ABOUT THE STRUCTURE OF THE DISCRETE AND CONTINUOUS ERINGEN'S NONLOCAL ELASTICA

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Abstract. The Eringen's nonlocal elastica equation does not possess a Lagrangian formulation. In this article, we find a variational integrating factor which enables us to provide a Lagrangian and Hamiltonian structure associated to this equation. We then derive a discrete version of the Eringen's nonlocal elastica preserving the Lagrangian and Hamiltonian structure. We also give an expression of the solutions in term of elliptic integrals of the first kind.

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1. Introduction

In two letters to one of the authors (J.C) [4], N. Challamel raised a number of issues concerning the continuous Eringen's nonlocal elastica equation defined by

$$(1)\qquad \left(1-\beta\ell^2\cos(x)\right)\ddot{x}+\beta\left(1+\ell^2\dot{x}^2\right)\sin(x)=0,\qquad \text{with}\qquad \dot{x}(0)=\dot{x}(1)=0,$$

where $\ell^2=\dfrac{a^2}{12L^2}=\dfrac{1}{12N^2}$ and $\dot{x}=\dfrac{dx}{ds}$ and its discrete analogue defined by N. Challamel and al. in [5] by

$$(2)\qquad x_{i+1}-2x_i+x_{i-1}=-\frac{\beta}{n^2}\sin(x_i),$$

with the boundary conditions $x_1=x_0$ and $x_{n-1}=x_n$, using the continualization method as exposed for example in [5].

Putting apart the boundary conditions, we are interested in the algebro-geometric structure of these two dynamical systems and their relation.

Deriving a discrete analogue of a continuous differential equation is always a challenge and is not only a question of discretizing the differential equation using classical tools of numerical analysis. Indeed, doing such a discretization destroyes in general the basic algebraic, geometric or qualitative properties of the equations and solutions of the continuous model. An example of well defined discrete analogue is provided by the construction of variational integrators for Euler-Lagrange equations. Indeed, in this case, variational integrators are designed in order to preserve the variational structure at the discrete level and as a consequence most of the qualitative properties of the solutions.

The main point is that the Eringen's nonlocal elastica does not possess a specific geometrical or algebraic structure which can be used to constraint the discrete analogue one is looking for. For example, this equation does not possess a Lagrangian formulation. In this note, we provide a discrete analogue of the Eringen's nonlocal elastica by constructing first a **variational integrating factor**, i.e. a function a such that the equation multiplied by a possesses a Lagrangian formulation. Using this result, we are able to derive a Hamiltonian function and to exhibit an **explicit first integral** for the Eringen's nonlocal elastica.

This result can be used to solve three distinct problems:

- First, in [4], N. Challamel suggested that one can probably obtain **explicit formula for the solutions** of the Eringen's nonlocal elastica using **elliptic integrals**. In Section 6, taking benefit of the Hamiltonian structure, we prove that this is indeed the case using elliptic integrals of the first kind and simplifying previous result of M. Lembo [10, 11, 12].

- Second, taking benefit of the existence of a Lagrangian and Hamiltonian structure, we derive a **variational integrator** for the Eringen's nonlocal elastica, i.e. a numerical integrator preserving the Lagrangian and Hamiltonian structure at the discrete level. A classical property of variational integrators is that they preserve very well energy, i.e. the evaluation of the Hamiltonian on solutions. This property induce a very good preservation of the first integral at the discrete level.

- Third, the variational integrator is implicit due to the presence of the integrating factor. We obtain an explicit numerical scheme taking into account the value of the first integral. We call this new numerical integrator a **topological integrator**.

The plan of the paper is as follows:

In Section 2, we introduce a family of ordinary differential equations called the **Eringen's family** generalizing the classical Eringen's nonlocal elastica for which we explicit the **necessary**

and sufficient Helmholt'z conditions for the existence of a Lagrangian variational formulation (see [?]). In particular, we prove that the Eringen's nonlocal elastica does not possess a variational formulation giving a formal proof of arguments and statements given by N. Challamel and al. in [6].

In Section 3, we characterize the subfamily of the Eringen's family for which an integrating exists, i.e. a function such that the given equation multiplied by this function possesses a Lagrangian variational formulation. In particular, we are able to provide an **explicit integrating factor** for the Eringen's nonlocal elastica.

In Section 4, we derive the Hamiltonian associated to the modified Eringen's nonlocal elastica. We deduce an **explicit first integral**. The first integral is then used to provide **explicit formula for the solutions** in term of **Ellpitic integrals** in Section 6.

Section 7 deals with the construction of an efficient numerical scheme for the Eringen's nonlocal elastica. We use the formalism of discrete embedding in order to derive **variational integrators** for the modified Eringen's nonlocal elastica. Variational integrators are in this case implicit. However, a slight modification of the construction lead to an explicit scheme called a **topological integraor**. These two scheme are implemented and compared with the classical Euler scheme as well as the Challamel and al. discrete Eringen's nonlocal elastica defined in [6] here called **Challamel's integrator**. In particular, we prove that variational integrators and the corresponding discrete Hamiltonian versions are more efficient than the other numerical scheme.

Section 8 discusses some perspectives opened by this work.

2. Explicit Helmholtz's conditions for a class of second order differential equations

We denote by $\mathscr{E}$ the three parameter family of second order differential equations defined by

$$a(x)\ddot{x} + b(x)(\dot{x})^2 + c(x) = 0, \tag{3}$$

where a, b, c are real functions. We call **Eringen's family** the previous set of second order differential equations.

This terminology is suggested by the fact that the Eringen's nonlocal elastica belongs to $\mathscr{E}$ with

$$a(x) = 1 - \beta l^2 \cos x, \quad b = a', \quad c = \frac{1}{l^2} a'. \tag{4}$$

An element of $\mathscr{E}$ is denoted by $\mathscr{E}_{a,b,c}$.

A natural question is to characterize the sub-family of equations which are Lagrangian. This can be done using the **Helmholtz's criterion**: Let O be the differential operator associated to (3) and given by

$$O = a(\cdot)\frac{d^2}{dt^2} + b(\cdot)\left(\frac{d}{dt}\right)^2 + c(\cdot) \tag{5}$$

and denotes by $DO(x).h$ the Frechet derivative of O along the direction h defined by

$$DO(x).h = \lim_{\epsilon \to 0} \frac{O(x + \epsilon h) - O(x)}{\epsilon}. \tag{6}$$

Then (3) is Lagrangian if and only if $DO(x)$ is self adjoint.

A proof can be found in the book of J-P. Olver ([13],p.377-379 for a historical about the Helmholtz's problem of the calculus of variations and Theorem 5.92 p.364).

Applying the previous result, one can obtain explicit conditions ensuring that (3) is Lagrangian.

Lemma 2.1. *The second order differential equation* (3) *is Lagrangian if and only if* $a' = 2b$.

Proof. The Frechet derivative of the differential operator is given by

$$DO(x).h = a\ddot{h} + a'\ddot{x}h + 2b\dot{x}\dot{h} + 2b'(\dot{x})^2h + c'h. \tag{7}$$

The adjoint is then given by

$$\begin{array}{rcl} DO(x)^\star.h & = & \dfrac{d^2}{dt^2}(ah) + a'\ddot{x}h - \dfrac{d}{dt}(2b\dot{x}h) + 2b'(\dot{x})^2h + c'h, \\ & = & \ddot{a}h + 2\dot{a}\dot{h} + a\ddot{h} + a'\ddot{x}h - 2\dot{b}\dot{x}h - 2b\dot{x}\dot{h} + 2b'(\dot{x})^2h + c'h. \end{array} \tag{8}$$

Using the fact that for an arbitrary function f, we have $\dot{f}(x) = \dot{x}f'(x)$ and $\ddot{f}(x) = \ddot{x}f'(x) + (\dot{x})^2 f''(x)$, we obtain

$$\begin{array}{rcl} DO(x)^\star.h & = & (\ddot{x}a' + (\dot{x})^2a'')h + 2\dot{x}a'\dot{h} + a\ddot{h} + a'\ddot{x}h - 2b'(\dot{x})^2h - 2b\ddot{x}h - 2b\dot{x}\dot{h} + 2b'(\dot{x})^2h + c'h, \\ & = & a\ddot{h} + (2\dot{x}a' - 2b\dot{x})\dot{h} + (\ddot{x}a' + (\dot{x})^2a'' + a'\ddot{x} - 2b'(\dot{x})^2 - 2b\ddot{x} + 2b'(\dot{x})^2 + c')h \end{array} \tag{9}$$

The self adjoint property gives the following set of relations

$$\left\{ \begin{array}{lcl} 2\dot{x}(a' - 2b) & = & 2b\dot{x}, \\ \ddot{x}a' + (\dot{x})^2a'' + a'\ddot{x} - 2b'(\dot{x})^2 - 2b\ddot{x} + 2b'(\dot{x})^2 + c' & = & a'\ddot{x} + 2b'(\dot{x})^2 + c', \end{array} \right. \tag{10}$$

which leads to

$$\left\{ \begin{array}{lcl} a' - b & = & b, \\ \ddot{x}(a' - 2b) + (\dot{x})^2(a'' - 2b') & = & 0, \end{array} \right. \tag{11}$$

Of course, the first equation

$$a' - 2b = 0, \tag{12}$$

implies the second one. This concludes the proof. □

Applying this result to the Eringen's nonlocal elastica we deduce that:

Lemma 2.2. *The Eringen's nonlocal elastica equation does not possess a Lagrangian formulation*

Proof. As $b = a'$ the Helmholtz's condition reduces to $a' = 0$ which is not true. □

This result has been stated in ([6],p.132). The previous result can be considered as a complete proof of this statement.

