \\ \frac{dM_3}{dS} + M_2\kappa_1 - M_1\kappa_2 &= 0 \end{aligned}$$

Constitutive laws

St. Venant Kirshhoff energy per unit length

$$\psi = \frac{1}{2}\left(EI_1\,\kappa_1^2 + EI_2\,\kappa_2^2 + GI_3\,\kappa_3^2\right)$$

Linear constitutive laws

$$M_1 = EI_1\,\kappa_1 \qquad M_2 = EI_2\,\kappa_2 \qquad M_3 = GI_3\,\kappa_3$$

$$\frac{dM}{dS} = 0 \Longrightarrow$$

Equilibrium

$$\begin{aligned} EI_1\frac{d\kappa_1}{dS} + (GI_3 - EI_2)\kappa_2\kappa_3 &= 0 \\ EI_2\frac{d\kappa_2}{dS} + (EI_1 - GI_3)\kappa_1\kappa_3 &= 0 \\ GI_3\frac{d\kappa_3}{dS} + E(I_2 - I_1)\kappa_1\kappa_2 &= 0 \end{aligned}$$

Dimensionless procedure

Dimensionless parameters

$$\varrho = \sqrt{\frac{I_1+I_2}{A}} \qquad s = \frac{S}{\varrho} \qquad \ell = \frac{L}{\varrho}$$

$$g = \frac{E}{G} \qquad e = \frac{I_1}{I_2}$$

Kinematical variables

$$\varphi_i(s) = \frac{1}{\varrho}\,\underline{\varphi}_i(S)$$

$$\kappa_i(s) = \varrho\,\underline{\kappa}_i(S)$$

with $\quad I_1 \leq I_2 < I_3, \quad 0 < e \leq 1, \quad 2 \lesssim g \lesssim 3$

Dimensionless variables

Dimensionless parameters

$$g = \frac{E}{G} \qquad e = \frac{I_1}{I_2}$$

$$2 \lesssim g \lesssim 3 \qquad 0 < e \leq 1$$

Bending stiffness

$$r_1 := \frac{eg}{1+e}$$

$$r_2 := \frac{g}{1+e}$$

Moments

$$M_1 = r_1 \kappa_1 \qquad M_2 = r_2 \kappa_1 \qquad M_3 = \kappa_3$$

Energy per unit length

$$\Psi(s) = \frac{1}{2}\left(r_1\, \kappa_1^2 + r_2\, \kappa_2^2 + \kappa_3^2\right)$$

Curvature formulation

$$\begin{aligned} r_1\, \kappa_1'(s) - (r_2 - 1)\, \kappa_2(s)\kappa_3(s) &= 0 \\ r_2\, \kappa_2'(s) + (r_1 - 1)\, \kappa_1(s)\kappa_3(s) &= 0 \\ \kappa_3'(s) + (r_2 - r_1)\, \kappa_1(s)\kappa_2(s) &= 0 \end{aligned}$$

Non linear first order

Similar to Euler's rotation

D.F. Lawden 2013

Geometrical regimes (material + cross section shapes)

Dimensionless parameters

$$g = \frac{E}{G} \qquad e = \frac{I_1}{I_2}$$

$$0 < e \leq 1 \qquad 2 \lesssim g \lesssim 3$$

Bending stiffness

$$r_1 := \frac{eg}{1+e}$$

$$r_2 := \frac{g}{1+e}$$

Cross section properties

$r_1 = r_2$	*i.e.*	$e = 1$	Symetric cross-section
$1 < r_1$	*i.e.*	$e > \dfrac{1}{g-1}$	Thick cross-section
$r_1 < 1$	*i.e.*	$e < \dfrac{1}{g-1}$	Thin cross-section (ribbon-like rod)

Homogeneous solutions

Equilibrium

$$\begin{aligned} r_1\,\kappa_1'(s) - (r_2 - 1)\,\kappa_2(s)\kappa_3(s) &= 0 \\ r_2\,\kappa_2'(s) + (r_1 - 1)\,\kappa_1(s)\kappa_3(s) &= 0 \\ \kappa_3'(s) + (r_2 - r_1)\,\kappa_1(s)\kappa_2(s) &= 0 \end{aligned}$$

Bending % d_1

$$\begin{aligned} \kappa_1(s) &= \kappa_{10} \\ \kappa_2(s) &= 0 \\ \kappa_3(s) &= 0 \end{aligned}$$

Bending % d_2

$$\begin{aligned} \kappa_1(s) &= 0 \\ \kappa_2(s) &= \kappa_{20} \\ \kappa_3(s) &= 0 \end{aligned}$$

Pure torsion

$$\begin{aligned} \kappa_1(s) &= 0 \\ \kappa_2(s) &= 0 \\ \kappa_3(s) &= \kappa_{30} \end{aligned}$$

$e \neq 1$ and $r_1 \neq 1$

Trivial solutions

Equilibrium

$$
\begin{aligned}
r_1\, \kappa_1'(s) - (r_2 - 1)\, \kappa_2(s)\kappa_3(s) &= 0\\
r_2\, \kappa_2'(s) + (r_1 - 1)\, \kappa_1(s)\kappa_3(s) &= 0\\
\kappa_3'(s) + (r_2 - r_1)\, \kappa_1(s)\kappa_2(s) &= 0
\end{aligned}
$$

