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Covariant Thermodynamic

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Context : Finite transformations of solids

- Forming processes
- Biomechanics
- Crash test
- Explosion
- …

Standard materials: derive constitutive models with thermodynamic considerations.

Proposition: the deforming material should see the same constitutive model as the observer in the laboratory : invariance that includes deforming observers. ⇒ A space-time context

⇒ Covariant thermodynamic

Outline

- Context
- Space-time and geometric context
- Thermodynamics
  - Conservation of the energy-momentum tensor
  - Entropy
  - Second principle of thermodynamics
  - Clausius-Duhem Inequality
- Constitutive model

Relativity

Special Relativity

- Galilean's frames
- Lorentz transformations
- No curvature
- No gravitation

Special and General Relativity (overlap)

- Non Galilean's frames
- Covariance
- No curvature
- No gravitation

General Relativity

- Non Galilean's frames
- Covariance
- Curved space
- Gravitation

- Space-time domain Ω
- Four-dimensional differentiable manifold $\mathcal{M}$
- Ambient metric tensor **g** of signature (1,-1,-1,-1)
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- M an event M : a point in Ω
- $\{x^\mu\}, \mu = 1, 2, 3, 4$ a local coordinate system in an open neighborhood containing M.
- World lines in Ω; a flow; tangent vector:

$$u^\mu = \frac{dx^\mu}{ds} \quad \text{with} \quad ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu$$

Geometric context

- Invariance with respect to deforming observers

  ⇒ fiber bundle structure

- 4D basis in the tangent space at M: $(\partial/\partial x^{\mu})$ defines an observer

$$e_{\mu} = \left(\frac{\partial x^{\kappa}}{\partial y^{\nu}}\right)\frac{\partial}{\partial x^{\kappa}} = X^{\kappa}_{\mu}(x)\frac{\partial}{\partial x^{\kappa}}$$

- Basis transformation : X^{ν}_{μ} where is non singular and belongs to the group $GL(4,\mathbb{R})$

- $\mathcal{M}\times GL(4,\mathbb{R})$ principal fiber bundle over $\mathcal{M}$

- Projection on time of a vector α : $\alpha_\nu u^\nu$
- Projection on space of a vector α :

$$\underline{\alpha}^\mu = \alpha^\mu - \alpha_\nu u^\nu u^\mu = \alpha_\nu (g^{\mu\nu} - u^\mu u^\nu) = \alpha_\nu \underline{g}^{\mu\nu}$$

- Lie derivative in the velocity field: $\mathscr{L}_u(.)$
- Covariant derivative (metric connection): $\nabla_\mu(.)$
- Define the rate of deformation **d**: $\boldsymbol{d} = \underline{\boldsymbol{d}}$

and note that $d_{\mu\nu} = \frac{1}{2}\mathscr{L}_u(g_{\mu\nu})$ and that

$$d^\mu_{\ \nu} = \frac{1}{2}\left(\nabla_\nu u^\mu + \nabla_\mu u^\nu\right)$$

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- Introduce the energy-momentum tensor **T** with de decomposition:

where: $$T^{\mu\nu} = \mathcal{U} u^{\mu} u^{\nu} + q^{\mu} u^{\nu} + q^{\nu} u^{\mu} + \underline{T}^{\mu\nu}$$

$$\mathcal{U} = T^{\alpha\beta} u_{\alpha} u_{\beta} = \widetilde{\rho}_c \left(c^2 + e_{int}\right)$$ is the projection on time

$\widetilde{\rho}_c$ is the rest mass density

e_{int} is the specific internal energy.

$$q^{\mu} = T^{\alpha\beta} \underline{g}^{\mu}{}_{\alpha} u_{\beta}$$ is the projection on time and space

$$\underline{T}^{\mu\nu} = T^{\alpha\beta} \underline{g}^{\mu}_{\alpha} \underline{g}^{\nu}_{\beta}$$ is the projection on space,

identified with a stress tensor

The conservation is: $$\nabla_{\nu} T^{\mu\nu} = 0$$

Entropy

- Define a time like vector: $\Psi^\mu = \widetilde{\rho}_c \Psi u^\mu$

- Define the difference:

$$\theta S^\mu = T^{\mu\nu} u_\nu - \Psi^\mu = T^{\mu\nu} u_\nu - \widetilde{\rho}_c \Psi u^\mu$$

$$\mathcal{U} = T^{\alpha\beta} u_\alpha u_\beta = \widetilde{\rho}_c \left(c^2 + e_{int}\right)$$