3. Integrating factor and the Helmholtz's conditions

The Helmholtz's conditions are deeply related to the presentation of the equation. In particular, even if $O(x) = 0$ does not satisfy the conditions, the equation $\alpha O(x) = 0$ with α a suitable function of x which is not zero almost everywhere, although equivalent to the initial equation can possess a Lagrangian formulation. The function α is then called an **integrating factor**.

Using Lemma 2.1, we have the following characterization of admissible integrating factors:

Lemma 3.1. *Let us consider a differential equation of the form* (3) *such that* $a' - 2b = f$ *with* $f \neq 0$. *Let* α *be an almost everywhere non zero function. Then the differential equation* (3) *admits* α *as an integrating factor if*

$$\alpha' a + \alpha f = 0. \tag{13}$$

Applying this result on the Eringen's nonlocal elastica, we obtain the following result:

Theorem 3.2. *The Eringen's nonlocal elastica possesses a unique (up to multiplication by a constant) integrating factor given by the function* a. *A possible Lagrangian is given by*

$$L(x, v) = \frac{1}{2}a^2v^2 + \frac{1}{2l^2}(a')^2 + \frac{1}{l^2}a''. \tag{14}$$

Proof. In the Eringen's nonlocal elastica case, we have $f = -a'$ so that (13) is equivalent to

$$\alpha' a - \alpha a' = 0. \tag{15}$$

As a is almost everywhere non zero, this equation can be solved explicitly ans gives

$$\alpha = Ca, \tag{16}$$

where C is a constant. As a consequence, α is an admissible function and the a-deformation of the Eringen's nonlocal elastica equation possesses a Lagrangian formulation.

Using the proposed Lagrangian, we obtain

$$\frac{\partial L}{\partial v} = a^2 v, \quad \frac{\partial L}{\partial x} = a' a v^2 + \frac{1}{l^2} a' a'' + \frac{1}{l^2} a^{(3)}, \tag{17}$$

where $a^{(3)}$ denotes the third derivative of a with respect to x. Using the fact that

$$a^{(3)} = -a', \tag{18}$$

we obtain for the Euler-Lagrange equation

$$\frac{d}{dt}\left(a^2 \dot{x}\right) = a' a (\dot{x})^2 + \frac{1}{l^2} a' a'' - \frac{1}{l^2} a', \tag{19}$$

which gives using the fact that $a'' = 1 - a$ that

$$\begin{array}{rcl} \frac{d}{dt}\left(a^2 \dot{x}\right) & = & a' a (\dot{x})^2 - \frac{1}{l^2} a' a \\ & = & a\left(a' (\dot{x})^2 - \frac{1}{l^2} a'\right). \end{array} \tag{20}$$

As

$$\begin{array}{rcl} \frac{d}{dt}\left(a^2 \dot{x}\right) & = & 2(\dot{x})^2 a' a + a^2 \ddot{x}, \\ & = & a\left(2(\dot{x})^2 a' + a\ddot{x}\right), \end{array} \tag{21}$$

we obtain, introducing this expression in equation (20) that

$$a\left((\dot{x})^2 a' + a\ddot{x} + \frac{1}{l^2} a'\right) = 0, \tag{22}$$

which concludes the proof. □

4. A Hamiltonian associated to the modified Eringen's nonlocal elastica equation

The classical way of constructing a Hamiltonian formulation associated to the Lagrangian one via the Legendre transform [2] gives the following result:

Theorem 4.1. *The Hamiltonian system corresponding to the Eringen's equation is given by*

$$\left\{ \begin{array}{rcl} \dot{x} & = & \frac{\partial H}{\partial p} = \frac{p}{a^2}, \\ \dot{p} & = & -\frac{\partial H}{\partial x} = a\left(a' \frac{p^2}{a^4} - \frac{1}{l^2} a'\right), \end{array} \right. \tag{23}$$

with the Hamiltonian function

$$H(x,p) = \frac{1}{2a^2} p^2 - \frac{1}{2l^2} (a')^2 - \frac{1}{l^2} a''. \tag{24}$$

Proof. The variable p corresponding to the momentum is defined by

$$p = \frac{\partial L}{\partial v} = a^2 v, \tag{25}$$

which is one to one as long as $a \neq 0$.

The Legendre transform gives for the Lagrangian L given in Lemma 3.2 the following Hamiltonian:

$$\begin{array}{rcl} H(x,p) & = & pv - L(x,v), \\ & = & \frac{p^2}{a^2} - \frac{1}{2} a^2 \frac{p^2}{a^4} - \frac{1}{2l^2} (a')^2 - \frac{1}{l^2} a'', \\ & = & \frac{1}{2a^2} p^2 - \frac{1}{2l^2} (a')^2 - \frac{1}{l^2} a''. \end{array} \tag{26}$$

A simple computation gives the equation of motion. This concludes the proof. □

It must be noted that the Hamiltonian function depends on the parameters β and l and must be understood as

$$H_{\beta,l}(x,p) = \frac{1}{2a_{\beta,l}^2}p^2 - \frac{1}{2l^2}(a'_{\beta,l})^2 - \frac{1}{l^2}a''_{\beta,l}, \tag{27}$$

with

$$a_{\beta,l}(x) = 1 - \beta l^2 \cos(x). \tag{28}$$

As we have

$$a'_{\beta,l}(x) = \beta l^2 \sin(x), \quad \text{and} \quad a''_{\beta,l}(x) = \beta l^2 \cos(x), \tag{29}$$

we have explicitly the Hamiltonian

$$H_{\beta,l}(x,p) = \frac{1}{2(1-\beta l^2\cos(x))^2}p^2 - \frac{1}{2}\beta^2 l^2 (\sin(x))^2 - \beta\cos(x). \tag{30}$$

Taking $l = 0$ in the previous Hamiltonian, we obtain the classical **simple pendulum** equation:

Corollary 4.2. *Let $\ell = 0$, then the Hamiltonian system Eqs.* (23) *corresponds to the simple pendulum motion:*

$$\begin{cases} \dot{x} = p, \\ \dot{p} = -\beta\sin(x). \end{cases} \tag{31}$$

with the Hamiltonian

$$H_{\beta,0}(x,p) = \frac{p^2}{2} - \beta\cos(x). \tag{32}$$

This property can be used to deduce interesting **qualitative properties** if the Eringen's nonlocal elastica using perturbation theory and the fact that the Hamiltonian is a **constant of motion** on the solutions of the system.

The shape of the energy manifold looks as follows:

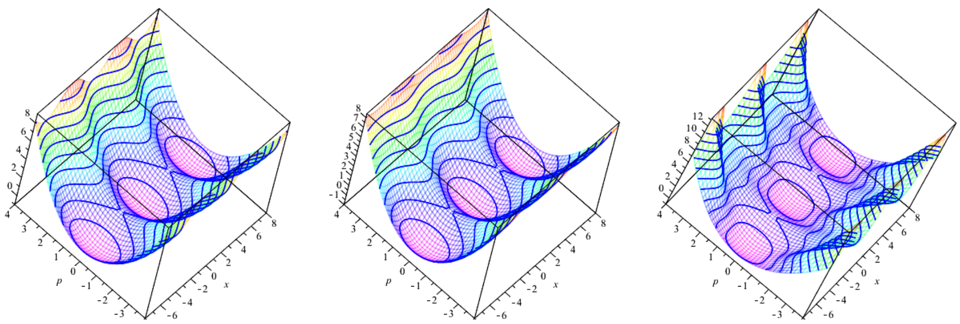


FIGURE 1. Energy manifold for $\beta = 1$ and $l = 0$, $l = 0.2$ and $l = 0.5$

5. Qualitative behavior of the Eringen's nonlocal elastica solutions

As already reminded in Section 4, the main consequence of the existence of a Hamiltonian structure given by Theorem 4.1 is the fact that it provides a constant of motion, i.e. that for all solutions $(x(t), p(t))$ of the Hamiltonian equation (23), we have

$$H_{\beta,l}(x(t),p(t)) = H_{\beta,l}(x(0),p(0)), \tag{33}$$

for all $t \in \mathbb{R}$.

Giving the Legendre transform, it means that we have the following result:

Lemma 5.1. *Let x be a solution of the Eringen's nonlocal elastica equation, then the function $H(x, a^2\dot{x})$ is constant.*

Proof. This follows directly from the fact that for any solution (x_t, p_t) of the Hamiltonian system, we have $H(x_t, p_t)$ which is a constant function. As $p_t = a^2\dot{x}$ by the Legendre transform and x_t is by construction a solution of the Eringen's nonlical elastica, we obtain the result. □

A natural idea is then to look at the **level sets** of the function $I_{\beta,l} : \mathbb{R}^2 \mapsto \mathbb{R}$ defined as

$$I_{\beta,l}(x, v) = H_{\beta,l}(x, a^2(x)v), \tag{34}$$

in order to have a global view of the qualitative behavior of the solutions of the Eringen's nonlocal elastica.