$1 < r_1 = r_2$ (*i.e.* $e = 1$)

$$
\begin{aligned}
\kappa_1(s) &= \kappa_{10} \cos\left(\frac{g-2}{g}\kappa_{30}\, s\right) + \kappa_{20} \sin\left(\frac{g-2}{g}\kappa_{30}\, s\right)\\
\kappa_2(s) &= \kappa_{20} \cos\left(\frac{g-2}{g}\kappa_{30}\, s\right) - \kappa_{10} \sin\left(\frac{g-2}{g}\kappa_{30}\, s\right)\\
\kappa_3(s) &= \kappa_{30}
\end{aligned}
$$

$r_1 = r_2 = 1$

$$
\begin{aligned}
\kappa_1(s) &= \kappa_{10}\\
\kappa_2(s) &= \kappa_{20}\\
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\end{aligned}
$$

Trivial solutions

Equilibrium

$$\begin{aligned} r_1\,\kappa_1'(s) - (r_2-1)\,\kappa_2(s)\kappa_3(s) &= 0 \\ r_2\,\kappa_2'(s) + (r_1-1)\,\kappa_1(s)\kappa_3(s) &= 0 \\ \kappa_3'(s) + (r_2-r_1)\,\kappa_1(s)\kappa_2(s) &= 0 \end{aligned}$$

$1 < r_1 = r_2$ (*i.e.* $e = 1$)

$$\begin{aligned} \kappa_1(s) &= \kappa_{10}\cos\left(\frac{g-2}{g}\kappa_{30}\,s\right) + \kappa_{20}\sin\left(\frac{g-2}{g}\kappa_{30}\,s\right) \\ \kappa_2(s) &= \kappa_{20}\cos\left(\frac{g-2}{g}\kappa_{30}\,s\right) - \kappa_{10}\sin\left(\frac{g-2}{g}\kappa_{30}\,s\right) \\ \kappa_3(s) &= \kappa_{30} \end{aligned}$$

$r_1 = r_2 = 1$

$$\begin{aligned} \kappa_1(s) &= \kappa_{10} \\ \kappa_2(s) &= \kappa_{20} \\ \kappa_3(s) &= \kappa_{30} \end{aligned}$$

$r_1 = 1 < r_2$

$$\begin{aligned} \kappa_1(s) &= \kappa_{10}\cos\left((g-2)\kappa_{20}\,s\right) + \kappa_{30}\sin\left((g-2)\kappa_{20}\,s\right) \\ \kappa_2(s) &= \kappa_{20} \\ \kappa_3(s) &= \kappa_{30}\cos\left((g-2)\kappa_{20}\,s\right) - \kappa_{10}\sin\left((g-2)\kappa_{20}\,s\right) \end{aligned}$$

Invariants

$\boldsymbol{M}$ is ***uniform*** all along the rod $||\boldsymbol{M}|| := M$ is constant.

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First invariant: Moment

$$M^2 = \Big(r_1\,\kappa_1(s)\Big)^2 + \Big(r_2\,\kappa_2(s)\Big)^2 + \kappa_3(s)^2$$

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First invariant: Moment

$$M^2 = \Big(r_1\,\kappa_1(s)\Big)^2 + \Big(r_2\,\kappa_2(s)\Big)^2 + \kappa_3(s)^2$$

Second invariant: Energy per unit length

$$\psi = \frac{1}{2}\Big(r_1\,\kappa_1(s)^2 + r_2\,\kappa_2(s)^2 + \kappa_3(s)^2\Big)$$

Invariants

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Second invariant: Energy per unit length

$$\psi = \frac{1}{2}\Big(r_1\,\kappa_1(s)^2 + r_2\,\kappa_2(s)^2 + \kappa_3(s)^2\Big)$$

$$\mu^2 := 2\psi$$

Geometrical interpretation

$$M^2 = M_1^2 + M_2^2 + M_3^2$$

Sphere

$$\mu^2 = \frac{1}{r_1}M_1^2 + \frac{1}{r_2}M_2^2 + M_3^2$$

Ellipsoid

In (M_1, M_2, M_3) Configuration

Geometrical interpretation

$$M^2 = M_1^2 + M_2^2 + M_3^2$$

Sphere

$$\mu^2 = \frac{1}{r_1} M_1^2 + \frac{1}{r_2} M_2^2 + M_3^2$$

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In (M_1, M_2, M_3) Configuration

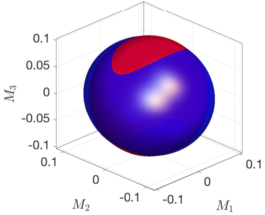
Solutions $\Longleftrightarrow$ Surfaces intersections

Existence condition

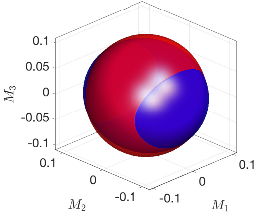
$$\min_{i=1,2} (r_i, 1) \leq \eta := \frac{M}{\mu} \leq \max_{i=1,2} (r_i, 1)$$

Geometrical representation with fixed energy μ

Thick



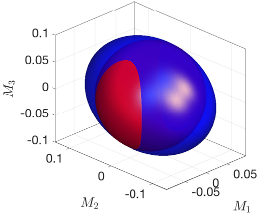
$$1 \leq \eta \leq \sqrt{r_1} < \sqrt{r_2}$$