- Then, with:

 the projection of **S** on time is

$$S^\mu u_\mu = \frac{\widetilde{\rho}_c \left(c^2 + e_{int}\right) - \widetilde{\rho}_c \Psi}{\theta} = \widetilde{\rho}_c \eta$$

$$\theta\eta = c^2 + e_{int} - \Psi$$

- and: $\Psi = c^2 + e_{int} - \theta\eta$

- Ψ, the free energy; θ the temperature; η the specific entropy

- The projection on space of **S**: $\theta S^\mu = T^{\mu\nu} u_\nu - \widetilde{\rho}_c \Psi u^\mu$

$$\underline{S}^\nu = S_\mu \underline{g}^{\mu\nu} = \frac{T^\kappa_\mu u_\kappa - \widetilde{\rho}_c \Psi u_\mu}{\theta} \underline{g}^{\mu\nu} = \frac{q^\nu}{\theta}$$

with $q^\nu = T^{\alpha\beta} \underline{g}^\nu{}_\alpha u_\beta$

- $$S^\mu = \widetilde{\rho}_c \eta u^\mu + \underline{S}^\mu = \widetilde{\rho}_c \eta u^\mu + \frac{q^\mu}{\theta} = \frac{T^{\mu\nu} u_\nu - \widetilde{\rho}_c \Psi u^\mu}{\theta}$$

- Note that (Vallée 1981) :

$$S^\mu = \frac{T^{\mu\nu} u_\nu - \widetilde{\rho}_c \Psi u^\mu}{\theta} = (\widetilde{\rho}_c \eta - \frac{\mathcal{U}}{\theta}) u^\mu + T^{\mu\nu} \frac{u_\nu}{\theta}$$

- The space-time second principle of thermodynamics:

$$\nabla_\mu S^\mu \geq 0$$

- Then

$$\nabla_\mu \left(\widetilde{\rho}_c \eta u^\mu + \frac{q^\mu}{\theta} \right) \geq 0$$

- and

$$\theta \widetilde{\rho}_c \mathscr{L}_u(\eta) + \nabla_\mu q^\mu - \frac{q^\mu}{\theta} \nabla_\mu \theta \geq 0$$

- Consider the conservation of energy-momentum and the second principle:

$$-\left\{\begin{array}{ccc}\theta\nabla_\mu S^\mu & \geq & 0\\ u_\mu\nabla_\nu T^{\mu\nu} & = & 0\end{array}\right. \rightarrow \theta\nabla_\mu S^\mu - u_\mu\nabla_\nu T^{\mu\nu} \geq 0$$

- This leads to

$$\underline{T}^{\mu\nu}\underline{d}_{\mu\nu} - \widetilde{\rho}_c\big(\mathscr{L}_u(\Psi) + \eta\mathscr{L}_u(\theta)\big) - \frac{q^\mu}{\theta}\nabla_\mu\theta + q_\mu a^\mu \geq 0$$

- with the mass conservation and:

$$T^{\mu\nu} = \mathcal{U}u^\mu u^\nu + q^\mu u^\nu + q^\nu u^\mu + \underline{T}^{\mu\nu}$$