In the following, we provide the level set of $H_{1,l}$ and $I_{1,l}$ for different values of l and we compare with the phase portrait of the Eringen's nonlocal elastica showing the strong influence of the first integral on the dynamics.

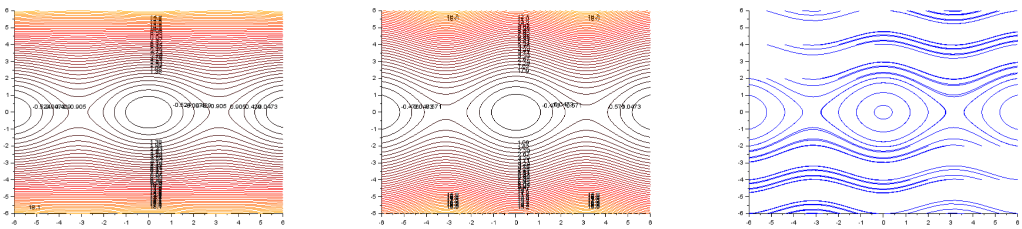


FIGURE 2. Level sets of $H_{\beta,l}$, $I_{\beta,l}$ and phase portrait of the Eringen's nonlocal elastica for $(\beta, l) = (1, 0.2)$

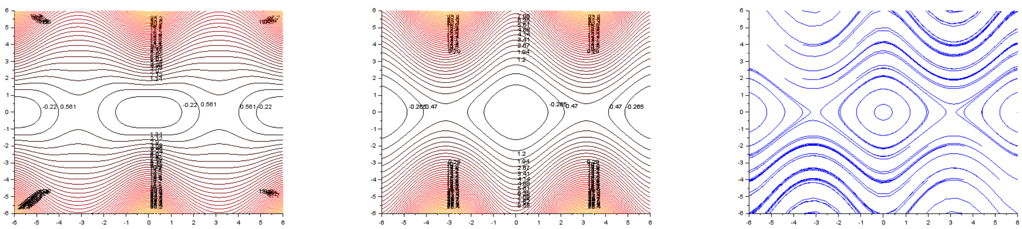


FIGURE 3. Level sets of $H_{\beta,l}$, $I_{\beta,l}$ and phase portrait of the Eringen's nonlocal elastica for $(\beta, l) = (1, 0.5)$

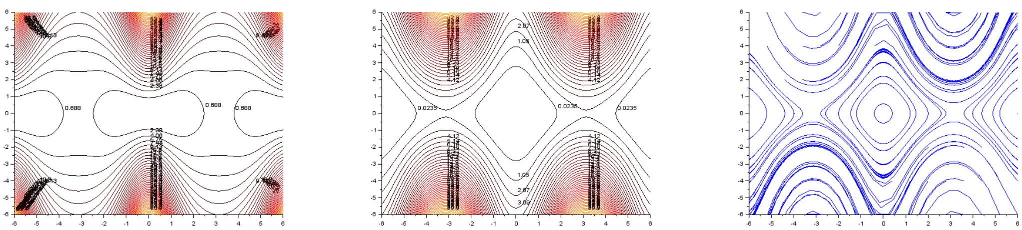


FIGURE 4. Level sets of $H_{\beta,l}$, $I_{\beta,l}$ and phase portrait of the Eringen's nonlocal elastica for $(\beta, l) = (1, 0.7)$

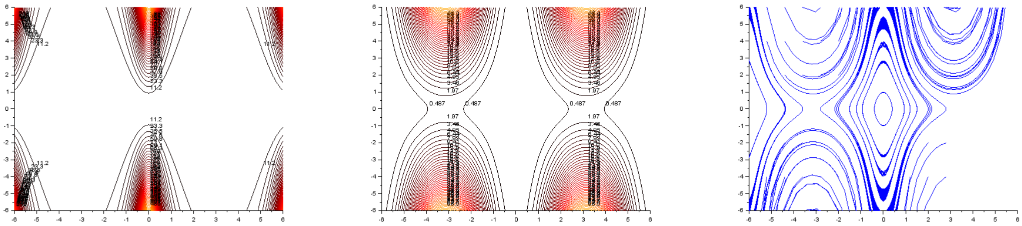


FIGURE 5. Level sets of $H_{\beta,l}$, $I_{\beta,l}$ and phase portrait of the Eringen's nonlocal elastica for $(\beta,l)=(1,0.9)$

Another representation can be obtained taking into account the 2π-periodicity of the solutions of the equations. In that case, the phase portrait must be given on $\mathbb{R}/2\pi\mathbb{Z}\times\mathbb{R}$, i.e. $S^1\times\mathbb{R}$ where S^1 is the unit circle, i.e. a cylinder, which is the classical phase space of the simple pendulum equation.

FIGURE 6. Phase portrait of the Eringen's nonlocal elastica for $\beta=1$, $l=0$, $l=0.2$, $l=0.5$, $l=0.7$ and $l=0.9$

6. Explicit computation of the solutions of the Eringen's nonlocal elastica equation

An important problem is to obtain **explicit forms** for the solutions of the Eringen's nonlocal elastica equation. In [4], N. Challamel suggests that one can probably obtain such solutions using **elliptic integrals**.

Taking benefit of the Hamiltonian structure of the equation and as a consequence, of the existence of a constant of motion, we derive explicit expression for the solutions of the Eringen's nonlocal elastica equation. They are indeed expressed using **elliptic integrals of the first kind** as suggested by N. Challamel and the fact that for $l=0$ the system reduces to the simple pendulum equation.

6.1. Eringen's solutions from Hamiltonian function.

6.1.1. *The simple case.* Let us begin with the classical case of the simple pendulum equation, i.e. for the Hamiltonian $H_{\beta,0}(x,p)=\frac{p^2}{2}-\beta\cos(x)$ which is associated to $H_{\beta,\ell}$ with $l=0$. As $H_{\beta,0}$ is a constant of motion, we have

$$\begin{aligned}
\frac{p^2}{2}-\beta\cos(x)=c \quad &\Longrightarrow \quad p^2=2\left(c+\beta\cos(x)\right),\\
&\Longleftrightarrow \quad \dot{x}^2=2(c+\beta)\left[1-\frac{2\beta}{c+\beta}\sin^2(\theta/2)\right],\\
&\Longleftrightarrow \quad \frac{dx}{dt}=\pm\sqrt{2(c+\beta)}\sqrt{1-\frac{2\beta}{c+\beta}\sin^2(\theta/2)},\\
&\Longleftrightarrow \quad dt=\pm\frac{1}{\sqrt{2(c+\beta)}}\frac{dx}{\sqrt{1-\frac{2\beta}{c+\beta}\sin^2(x/2)}}.
\end{aligned}$$

Setting $k^2 = \frac{2\beta}{c+\beta}$ and integrating both sides gives

$$t = \pm\frac{1}{\sqrt{2(c+\beta)}}\int_{x_0}^{x(t)} \frac{dz}{\sqrt{1-k^2\sin^2(z/2)}} := E_{\beta,k,c}(x). \tag{35}$$

The right-hand side of Eq. (35) is an **incomplete elliptic integral of the first kind**.

$$E_{\beta,x_0,c}\big(x(t)\big) := \int_{x_0}^{x(t)} \pm\frac{1}{\sqrt{2}}\frac{dz}{\sqrt{c+\beta\cos(z)}} = \int_{\phi_0}^{\phi(t)} \mp\frac{1}{\sqrt{2}}\frac{1}{\sqrt{c+\beta u}}\frac{du}{\sqrt{1-u^2}} = t, \tag{36}$$

As the simple pendulum case is the simplest one for which the computations can be made, we have no hope to obtain a simpler form in the general case than one using elliptic integrals. This is done in the following Section.

6.1.2. *The general case.* The same procedure can be followed for the general case, i.e. one considers the Hamiltonian $H_{\beta,\ell}$. The result can be resumed as follows:

Lemma 6.1. *Let $(x_0, v_0) \in \mathbb{R}^2$ be given. We denote by c the quantity $H_{\beta,l}(x_0, a^2(x_0)v_0) = c_0$ and $\beta l^2 = \lambda$, $2c_0 + \beta^2 l^2 = k_0$, $\beta^2 l^2 = \gamma$ then for all $t \in \mathbb{R}$, the solution $x(t)$ satisfies*

$$t = E_{\beta,l,x_0,c_0}\big(x(t)\big) := \int_{\cos(x_0}^{\cos(x(t))} \pm\frac{\lambda u - 1}{\sqrt{k_0 + 2\beta u - \gamma u^2}}\frac{du}{\sqrt{1-u^2}}.$$

This kind of integral is called an elliptic integral.