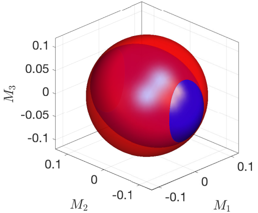
$$1 < \sqrt{r_1} \leq \eta \leq \sqrt{r_2}$$

$$\eta = \frac{M}{\mu}$$

Thin



$$\sqrt{r_1} \leq \eta \leq 1 < \sqrt{r_2}$$



$$\sqrt{r_1} < 1 \leq \eta \leq \sqrt{r_2}$$

Analytical solutions

$$\begin{aligned} r_1\,\kappa_1'(s) - (r_2-1)\,\kappa_2(s)\kappa_3(s) &= 0 \\ r_2\,\kappa_2'(s) + (r_1-1)\,\kappa_1(s)\kappa_3(s) &= 0 \\ \kappa_3'(s) + (r_2-r_1)\,\kappa_1(s)\kappa_2(s) &= 0 \end{aligned}$$

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- **For thick rod**

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- **For thick rod**

$1 \le \eta \le \sqrt{r_1} < \sqrt{r_2}$

$$
\begin{aligned}
\kappa_1(s) &= \overline{\kappa}_1\,\mathrm{sn}(\lambda(s+s_0)\mid m)\\
\kappa_2(s) &= \overline{\kappa}_2\,\mathrm{cn}(\lambda(s+s_0)\mid m)\\
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$$\begin{aligned} \kappa_1(s) &= \overline{\kappa}_1\,\mathrm{sn}(\lambda(s+s_0)\mid m)\\ \kappa_2(s) &= \overline{\kappa}_2\,\mathrm{cn}(\lambda(s+s_0)\mid m)\\ \kappa_3(s) &= \pm\overline{\kappa}_3\,\mathrm{dn}(\lambda(s+s_0)\mid m) \end{aligned}$$

$1 < \sqrt{r_1} \leq \eta \leq \sqrt{r_2}$

$$\begin{aligned} \kappa_1(s) &= \overline{\kappa}_1\,\mathrm{sn}(\lambda(s+s_0)\mid m)\\ \kappa_2(s) &= \pm\overline{\kappa}_2\,\mathrm{dn}(\lambda(s+s_0)\mid m)\\ \kappa_3(s) &= \overline{\kappa}_3\,\mathrm{cn}(\lambda(s+s_0)\mid m) \end{aligned}$$

- **For thin rod**

$\sqrt{r_1} \leq \eta \leq 1 < \sqrt{r_2}$

$$\begin{aligned} \kappa_1(s) &= \pm\overline{\kappa}_1\,\mathrm{dn}(\lambda(s+s_0)\mid m)\\ \kappa_2(s) &= \overline{\kappa}_2\,\mathrm{cn}(\lambda(s+s_0)\mid m)\\ \kappa_3(s) &= \overline{\kappa}_3\,\mathrm{sn}(\lambda(s+s_0)\mid m) \end{aligned}$$

$\sqrt{r_1} < 1 \leq \eta \leq \sqrt{r_2}$

$$\begin{aligned} \kappa_1(s) &= \overline{\kappa}_1\,\mathrm{cn}(\lambda(s+s_0)\mid m)\\ \kappa_2(s) &= \pm\overline{\kappa}_2\,\mathrm{dn}(\lambda(s+s_0)\mid m)\\ \kappa_3(s) &= \overline{\kappa}_3\,\mathrm{sn}(\lambda(s+s_0)\mid m) \end{aligned}$$

Analytical solutions

$$
\begin{aligned}
r_1\,\kappa_1'(s) - (r_2-1)\,\kappa_2(s)\kappa_3(s) &= 0\\
r_2\,\kappa_2'(s) + (r_1-1)\,\kappa_1(s)\kappa_3(s) &= 0\\
\kappa_3'(s) + (r_2-r_1)\,\kappa_1(s)\kappa_2(s) &= 0
\end{aligned}
$$

- **For thick rod**

$1 \le \eta \le \sqrt{r_1} < \sqrt{r_2}$

$$
\begin{aligned}
\kappa_1(s) &= \overline{\kappa}_1\,\mathrm{sn}(\lambda(s+s_0)\mid m)\\
\kappa_2(s) &= \overline{\kappa}_2\,\mathrm{cn}(\lambda(s+s_0)\mid m)\\
\kappa_3(s) &= \pm\overline{\kappa}_3\,\mathrm{dn}(\lambda(s+s_0)\mid m)
\end{aligned}
$$

$1 < \sqrt{r_1} \le \eta \le \sqrt{r_2}$

$$
\begin{aligned}
\kappa_1(s) &= \overline{\kappa}_1\,\mathrm{sn}(\lambda(s+s_0)\mid m)\\
\kappa_2(s) &= \pm\overline{\kappa}_2\,\mathrm{dn}(\lambda(s+s_0)\mid m)\\
\kappa_3(s) &= \overline{\kappa}_3\,\mathrm{cn}(\lambda(s+s_0)\mid m)
\end{aligned}
$$

- **For thin rod**

$\sqrt{r_1} \le \eta \le 1 < \sqrt{r_2}$

$$
\begin{aligned}
\kappa_1(s) &= \pm\overline{\kappa}_1\,\mathrm{dn}(\lambda(s+s_0)\mid m)\\
\kappa_2(s) &= \overline{\kappa}_2\,\mathrm{cn}(\lambda(s+s_0)\mid m)\\
\kappa_3(s) &= \overline{\kappa}_3\,\mathrm{sn}(\lambda(s+s_0)\mid m)
\end{aligned}
$$