$$\Psi = c^2 + e_{int} - \theta\eta$$

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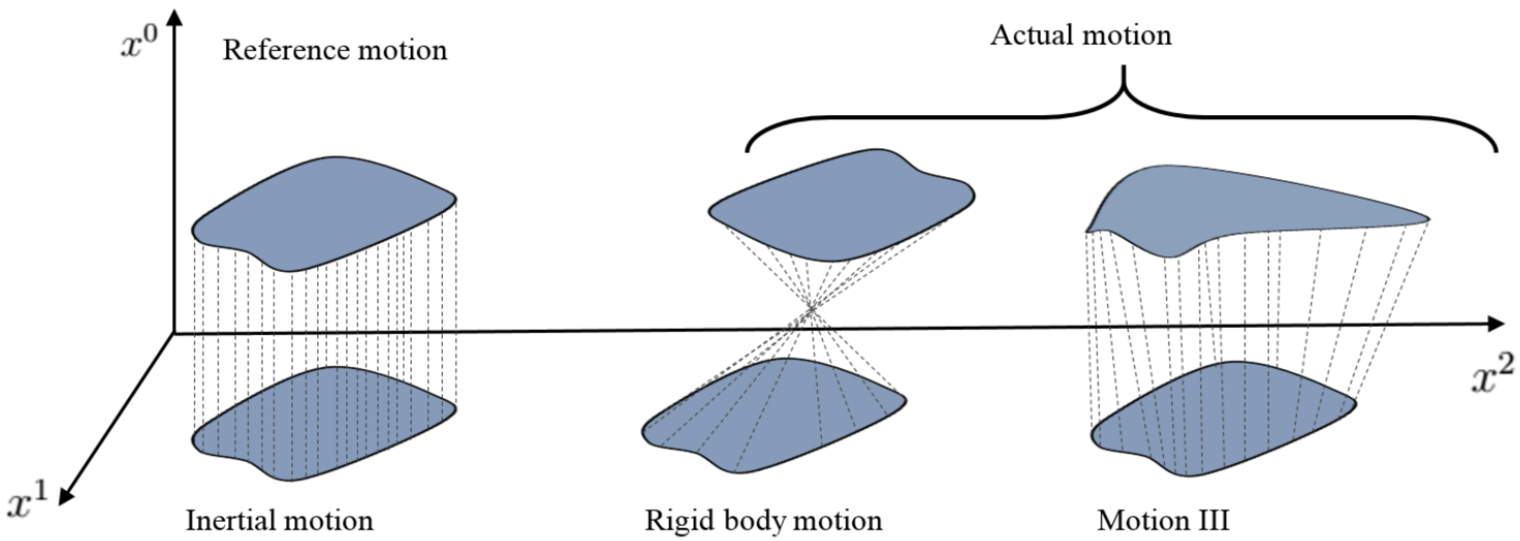
- The transformation: $x^{\mu} = \phi(X^{\nu})$
- The Cauchy deformation tensor : $b_{\mu\nu} = g_{\alpha\beta} \dfrac{\partial X^{\alpha}}{\partial x^{\mu}} \dfrac{\partial X^{\beta}}{\partial x^{\nu}}$
- The Euler strain tensor : $e_{\mu\nu} = \dfrac{1}{2}(g_{\mu\nu} - b_{\mu\nu})$



Constitutive models

- To construct a constitutive model, we choose:

$$\Psi(\theta, \underline{I_I}, \underline{I_{II}}) = -\frac{\mathcal{C}}{2\theta_0}(\theta-\theta_0)^2 - 3\frac{\kappa\alpha}{\widetilde{\rho}_c}(\theta-\theta_0)\underline{I_I} + \frac{\lambda}{2\widetilde{\rho}_c}(\underline{I_I})^2 + \frac{\mu}{\widetilde{\rho}_c}\underline{I_{II}}$$

- The invariants of **e** :

$$\underline{I_I} = \underline{e}_{\mu\nu}\underline{g}^{\mu\nu}$$

$$\underline{I_{II}} = \underline{e}^{\mu\nu}\underline{e}_{\mu\nu}$$

$$\underline{T}^{\mu\nu}\underline{d}_{\mu\nu} - \widetilde{\rho}_c\big(\mathscr{L}_u(\Psi) + \eta\mathscr{L}_u(\theta)\big) - \frac{q^\mu}{\theta}\nabla_\mu\theta + q_\mu a^\mu \geq 0$$

$$\widetilde{\rho}_c\left(\frac{\partial\Psi}{\partial\theta} + \eta\right)\mathscr{L}_u(\theta) + q^\mu\left(\frac{\nabla_\mu\theta}{\theta} - a_\mu\right) + \widetilde{\rho}_c\frac{\partial\Psi}{\partial\underline{I}_I}\mathscr{L}_u(\underline{I}_I) + \widetilde{\rho}_c\frac{\partial\Psi}{\partial\underline{I}_{II}}\mathscr{L}_u(\underline{I}_{II})$$

$$+\widetilde{\rho}_c\left(\frac{\partial\Psi}{\partial\kappa\alpha}\mathscr{L}_u(\kappa\alpha) + \frac{\partial\Psi}{\partial\lambda}\mathscr{L}_u(\lambda) + \frac{\partial\Psi}{\partial\mu}\mathscr{L}_u(\mu)\right) - T_\sigma^{\mu\nu}\underline{d}_{\mu\nu} \leq 0$$

- With:

$$\mathscr{L}_u(\underline{I_I}) = \left(\underline{g}^{\mu\nu} - 2\underline{e}^{\mu\nu}\right)\underline{d}_{\mu\nu}$$

$$\mathscr{L}_u(\underline{I_{II}}) = 2(\underline{e}^{\mu\nu} - \underline{e}^{\mu}_{\beta}\,\underline{e}^{\beta\nu} - \underline{e}^{\nu}_{\beta}\,\underline{e}^{\mu\beta})\underline{d}_{\mu\nu}$$