Proof. Solving the equation $H_{\beta,l}(x,p) = c$ with respect to p, we have

$$\frac{1}{2(1-\beta l^2\cos(x))^2}p^2 - \frac{1}{2}\beta^2 l^2(\sin(x))^2 - \beta\cos(x) = c$$

then

$$p = \pm(1-\beta l^2\cos x)\sqrt{2c+\beta^2 l^2 + 2\beta\cos(x) - \beta^2 l^2(\cos(x))^2}, \tag{37}$$

where we use $1-(\sin(x))^2 = (\cos(x))^2$. Taking into account the Hamiltonian system, we have $p = (1-\beta l^2\cos(x))^2\dot{x}$ and substituting into (37),

$$\dot{x} = \pm\frac{1}{(1-\beta l^2\cos x)}\sqrt{2c+\beta^2 l^2 + 2\beta\cos(x) - \beta^2 l^2(\cos(x))^2},$$

$$\Updownarrow$$

$$dt = \pm\frac{(1-\beta l^2\cos x)}{\sqrt{2c+\beta^2 l^2 + 2\beta\cos(x) - \beta^2 l^2(\cos(x))^2}}dx$$

Integrating both side we obtain, using the function for incomplete elliptic integral of the first kind,

$$t = E_{\beta,l,x_0,c}\big(x(t)\big) := \int_{x_0}^{x(t)} \pm\frac{(1-\lambda\cos(z))}{\sqrt{k+2\beta\cos(z) - \gamma(\cos(z))^2}}dz, \tag{38}$$

Setting $u = \cos z$ then $du = -\sin z dz = -\sqrt{1-u^2}dz$, then the previous integral becomes

$$t = E_{\beta,l,x_0,c}\big(x(t)\big) := \int_{u_0}^{u(t)} \pm\frac{\lambda u - 1}{\sqrt{k+2\beta u - \gamma u^2}}\frac{du}{\sqrt{1-u^2}},$$

where $u(t) = \cos(x(t))$ and $u_0 = \cos(x_0)$. This concludes the proof. ☐

6.2. **Eringen's solutions from canonical variables.** There is in fact another way to obtain solutions Eringen equation by considering a suitable change of variables the so-called canonical variables (see [15]).

6.2.1. *Canonical variables for the original Eringen's equation.* Consider the Eringen's equation

$$a(x)\ddot{x} + b(x)\dot{x}^2 + kb(x) = 0. \tag{39}$$

where $a(x) = 1 - \beta l^2 \cos(x)$, $b(x) = a'(x)$ and $k = \frac{1}{l^2}$. By a suitable change of variables $(t, x) = (r, w)$, we have that

$$\dot{x} = \dot{w} = \frac{1}{\dot{r}}, \quad \ddot{x} = -\frac{\ddot{r}}{\dot{r}^3}.$$

Rewriting equation in terms of r and w becomes

$$-a(w)\frac{\ddot{r}}{\dot{r}^3} + b(w)\frac{1}{\dot{r}^2} + kb(w) = 0 \Rightarrow -a(w)\ddot{r} + b(w)\dot{r} + kb(w)\dot{r}^3 = 0$$

Sitting $z = \dot{r}$, the last equation becomes separable first-order ODE

$$\dot{z} = \frac{b(w)}{a(w)}\left(z + kz^3\right),$$

its solution is given by

$$\frac{z}{\sqrt{1 + kz^2}} = ca(w), \quad c > 0$$

Solving the last equation for z yields,

$$z = \dot{r} = \pm\frac{ca(w)}{\sqrt{1 - k^2c^2a^2(w)}}$$

Finally, returning to the original variables we have that $\dot{r} = dt/dx$ and integrating the last equation gives

$$t = \pm\int \frac{ca(x)}{\sqrt{1 - k^2c^2a(x)}}dx + c_1$$

which is an elliptic integral.

6.2.2. *Canonical variables with the Lagrangian L.* We can also find the solutions of Eringen's equation using the corresponding Lagrangian L as given in Section 3:

$$L(x, v) = \frac{1}{2}a^2v^2 + \frac{1}{2l^2}(a')^2 + \frac{1}{l^2}a''$$

It is obvious that the Lagrangian L does not depend on t so the operator $\boldsymbol{X} = \frac{\partial}{\partial t}$ called a variational symmetry [13]. The canonical variables corresponds to the operator $\boldsymbol{X}$ are given by $(t, x) = (r, w)$ (see [15]) so that, under such variables one can write the Lagrangian L in term of r and w (see [15, page 65]), we have

$$v = \dot{x} = \dot{w} = \frac{1}{\dot{r}}$$

and the Lagrangian L becomes

$$\begin{aligned}\tilde{L}(w, \dot{r}) = \frac{1}{\dot{w}}L(x, v) = \frac{1}{v}L(x, v) &= \frac{1}{2}a^2v + \frac{1}{2l^2v}(a')^2 + \frac{1}{l^2v}a'' \\ &= \frac{1}{2}a^2(w)\dot{r} + \frac{1}{2l^2\dot{r}}(a'(w))^2 + \frac{1}{l^2\dot{r}}a''(w).\end{aligned}$$

Therefore, the corresponding Euler Lagrange equation is given by

$$\frac{d}{dw}\left(\frac{\partial\tilde{L}}{\partial\dot{r}}\right) = \frac{\partial\tilde{L}}{\partial r}$$

As $\tilde{L}$ does not depend on r, we have the following first integral:

$$I(w, \dot{r}) = \frac{\partial\tilde{L}}{\partial\dot{r}} = \frac{1}{2}a^2(w) - \frac{1}{2l^2\dot{r}^2}(a'(w))^2 - \frac{1}{l^2\dot{r}^2}a''(w).$$

Rewriting I in term of the original variables becomes

$$I(x, v) = \frac{1}{2}a^2 - \frac{1}{2l^2}(a')^2\dot{x}^2 - \frac{1}{l^2}a''\dot{x}^2.$$

Since $I(x,v)$ is a constant of motion one can write $I(x,v)=c$, then we obtain

$$\dot{x}=\pm\frac{\sqrt{\frac{1}{2}a^2-c}}{\sqrt{\frac{1}{2l^2}(a')^2+\frac{1}{l^2}a''}}\Longrightarrow t=\int\pm\frac{\sqrt{\frac{1}{2l^2}(a')^2+\frac{1}{l^2}a''}}{\sqrt{\frac{1}{2}a^2-c}}dx+c_1$$

This kind of integral is also an elliptic one.

7. Variational and Topological numerical integrator for the Eringen's nonlocal elastica

The existence of the first integral $I_{\beta,l}$ for the Eringen's nonlocal elastica can be used to design numerical integrators preserving this first integral. Such a numerical integrator is reminiscent of geometric numerical integration and can be called topological numerical integrator as the preservation of the first integral ensure that the topological properties of the solutions constrained by the first integral are preserved. To construct such a topological numerical integrator an idea is first to use the existence of an integrating factor and the variational structure which is associated. We then first construct a variational integrator for the modified Eringen's nonlocal elastica using the **discrete embedding formalism** ([8],[7]). Having this numerical integrator, we are able to propose a discrete dynamical systems which can be called "discrete Eringen's nonlocal elastica" as the fundamental properties of this discrete system are similar to the continuous case from the point of view of first integral and existence of an underlying variational structure up to an integrating factor.

7.1. **Using a classical Euler scheme.** In this Section, we provide some simulations of the Eringen's nonlocal elastica using an explicit Euler scheme. The quality of the numerical scheme is measured by the quality of the preservation of the first integral at the discrete level. As we have seen in Section 5, the qualitative properties of the solutions are controlled by the level surface of the first integral.

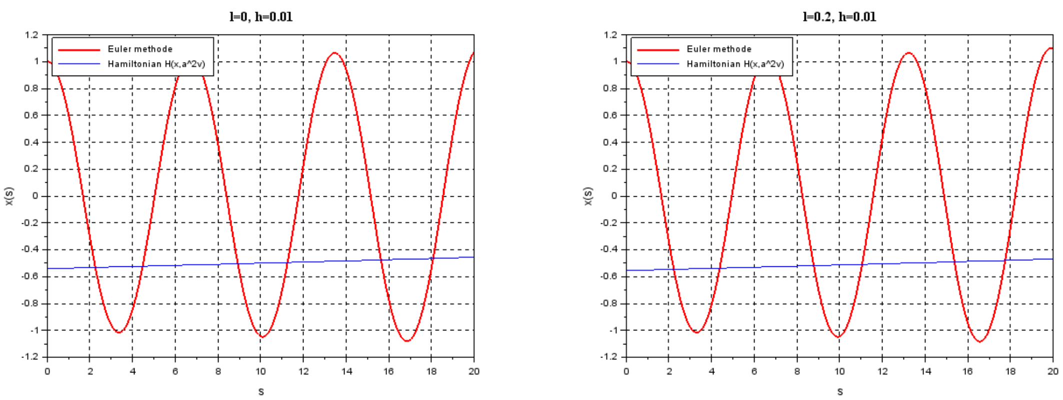


Figure 7. Numerical solution of the Eringen's nonlocal elastica for $l=0$ and $l=0.2$ and the corresponding evaluation of the first integral - Euler scheme - $h=0.01$

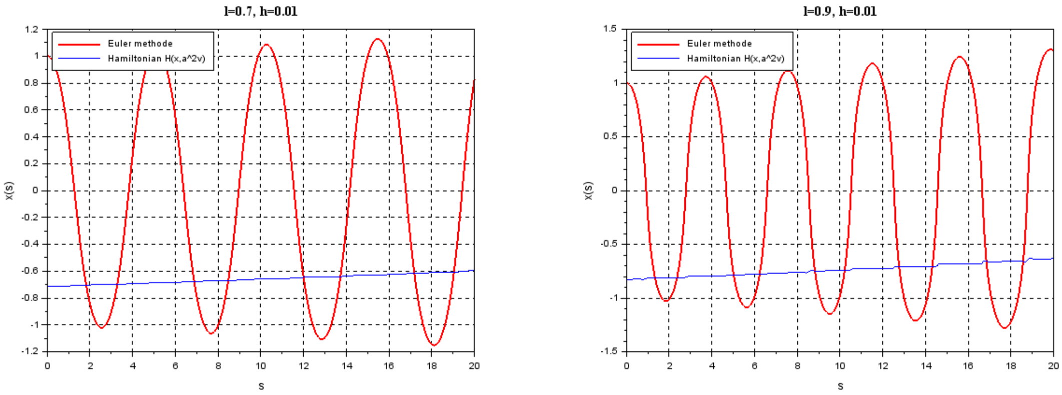


FIGURE 8. Numerical solution of the Eringen's nonlocal elastica for $l = 0.7$ and $l = 0.9$ and the corresponding evaluation of the first integral - Euler scheme - $h = 0.01$

As we can see, a rapid divergence of the values of the first integral is observed.