$\sqrt{r_1} < 1 \le \eta \le \sqrt{r_2}$

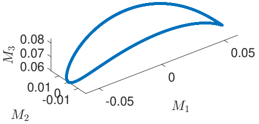
$$
\begin{aligned}
\kappa_1(s) &= \overline{\kappa}_1\,\mathrm{cn}(\lambda(s+s_0)\mid m)\\
\kappa_2(s) &= \pm\overline{\kappa}_2\,\mathrm{dn}(\lambda(s+s_0)\mid m)\\
\kappa_3(s) &= \overline{\kappa}_3\,\mathrm{sn}(\lambda(s+s_0)\mid m)
\end{aligned}
$$

$\overline{\kappa}_i, \lambda, m$ controlled by

Structure r_1, r_2
Load parameters μ, η

Curvature analysis w.r.t η (fixed μ)

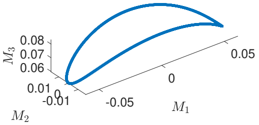
Thick



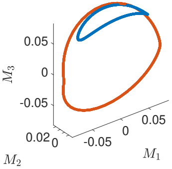
η fixed

Curvature analysis w.r.t η (fixed μ)

Thick



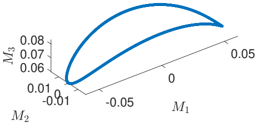
η fixed



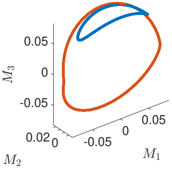
increasing η

Curvature analysis w.r.t η (fixed μ)

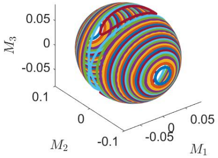
Thick



η fixed



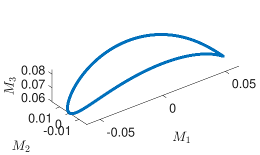
increasing η



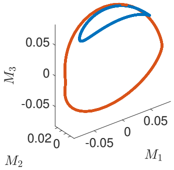
various regime

Curvature analysis w.r.t η (fixed μ)

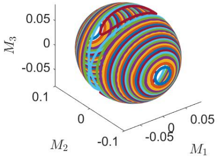
Thick



η fixed

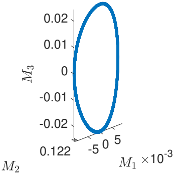


increasing η



various regime

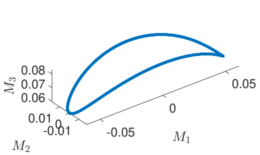
Thin



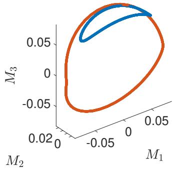
η fixed

Curvature analysis w.r.t η (fixed μ)

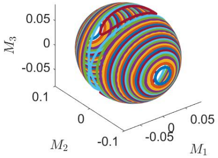
Thick



η fixed

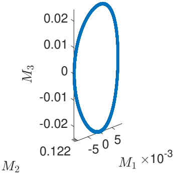


increasing η

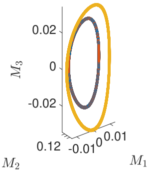


various regime

Thin



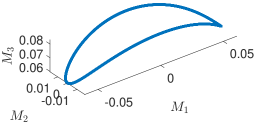
η fixed



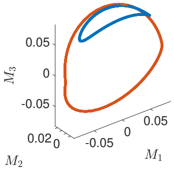
increasing η

Curvature analysis w.r.t η (fixed μ)

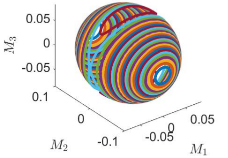
Thick



η fixed

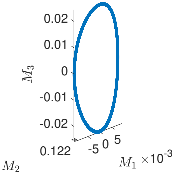


increasing η

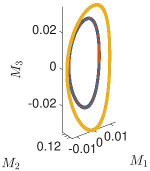


various regime

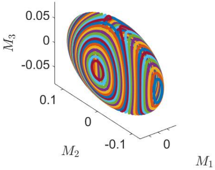
Thin



η fixed



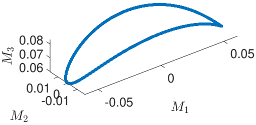
increasing η



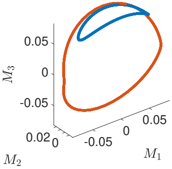
various regime

Curvature analysis w.r.t η (fixed μ)