- and the fact that : $\mathscr{L}_u(\underline{e}_{\mu\nu}) = \frac{1}{2}\mathscr{L}_u(\underline{g}_{\mu\nu}) = \underline{d}_{\mu\nu}$

$$q^{\mu}\left(\nabla_{\mu}\theta - \theta a_{\mu}\right) \leq 0$$

- it comes:

$$\eta = 3\frac{\kappa\alpha}{\widetilde{\rho}_c}\underline{e}_{\mu\nu}\underline{g}^{\mu\nu} + \frac{\mathcal{C}}{\theta_0}(\theta - \theta_0)$$

$$T^{\mu\nu}_{\sigma} = -3\kappa\alpha(\theta-\theta_0)\underline{g}^{\mu\nu} + \lambda\underline{e}_{\gamma\beta}\underline{g}^{\gamma\beta}\underline{g}^{\mu\nu} + 2\mu\underline{e}^{\mu\nu} - 3\kappa\alpha(\theta-\theta_0)\left(\underline{e}_{\gamma\beta}\underline{g}^{\gamma\beta}\underline{g}^{\mu\nu} - 2\underline{e}^{\mu\nu}\right)$$
$$+\lambda\left(-2\underline{e}_{\gamma\beta}\underline{g}^{\gamma\beta}\underline{e}^{\mu\nu} + \frac{\left(\underline{e}_{\gamma\beta}\underline{g}^{\gamma\beta}\right)^2}{2}\underline{g}^{\mu\nu}\right) + 2\mu\left(\underline{e}_{\gamma\beta}\underline{e}^{\gamma\beta}\underline{g}^{\mu\nu} - 2\left(\underline{e}^{\mu}_{\beta}\underline{e}^{\beta\nu}\right)^{sym}\right)$$

Conclusion

- A covariant thermodynamic.
- To derive constitutive models.
- Advantage: no need to consider a material (Lagrangean) configuration, the derivation is possible for all descriptions.
- Perspective : model dissipative effects

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Thank you

- The four-dimensional Cauchy deformation tensor $\boldsymbol{b}$:

$$b_{\mu\nu} = (F^{\alpha}_{\mu})^{-1}(F^{\beta}_{\nu})^{-1}\eta_{\alpha\beta}$$

$$\mathscr{L}_u(\underline{b}_{\mu\nu}) = 0$$

- The 4D Eulerian strain tensor $\boldsymbol{e}$:

$$e_{\mu\nu} = \frac{1}{2}(g_{\mu\nu} - b_{\mu\nu})$$

$$\mathscr{L}_u(\underline{e}_{\mu\nu}) = \underline{d}_{\mu\nu} = d_{\mu\nu}$$

- The four-acceleration $\boldsymbol{a}$:

$$a^{\mu} = u^{\lambda}\nabla_{\lambda}u^{\mu}$$

- The rate of deformation $\boldsymbol{d}$:

$$d_{\mu\nu} = \frac{1}{2}\mathscr{L}_u(g_{\mu\nu})$$

The energy-momentum tensor :

$$T^{\mu\nu} = \mathcal{U} u^\mu u^\nu + (q^\mu u^\nu + q^\nu u^\mu) + T_\sigma^{\mu\nu}$$

- $\mathcal{U}$ is a scalar density $\mathcal{U} = T^{\mu\nu} u_\mu u_\nu$
- q^μ is a vector $q^\mu = (\delta^\mu{}_\alpha - u^\mu u_\alpha) T^{\alpha\beta} u_\beta$
- $T_\sigma^{\mu\nu}$ is a second-rank tensor $T_\sigma^{\mu\nu} = (\delta^\mu{}_\alpha - u^\mu u_\alpha)(\delta^\nu{}_\beta - u^\nu u_\beta) T^{\alpha\beta}$

In the proper coordinate system :

$$\hat{T}^{\mu\nu} = \begin{pmatrix} \hat{\mathcal{U}} & \hat{q}^1 & \hat{q}^2 & \hat{q}^3 \\ \hat{q}^1 & \hat{T}_\sigma^{11} & \hat{T}_\sigma^{12} & \hat{T}_\sigma^{13} \\ \hat{q}^2 & \hat{T}_\sigma^{21} & \hat{T}_\sigma^{22} & \hat{T}_\sigma^{23} \\ \hat{q}^3 & \hat{T}_\sigma^{31} & \hat{T}_\sigma^{32} & \hat{T}_\sigma^{33} \end{pmatrix}$$

With physical considerations :

- the energy density $\mathcal{U} = \widetilde{\rho}_c \left(c^2 + e_{int}\right)$
- the energy flux density $\boldsymbol{q}$
- the stress tensor $\boldsymbol{T}_\sigma$