7.2. Variational integrator and the Eringen's nonlocal elastica.

7.2.1. *Reminder about discrete derivatives and integrals.* Let $[a, b]$ be a given interval and $N \in \mathbb{N}^*$. Let $h = 1/N$. We denote by $\mathbb{T}_h$ (or simply $\mathbb{T}$) a uniform time scale defined by $\mathbb{T}_h = \{t_i\}_{i=0,\dots,N}$ and $t_i = a + i * h$. We denote by $\mathbb{T}^- = \mathbb{T} \setminus \{t_0\}$ and $\mathbb{T}^+ = \mathbb{T} \setminus \{t_N\}$.

Let $g \in C([a, b], \mathbb{R})$ be a given function. The discrete associate to g is simply the function g restricted to $\mathbb{T}$. We also denote it by g in the following.

Let $f \in C(\mathbb{T}, \mathbb{R})$ and $t \in \mathbb{T}_{pm}$. The $\pm$-discrete integral of f over $[a, t]$ is the quantity denoted by $\displaystyle\int_{[a,b]} f(t)\Delta_\pm t$ and defined by

$$\int_{[a,t]} f(t)\Delta_\pm t = \sum_{t_i \in [a,t] \cap \mathbb{T}_\pm} g(t_i)h. \tag{40}$$

Of course $\int_[a, b]f(s)\Delta_- s$ corresponds to the right Riemann sum associated to $\mathbb{T}$ and $\int_[a, b]f(s)\Delta_+ s$ to the left one.

We denote by $\Delta_-[x]x$ the **backward discrete derivative** defined for all $x \in C(\mathbb{T}, \mathbb{R})$, $x(t_i) = x_i$, $i = 0, \dots, N$ by

$$\Delta_-[x](t_i) = \frac{x_i - x_{i-1}}{h}, \quad i = 1, \dots, N. \tag{41}$$

We denote by Δ_+ the **forward discrete derivative** defined for all $x \in C(\mathbb{T}, \mathbb{R})$ by

$$\Delta_+[x](t_i) = \frac{x_{i+1} - x_i}{h}, \quad i = 0, \dots, N - 1. \tag{42}$$

In the following, we simply denote $\Delta_\pm[x]_i$ for $\Delta_{pm}[x](t_i)$ when no confusion is possible.

The discrete derivatives and integrals satisfy a discrete analogue of the fundamental theorem of differential calculus, i.e.

$$\Delta_{pm}\left[\int_a^t f(s)\Delta_- s\right](t) = f(t), \ \forall\, t \in \mathbb{T}_\pm. \tag{43}$$

7.2.2. *Variational integrators and discrete embedding.* In this Section, we follow the discrete embedding formalism introduced in [?] an developed further in [8, 9] and [7].

We first define a discrete analogue of the Lagrangian functional

$$\mathscr{L}(x) = \int_a^b L(x(s), \dot{x}(s)ds, \tag{44}$$

with L given by Theorem 3.2. The $\pm$-discrete Lagrangian functional denoted by $\mathscr{L}_h$ is defined $C(\mathbb{T},\mathbb{R})$ by

$$\mathcal{L}_{\pm}^{h}[x] = \int_{\mathbb{T}_h} L\left(x, \Delta_{\pm} x\right) \Delta_{\pm} t. \tag{45}$$

The discrete Calculus of variations [7] leads to the discrete Euler-Lagrange equation defined by

$$\Delta_{\mp}\left[\frac{\partial L}{\partial v}(x, \Delta_{\pm}[x])\right](t) = \frac{\partial L}{\partial x}(x, \Delta_{\pm}[x])(t), \ \forall\, t \in \mathbb{T}^{+-} = \mathbb{T}^{+} \cap \mathbb{T}^{-}. \tag{46}$$

The particular feature of the previous numerical integrator is to provide a symplectic integrator which are known to possess very good properties of preservation of energy, i.e. of H.

7.2.3. *The discrete Eringen's nonlocal elastica.* We use the previous construction using the Lagrangian obtained in Theorem 3.2. The backward discrete Lagrangian functional is given

$$\mathscr{L}_{-}(x) = \int_{[a,b]} \left(\frac{1}{2}(a(x)\Delta x)^2 + \frac{1}{2l^2}(a'(x))^2 + \frac{1}{l^2}a''(x)\right)\Delta_{-} t = \sum_{i=1}^{N}\left(\frac{1}{2}(a_i(\Delta_{-}[x]_i)^2 + \frac{1}{2l^2}(a_i')^2 + \frac{1}{l^2}a_i''\right), \tag{47}$$

where $a_i = a(x_i)$, $a_i' = a'(x_i)$ and $a_i'' = a''(x_i)$.

The discrete Euler-Lagrange equation associated to L is given by the following Theorem:

Theorem 7.1. *The backward variational integrator associated to Eq.* (1) *is given by*

$$a_{i+1}^2(x_{i+1} - x_i) = a_i^2 x_i - a_i^2 x_{i-1} + a_i \beta \sin(x_i)\left(l^2(x_i - x_{i-1})^2 - h^2\right) \tag{48}$$

for $i = 1, \dots, N-1$

When $l = 0$ and $\beta = 1$, we obtain the classical variational integrator for the simple pendulum:

$$x_{i+1} = 2x_i - x_{i-1} - h^2 \sin(x_i). \tag{49}$$

It must be noted that the implicit character of the numerical scheme is directly related to the term corresponding to the integrating factor.

Proof. The equation (52) can be found directly from (17) using the fact that

$$\frac{1}{l^2}a'(a'' - 1) = -\frac{1}{l^2}aa', \tag{50}$$

so that

$$\frac{\partial L}{\partial x} = aa'\left[v^2 - \frac{1}{l^2}\right] \tag{51}$$

We then obtain the discrete Euler-Lagrange equation

$$\Delta_{+}\left[a^2 \cdot \Delta_{-}[x]\right]_i = a_i a_i'\left[(\Delta_{-}[x])i^2 - \frac{1}{l^2}\right], \qquad i = 1, \dots, n-1. \tag{52}$$

Using the expression of Δ_+ and Δ_- a more explicit form can be obtained. We have

$$\begin{aligned} h\Delta_{+}\left[a^2 \cdot \Delta_{-}[x]\right]_i &= a_{i+1}^2 \Delta_{-}[x]_{i+1} - a_i^2 \Delta_{-}[x]_i, \\ &= h^{-1}\left(a_{i+1}^2 x_{i+1} - (a_{i+1}^2 + a_i^2)x_i + a_i^2 x_{i-1}\right). \end{aligned} \tag{53}$$

The right hand term is given by

$$a_i a_i'\left[(\Delta_{-}[x])i^2 - \frac{1}{l^2}\right] = a_i a_i' h^{-2}\left((x_i - x_{i-1})^2 - h^2 l^{-2}\right). \tag{54}$$

As a consequence, we obtain the following expression

$$a_{i+1}^2 x_{i+1} - (a_{i+1}^2 + a_i^2)x_i + a_i^2 x_{i-1} = a_i a_i'\left((x_i - x_{i-1})^2 - h^2 l^{-2}\right) \tag{55}$$

for $i = 1, \dots, N-1$ which can be rewritten as

$$a_{i+1}^2(x_{i+1} - x_i) = a_i^2 x_i - a_i^2 x_{i-1} + a_i a_i'\left((x_i - x_{i-1})^2 - h^2 l^{-2}\right) \tag{56}$$

for $i = 1, \dots, N-1$. This concludes the proof. □

In the same way, the forward variational integrator is given by:

Theorem 7.2. *The forward variational integrator associated to Eq.* (1) *is given by*

$$x_{i+1} = \frac{1}{a_i^2}\left[(a_i^2 + a_{i-1}^2)x_i - a_{i-1}^2 x_{i-1} + a_i l^2 \sin(x_i)(x_{i+1}-x_i)^2 - h^2 a_i \sin(x_i)\right], \quad i = 1,\dots,N-1. \tag{57}$$

for $i = 1,\dots,N-1$

When $\beta = 1$ and $l = 0$, we obtain again the classical variational integrator for the simple pendulum

$$x_{i+1} = 2x_i - x_{i-1} - h^2 \sin(x_i). \tag{58}$$

When $l \neq 0$, the numerical scheme is implicit but relies on finding roots of a polynomials of degree 2. Precisely, in order to find x_{i+1}, we have to solve the polynomial equation $P_i(x) = 0$ where the polynomial P_i is given by

$$P_i(x) = \alpha_i x^2 - x\beta_i + \gamma_i, \tag{59}$$

with

$$\alpha_i = a_i l^2 \sin(x_i), \quad \beta_i = 2a_i l^2 x_i \sin(x_i) + a_i^2, \quad \gamma_i = (a_i^2 + a_{i-1}^2)x_i - a_{i-1}^2 x_{i-1} - h^2 a_i \sin(x_i). \tag{60}$$