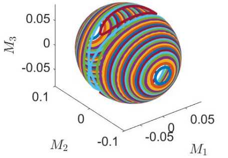
Thick



η fixed

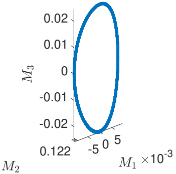


increasing η

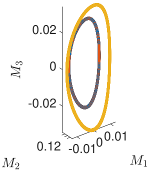


various regime

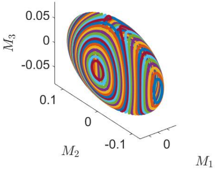
Thin



η fixed



increasing η

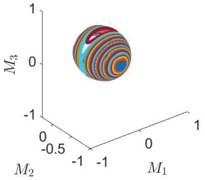


various regime

η controls the type of the solutions

Curvature analysis w.r.t μ

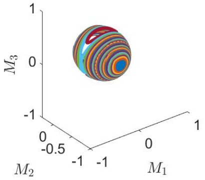
Thick



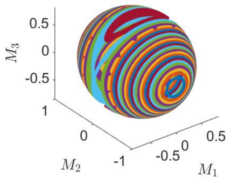
μ fixed

Curvature analysis w.r.t μ

Thick



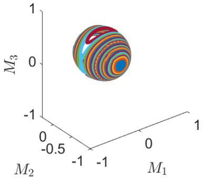
μ fixed



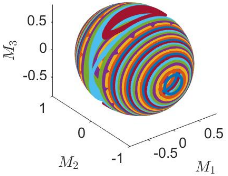
increasing μ

Curvature analysis w.r.t μ

Thick

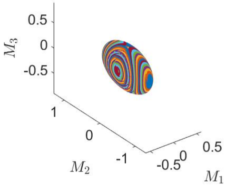


μ fixed



increasing μ

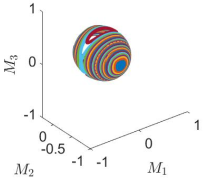
Thin



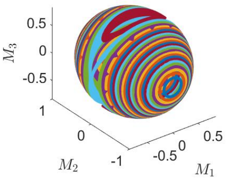
μ fixed

Curvature analysis w.r.t μ

Thick

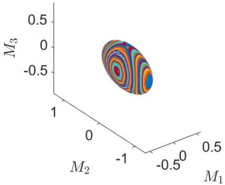


μ fixed

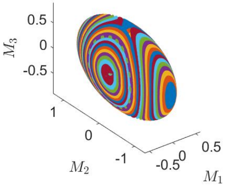


increasing μ

Thin



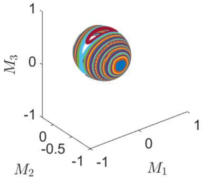
μ fixed



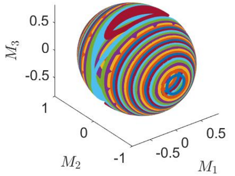
increasing μ

Curvature analysis w.r.t μ

Thick

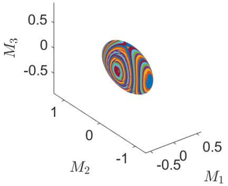


μ fixed

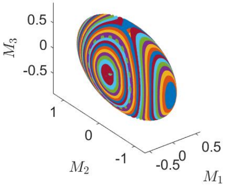


increasing μ

Thin



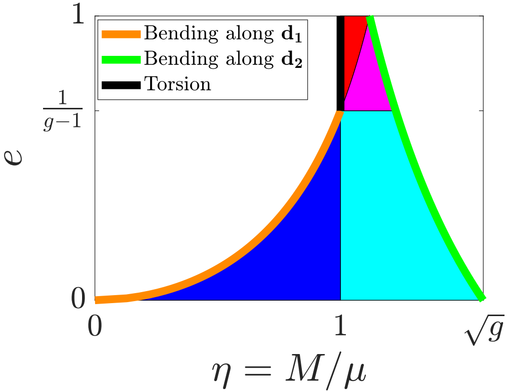
μ fixed



increasing μ

μ scaling parameter: size of the solution

Homogeneous solutions

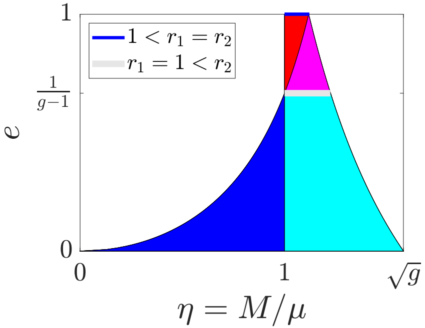


$$\kappa_1(s) = \frac{\mu}{\sqrt{r_1}}, \quad \kappa_2(s) = 0, \quad \kappa_3(s) = 0$$

$$\kappa_1(s) = 0, \quad \kappa_2(s) = \frac{\mu}{\sqrt{r_2}}, \quad \kappa_3(s) = 0$$

$$\kappa_1(s) = 0, \quad \kappa_2(s) = 0, \quad \kappa_3(s) = \mu$$

Trigonometric solutions



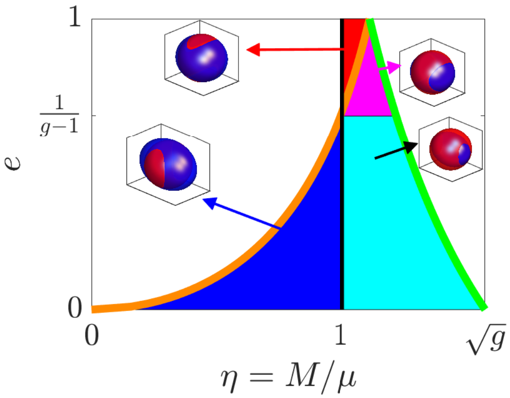
$$\kappa_1(s) = \kappa_{10}\cos\left(\frac{g-2}{g}\kappa_{30}\, s\right) + \kappa_{20}\sin\left(\frac{g-2}{g}\kappa_{30}\, s\right)$$

$$\kappa_2(s) = \kappa_{20}\cos\left(\frac{g-2}{g}\kappa_{30}\, s\right) - \kappa_{10}\sin\left(\frac{g-2}{g}\kappa_{30}\, s\right) \qquad \kappa_3(s) = \kappa_{30}$$

$$\kappa_1(s) = \kappa_{10}\cos\left((g-2)\kappa_{20}\, s\right) + \kappa_{30}\sin\left((g-2)\kappa_{20}\, s\right) \qquad \kappa_2(s) = \kappa_{20}$$

$$\kappa_3(s) = \kappa_{30}\cos\left((g-2)\kappa_{20}\, s\right) - \kappa_{10}\sin\left((g-2)\kappa_{20}\, s\right)$$

Jacobian solutions



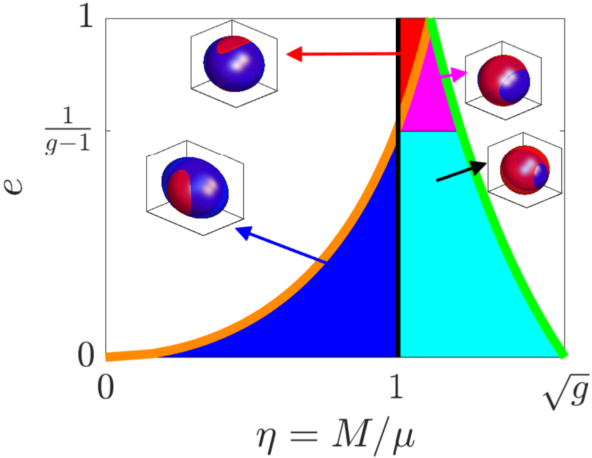
$$\kappa_1(s) = \overline{\kappa}_1 \mathrm{sn} \quad \kappa_2(s) = \overline{\kappa}_2 \mathrm{cn} \quad \kappa_3(s) = \overline{\kappa}_3 \mathrm{dn}$$

$$\kappa_1(s) = \overline{\kappa}_1 \mathrm{sn} \quad \kappa_2(s) = \overline{\kappa}_2 \mathrm{dn} \quad \kappa_3(s) = \overline{\kappa}_3 \mathrm{cn}$$

$$\kappa_1(s) = \overline{\kappa}_1 \mathrm{dn} \quad \kappa_2(s) = \overline{\kappa}_2 \mathrm{cn} \quad \kappa_3(s) = \overline{\kappa}_3 \mathrm{sn}$$

$$\kappa_1(s) = \overline{\kappa}_1 \mathrm{cn} \quad \kappa_2(s) = \overline{\kappa}_2 \mathrm{dn} \quad \kappa_3(s) = \overline{\kappa}_3 \mathrm{sn}$$

Jacobian solutions



Further analysis of rod shapes gives a better understanding for the role of η and e

Placement and directors

$\varphi' = \boldsymbol{d}_3$ $\boldsymbol{d}_i' = \boldsymbol{\kappa} \times \boldsymbol{d}_i$

Placement and directors

$$\boldsymbol{\varphi}' = \boldsymbol{d}_3 \qquad \boldsymbol{d}_i' = \boldsymbol{\kappa} \times \boldsymbol{d}_i$$

Deformed placement

$$\varphi_1' - \varphi_2\kappa_3 + \varphi_3\kappa_2 = 0$$
$$\varphi_2' - \varphi_3\kappa_1 + \varphi_1\kappa_3 = 0$$
$$\varphi_3' - \varphi_1\kappa_2 + \varphi_2\kappa_1 = 1$$

Directors

$$\boldsymbol{d}_1{}' = \kappa_3\boldsymbol{d}_2 - \kappa_2\boldsymbol{d}_3$$
$$\boldsymbol{d}_2{}' = \kappa_1\boldsymbol{d}_3 - \kappa_3\boldsymbol{d}_1$$
$$\boldsymbol{d}_3{}' = \kappa_2\boldsymbol{d}_1 - \kappa_1\boldsymbol{d}_2$$

Cartesian placement

$$\varphi_x = \boldsymbol{\varphi} \cdot \boldsymbol{e}_x$$
$$\varphi_y = \boldsymbol{\varphi} \cdot \boldsymbol{e}_y$$
$$\varphi_z = \boldsymbol{\varphi} \cdot \boldsymbol{e}_z$$

Placement and directors

$$\varphi' = \boldsymbol{d}_3 \qquad \boldsymbol{d}_i' = \boldsymbol{\kappa} \times \boldsymbol{d}_i$$

Deformed placement

$$\varphi_1' - \varphi_2\kappa_3 + \varphi_3\kappa_2 = 0$$
$$\varphi_2' - \varphi_3\kappa_1 + \varphi_1\kappa_3 = 0$$
$$\varphi_3' - \varphi_1\kappa_2 + \varphi_2\kappa_1 = 1$$

Directors

$$\boldsymbol{d}_1{}' = \kappa_3\boldsymbol{d}_2 - \kappa_2\boldsymbol{d}_3$$
$$\boldsymbol{d}_2{}' = \kappa_1\boldsymbol{d}_3 - \kappa_3\boldsymbol{d}_1$$
$$\boldsymbol{d}_3{}' = \kappa_2\boldsymbol{d}_1 - \kappa_1\boldsymbol{d}_2$$

Cartesian placement

$$\varphi_x = \boldsymbol{\varphi} \cdot \boldsymbol{e}_x$$
$$\varphi_y = \boldsymbol{\varphi} \cdot \boldsymbol{e}_y$$
$$\varphi_z = \boldsymbol{\varphi} \cdot \boldsymbol{e}_z$$

- First order O.D.E. with nonlinear coefficients

Hard to solve analytically ⟹ Numerical simulations.