Proof. We have

$$\begin{aligned} \Delta_-\left[a^2(x)\Delta_+[x]](t_i)\right] &= \frac{1}{h}\left(a_i^2\Delta_+[x]_i - a_{i-1}^2\Delta_+[x]_{i-1}\right), \\ &= \frac{1}{h^2}\left(a_i^2(x_{i+1}-x_i) - a_{i-1}^2(x_i - x_{i-1})\right), \\ &= \frac{1}{h^2}\left(a_i^2 x_{i+1} - (a_i^2 + a_{i-1}^2)x_i + a_{i-1}^2 x_{i-1}\right). \end{aligned} \tag{61}$$

Moreover, we have

$$\begin{aligned} \frac{\partial L}{\partial x}(x_i, \Delta_+[x]_i) &= l^2 \sin(x_i) a_i (\Delta_+[x]_i)^2 - \sin(x_i) a_i, \\ &= a_i \sin(x_i)(l^2(\Delta_+[x]_i)^2 - 1). \end{aligned} \tag{62}$$

□

In the forward and backward case the corresponding variational integrators are **implicit**. This little increase of the algorithmic complexity is the price to pay in order to obtain a variational integrator in this case. In Section 7.4, we look for a modification of the scheme which can lead to an explicit one.

7.3. **Simulations of the variational integrator for the Eriengen's nonlocal elastica.** In order to implement the variational integrator for the Eringen's nonlocal elastica, we need to solve the implicit equation. This is done numerically using a Newton-Raphson method.

Simulations of the variational integrator are provides in the following for $h = 0.1$ on the interval $[0, 20]$ with $x_0 = 1$, $x_1 = x_0$ for $l = 0$, 0.2, 0.5, 0.7 and 0.9.

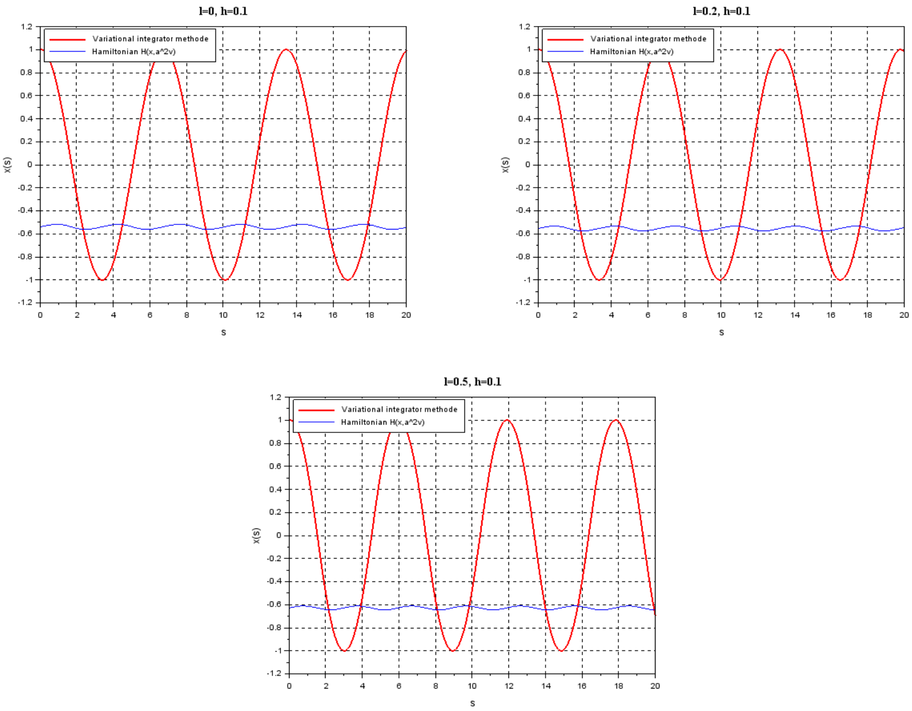


FIGURE 9. Numerical solution of the Eringen's nonlocal elastica for $l = 0$, $l = 0.2$, $l = 0.5$ and the corresponding evaluation of the first integral - Variational integrator - $h = 0.1$

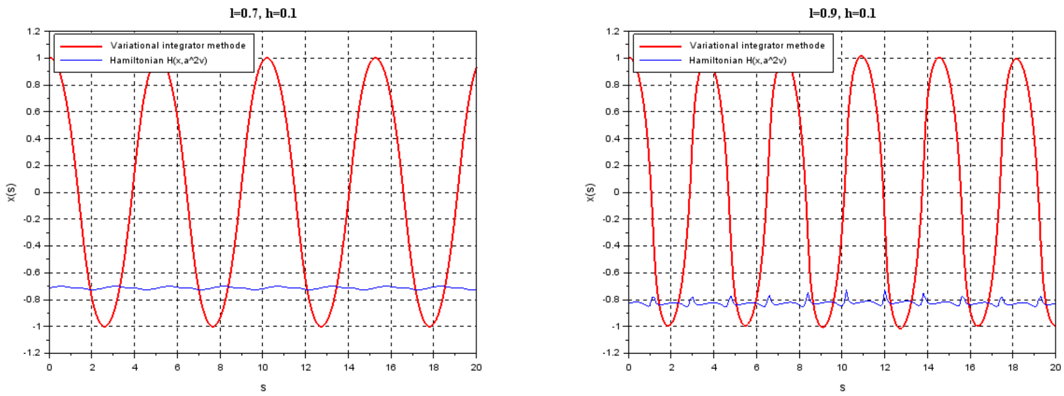


FIGURE 10. Numerical solution of the Eringen's nonlocal elastica for $l = 0.7$ and $l = 0.9$ and the corresponding evaluation of the first integral - Variational integrator - $h = 0.1$

We have then a very good preservation of the first integral and an accurate simulation of the behavior of the solutions.

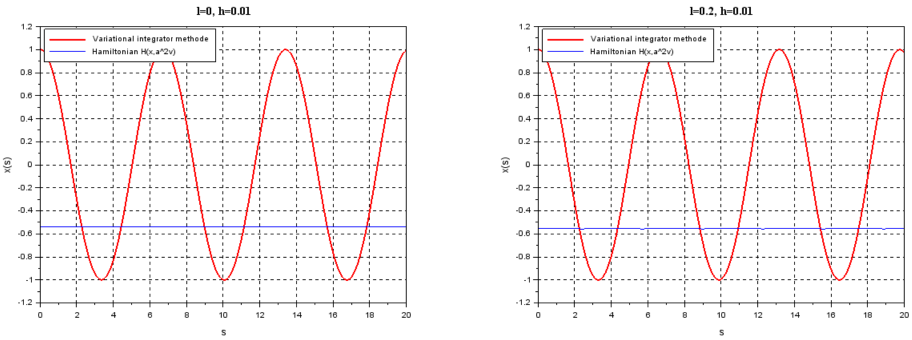


FIGURE 11. Numerical solution of the Eringen's nonlocal elastica for $l = 0$, $l = 0.2$ and the corresponding evaluation of the first integral - Variational integrator - $h = 0.01$

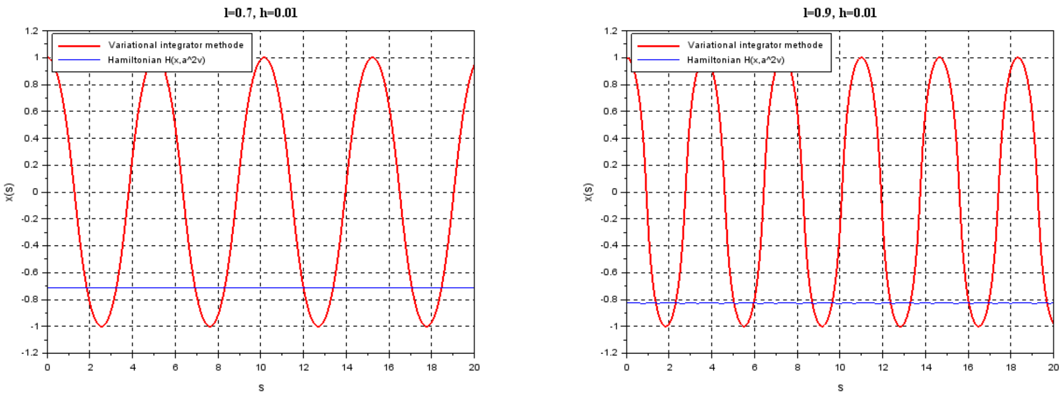


FIGURE 12. Numerical solution of the Eringen's nonlocal elastica for $l = 0.7$ and $l = 0.9$ and the corresponding evaluation of the first integral - Variational integrator - $h = 0.01$

7.4. **Topological integrator.** As we want to preserve the first integral, we have to satisfy the following equation for all $i = 0, \dots, N$ as precisely as possible in the backward case :

$$\frac{1}{2}a_i^2(\Delta_-[x]_i)^2 - \frac{1}{2}l^2(\sin(x_i))^2 - \cos(x_i) = c_0, \tag{63}$$

where c_0 is fixed as long as x_0 and x_1 are given. Another equivalent formulation is

$$\frac{1}{2}a_i^2(x_i - x_{i-1})^2 - h^2\frac{1}{2}l^2(\sin(x_i))^2 - h^2\cos(x_i) = h^2c_0, \tag{64}$$

In the forward case, we have to satisfy the equation

$$\frac{1}{2}a_i^2(\Delta_+[x]_i)^2 - \frac{1}{2}l^2(\sin(x_i))^2 - \cos(x_i) = c_0, \tag{65}$$

or equivalently that

$$\frac{1}{2}a_i^2(x_{i+1} - x_i)^2 - h^2\frac{1}{2}l^2(\sin(x_i))^2 - h^2\cos(x_i) = h^2c_0, \tag{66}$$

This last equation can be used to replace directly the term $(x_{i+1} - x_i)^2$ in the right hand side of the forward variational integrator. Indeed, multiplying the forward variational integrator by a_i, we obtain:

$$\begin{array}{rcl} a_i^3 x_{i+1} & = & a_i(a_i^2 + a_{i-1}^2)x_i - a_i a_{i-1}^2 x_{i-1} \\ & & +a_i^2 l^2 \sin(x_i)(x_{i+1} - x_i)^2 \\ & & -h^2 a_i^2 \sin(x_i), \quad i = 1, \dots, N-1. \end{array} \tag{67}$$

Replacing $a_i^2(x_{i+1} - x_i)^2$ by its expression, we have

$$\begin{array}{rcl} a_i^3 x_{i+1} & = & a_i(a_i^2 + a_{i-1}^2)x_i - a_i a_{i-1}^2 x_{i-1} \\ & & +l^2 \sin(x_i)\left(h^2 l^2 (\sin(x_i))^2 + 2h^2 \cos(x_i) + 2h^2 c_0\right) \\ & & -h^2 a_i^2 \sin(x_i), \quad i = 1, \dots, N-1. \end{array} \tag{68}$$

We then obtain the following topological integrator:

Lemma 7.1. *The topological integrator associated to the Eringen's nonlocal elastica is the explicit numerical scheme defined by*

$$\begin{array}{rcl} x_{i+1} & = & \dfrac{1}{a_i^3}\left[a_i(a_i^2 + a_{i-1}^2)x_i - a_i a_{i-1}^2 x_{i-1}\right. \\ & & +l^2 \sin(x_i)\left(h^2 l^2 (\sin(x_i))^2 + 2h^2 \cos(x_i) + 2h^2 c_0\right) \\ & & \left.-h^2 a_i^2 \sin(x_i)\right], \quad i = 1, \dots, N-1. \end{array} \tag{69}$$

Of course, recovering an explicit numerical scheme has a price: we have destroyed the discrete variational structure of the variational integrator. Nevertheless, as we will see in the next Section, we obtain a numerical integrator with good properties in particular for the preservation of the first integral.

7.5. **Simulations of the topological integrator.** Using the topological integrator which corresponds to the variational integrator associated to the modified equation, we obtain the following result for the same walues of l:

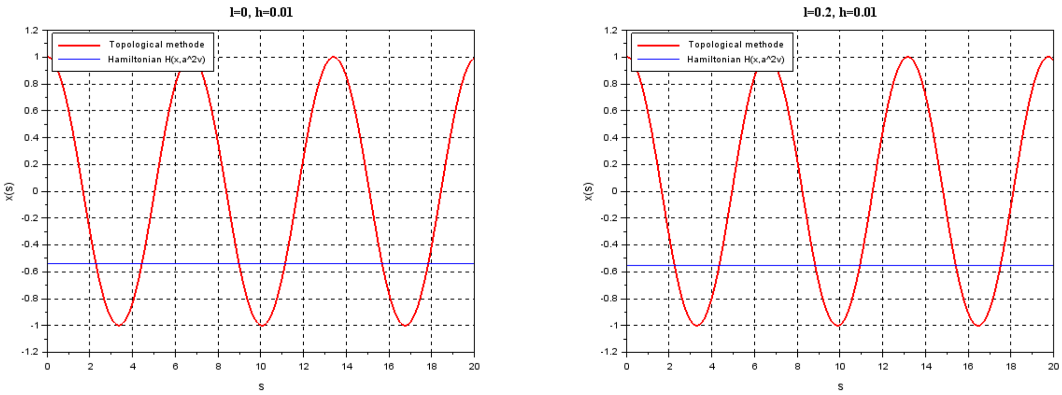


FIGURE 13. Numerical solution of the Eringen's nonlocal elastica for $l = 0$ and $l = 0.2$ and the corresponding evaluation of the first integral - Topological integrator - $h = 0.01$

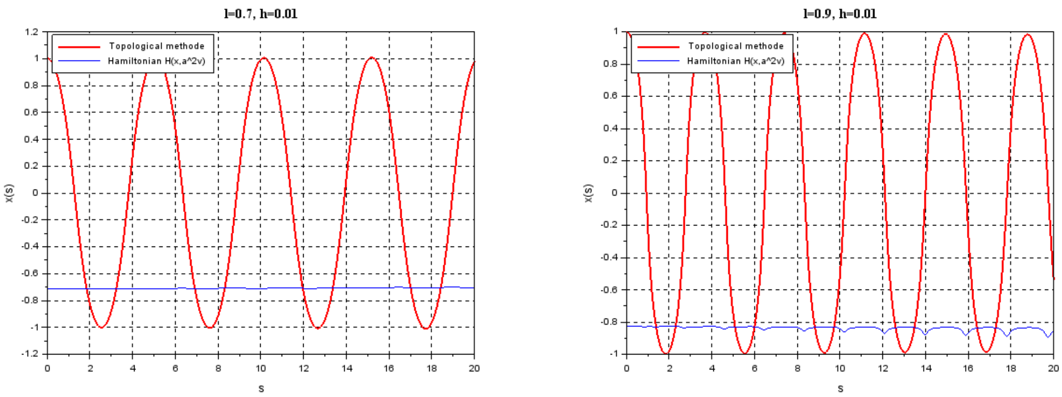


FIGURE 14. Numerical solution of the Eringen's nonlocal elastica for $l = 0.7$ and $l = 0.9$ and the corresponding evaluation of the first integral - Topological integrator - $h = 0.01$

We have a controled fluctuation around the exact value of the first integral $I_{1,l}$ which is a characteristic property of variational integrators due to their symplectic character.

7.6. **The Challamel's integrator.** In ([6], Equation (38) p.132), N. Challamel and al. introduce a discrete version of the Eringen's nonlocal elastica by rewritting first the second order equation as a two dimensional system of first order differential equations.

Definition 7.2 (Challamel's integrator)**.** *The Challamel's integrator is defined for* $i = 0, \ldots, n-1$*, by*

$$\begin{array}{rcl} x_{i+1} & = & x_i + h\kappa_i, \\ \kappa_{i+1} & = & \kappa_i - h\beta \sin(x_{i+1})(1 + l^2\kappa_i^2)a_{i+1}^{-1}. \end{array} \quad (70)$$

In the context of the study of Eringen's nonlocal elastica, they have to consider boundary conditions given by $\kappa_0 = 0$ and $\kappa_n = 0$.

Putting aside the boundary conditions, we look for the behavior of the previous integrator with respect to the first integral obtained for the continuous Eringen's nonlocal elastica.

7.7. **Simulations of the Challamel's integrator.** We implement the semi-implicit numerical scheme proposed by N. Challamel and al. in [6] called the Challamel's integrator in the following. We first take $h = 0.1$.

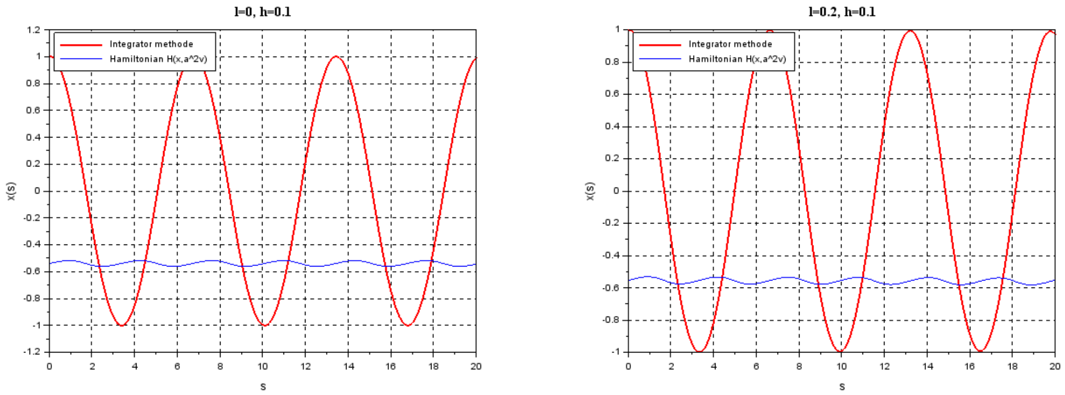


FIGURE 15. Numerical solution of the Eringen's nonlocal elastica for $l = 0$ and $l = 0.2$ and the corresponding evaluation of the first integral - Challamel's integrator - $h = 0.1$