Placement and directors

$\boldsymbol{\varphi}' = \boldsymbol{d}_3$ $\quad\quad$ $\boldsymbol{d}_i' = \boldsymbol{\kappa} \times \boldsymbol{d}_i$

Deformed placement

$$\varphi_1' - \varphi_2\kappa_3 + \varphi_3\kappa_2 = 0$$
$$\varphi_2' - \varphi_3\kappa_1 + \varphi_1\kappa_3 = 0$$
$$\varphi_3' - \varphi_1\kappa_2 + \varphi_2\kappa_1 = 1$$

Directors

$$\boldsymbol{d}_1{}' = \kappa_3\boldsymbol{d}_2 - \kappa_2\boldsymbol{d}_3$$
$$\boldsymbol{d}_2{}' = \kappa_1\boldsymbol{d}_3 - \kappa_3\boldsymbol{d}_1$$
$$\boldsymbol{d}_3{}' = \kappa_2\boldsymbol{d}_1 - \kappa_1\boldsymbol{d}_2$$

Cartesian placement

$$\varphi_x = \boldsymbol{\varphi} \cdot \boldsymbol{e}_x$$
$$\varphi_y = \boldsymbol{\varphi} \cdot \boldsymbol{e}_y$$
$$\varphi_z = \boldsymbol{\varphi} \cdot \boldsymbol{e}_z$$

- First order O.D.E. with nonlinear coefficients

Hard to solve analytically ⟹ Numerical simulations.

- For fixed $e, g; \mu$ and ℓ, rod behaviour depends on η.

Placement and directors

$\varphi' = \boldsymbol{d}_3$ $\qquad\qquad$ $\boldsymbol{d}_i' = \boldsymbol{\kappa} \times \boldsymbol{d}_i$

Deformed placement

$$\varphi_1' - \varphi_2\kappa_3 + \varphi_3\kappa_2 = 0$$
$$\varphi_2' - \varphi_3\kappa_1 + \varphi_1\kappa_3 = 0$$
$$\varphi_3' - \varphi_1\kappa_2 + \varphi_2\kappa_1 = 1$$

Directors

$$\boldsymbol{d}_1{}' = \kappa_3\boldsymbol{d}_2 - \kappa_2\boldsymbol{d}_3$$
$$\boldsymbol{d}_2{}' = \kappa_1\boldsymbol{d}_3 - \kappa_3\boldsymbol{d}_1$$
$$\boldsymbol{d}_3{}' = \kappa_2\boldsymbol{d}_1 - \kappa_1\boldsymbol{d}_2$$

Cartesian placement

$$\varphi_x = \boldsymbol{\varphi} \cdot \boldsymbol{e}_x$$
$$\varphi_y = \boldsymbol{\varphi} \cdot \boldsymbol{e}_y$$
$$\varphi_z = \boldsymbol{\varphi} \cdot \boldsymbol{e}_z$$

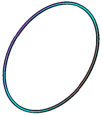
- First order O.D.E. with nonlinear coefficients

Hard to solve analytically ⟹ Numerical simulations.

- For fixed $e, g; \mu$ and ℓ, rod behaviour depends on η.
- Detailed study of this control parameter gives a better understanding of the geometry of rod shapes

η analysis: Trivial case

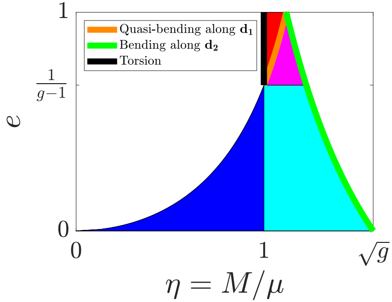
Thick



$\eta = 1$
Torsion

$\eta = \sqrt{r_1}$
Quasi-bending % $\boldsymbol{d}_1$

$\eta = \sqrt{r_2}$
Bending % $\boldsymbol{d}_2$



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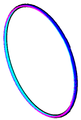

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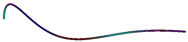

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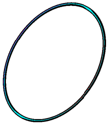
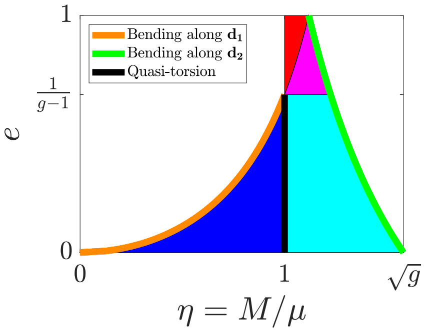


$\eta = \sqrt{r_2}$
Bending % $\boldsymbol{d}_2$



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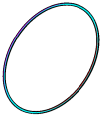

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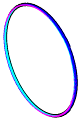

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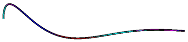

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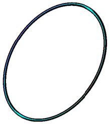
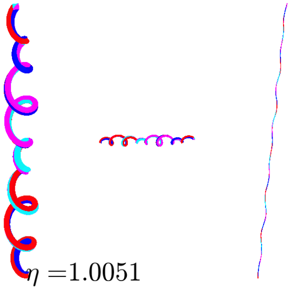


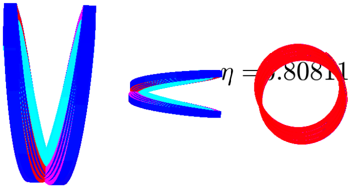
$\eta = \sqrt{r_2}$
Bending % $\boldsymbol{d}_2$

What is the behaviour of the rod by varying η for μ fixed???

η analysis: General rod behaviour

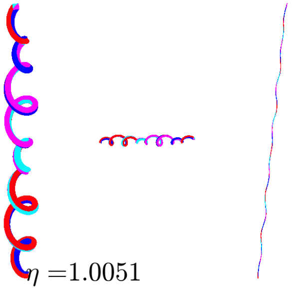


Thick



Thin

η analysis: General rod behaviour



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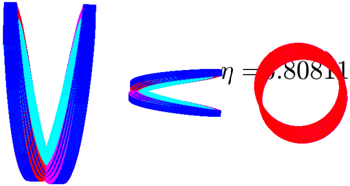
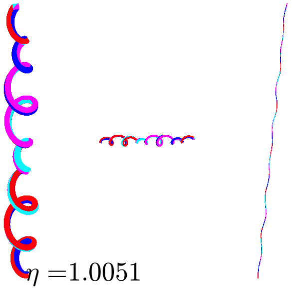


Thin

- η dictates shapes of the beam.

η analysis: General rod behaviour



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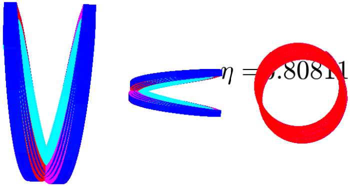
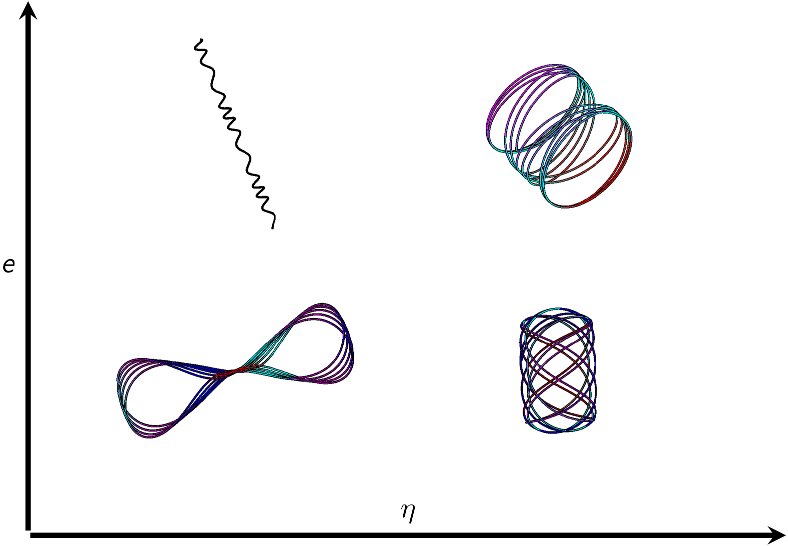


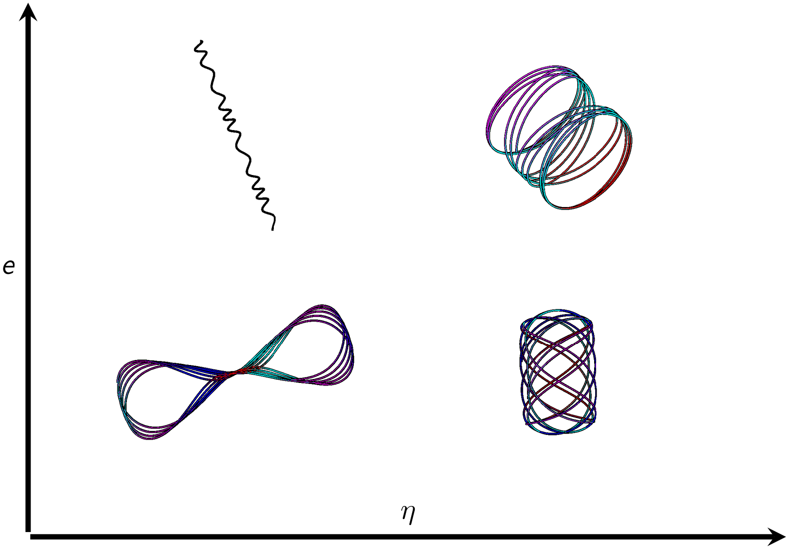
Thin

- η dictates shapes of the beam.
- Increasing η switch rod shape from torsion to bending and vice versa in a continuous way.

Influence of e on rod behaviour

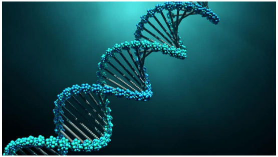


Influence of e on rod behaviour

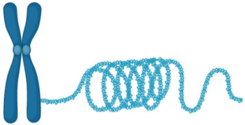


Pattern size ???

Perspectives

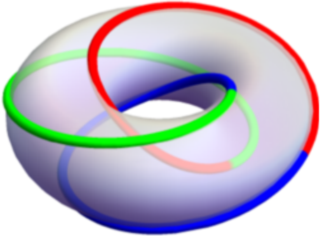


DNA helicoidal shapes



Chromosome condensation

Yarn balls shapes



Torus knots

Conclusion

- Kirshhoff rod subjected to pure moment.
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- η controls the shape of the pattern.
- μ controls the size of the pattern.

Independent of rod length.