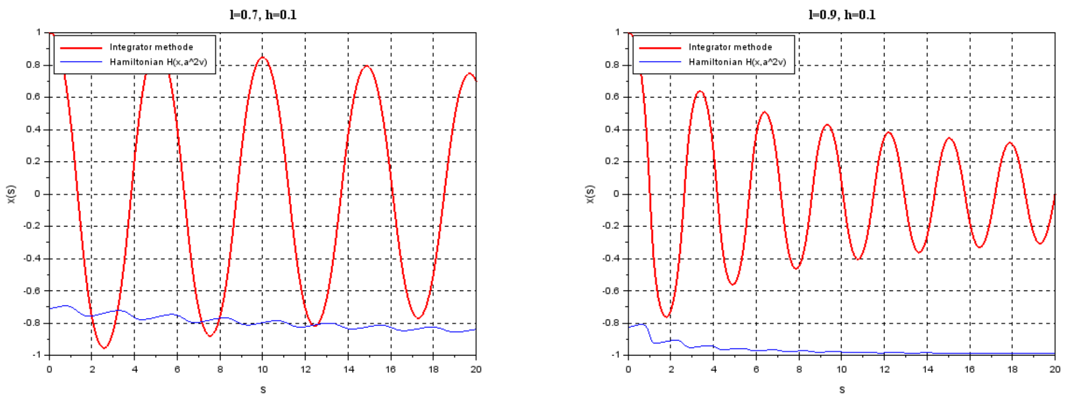


FIGURE 16. Numerical solution of the Eringen's nonlocal elastica for $l = 0.7$ and $l = 0.9$ and the corresponding evaluation of the first integral - Challamel's integrator - $h = 0.1$

As one can see, we have a bad behavior for the discrete model when l is greater than 0.5. For $h = 0.01$, we obtain the following results:

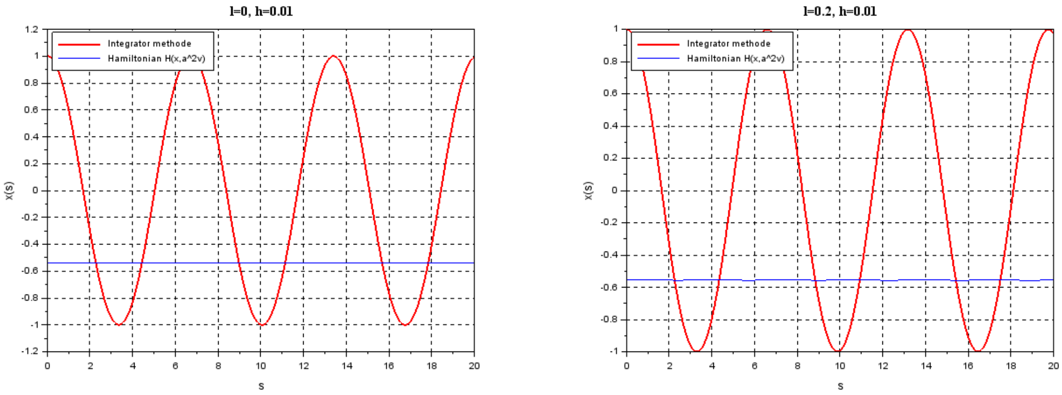


FIGURE 17. Numerical solution of the Eringen's nonlocal elastica for $l = 0$ and $l = 0.2$ and the corresponding evaluation of the first integral - Challamel's integrator - $h = 0.01$

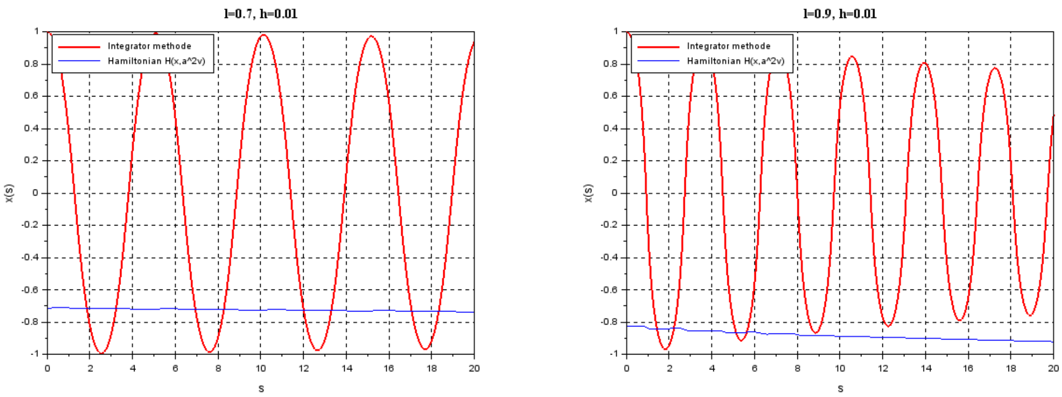


FIGURE 18. Numerical solution of the Eringen's nonlocal elastica for $l = 0.7$ and $l = 0.9$ and the corresponding evaluation of the first integral - Challamel's integrator - $h = 0.01$

Here again for large value of l the Challamel's integrator behaves badly with respect to the preservation of the first integral. The opportunity to consider this model as a good discrete analogue of the Eringen's nonlocal elastica is then questionable.

7.8. **Discrete Hamiltonian's Eringen's nonlocal elastica.** The Challamel's integrator looks for a two dimensional discrete equation associated to the original second order differential equation. Having in mind that the modified Eringen's nonlocal elastica is Lagrangian, a convenient procedure is to transform the classical Euler-Lagrange equation to its Hamiltonian form as done in Section 4.1. Using this structure, we can also derive a two dimensional discrete analogue of the modified Eringen's nonlocal elastica which preserve the Hamiltonian structure at the discrete level contrary to the Challamel's integrator. Again, we follow the discrete embedding formalism.

7.8.1. *Reminder about shifted and non shifted discrete Hamiltonian systems.* We remind some definitions and basic results about *discrete Hamiltonian systems* in the framework of the shifted or unshifted calculus of variations as exposed for example in [7]. We restrict ourself to uniform time scales, i.e. $\mathbb{T} = \{t_i\}_{i=0,\dots,n}$, $t_{i+1} - t_i = h$ for a given $h > 0$.

C. D. Ahlbrandt, M. Bohner, and J. Ridenhour in [1] introduced a notion of discrete Hamiltonian systems (on time scales) as follows:

Definition 7.3. *Let* $H : (t, q, p) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d \to H(t, q, p) \in \mathbb{R}$ *be a function of class* C^2 *in each of its variables. Let* $\mathbb{T} = \{t_i\}_{i=0,\dots,n}$ *be a time scale,* $t_{i+1} = t_i + h$, $i = 0, \dots, n-1$. *The*

Hamiltonian system associated to H on $\mathbb{T}_+$ is defined by

$$\begin{cases} \Delta_+ q = \partial_p H(t, q^\sigma, p), \\ \Delta_+ p = -\partial_q H(t, q^\sigma, p). \end{cases} \tag{71}$$

Using the shifted calculus of variations on time scales developed in [3], M. Bohner proved that the previous Hamiltonian systems on time scales can be obtained as critical points of shifted Lagrangian functionals on time scales. Precisely, we have:

Theorem 7.4. *The solutions of the Hamiltonian system* (71) *on $\mathbb{T}$ corresponds to critical points of the time scales functional*

$$\mathcal{L}^\sigma_{H,a,b,\mathbb{T}}(q,p) = \int_a^b [p\Delta_+ q - H(t, q^\sigma, p)]\,\Delta_+ t. \tag{$\mathcal{L}^\sigma_H$}$$

F. Pierret introduced in [14] a notion of Hamiltonian systems on time scales adapted to the framework of the non shifted calculus of variations. Precisely, we have:

Definition 7.5. *Let $H : (t, q, p) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d \to H(t, q, p) \in \mathbb{R}$ be a function of class C^2 in each of its variables. Let $\mathbb{T} = \{t_i\}_{i=0,\dots,n}$ be a time scale, $t_{i+1} = t_i + h$, $i = 0, \dots, n-1$ be a time scales . The Hamiltonian system associated to H on $\mathbb{T}$ is defined by*

$$\begin{cases} \Delta_+ q = \partial_p H(t, q, p), \\ \Delta_- p = -\Delta_-[\sigma]\partial_q H(t, q, p). \end{cases} \tag{72}$$

Here again, one can prove that Hamiltonian systems are critical point of Lagrangian functionals on time scales:

Theorem 7.6. *The solutions of the Hamiltonian systems* (72) *on $\mathbb{T}$ corresponds to critical points of the time scales functional*

$$\mathcal{L}_{H,a,b,\mathbb{T}}(q,p) = \int_a^b [p\Delta q - H(t, q, p)]\,\Delta t. \tag{$\mathcal{L}_H$}$$

We refer to the work of F. Pierret [14] for more details.

7.8.2. *Using non shifted discrete Hamiltonian systems for the Eringen's nonlocal elastica.* Let $p = a^2(x)\Delta_+[x]$ then a discrete Hamiltonian system associated to the modified Eringen's nonlocal elastica is given by:

$$\begin{array}{rcl} \Delta_-[p] & = & a\beta\sin(x)\left(l^2 a^{-4}p^2 - 1\right), \\ \Delta_+[x] & = & a^{-2}p. \end{array} \tag{73}$$

We then obtain

$$\begin{array}{rcl} p_i - p_{i-1} & = & ha_i\beta\sin(x_i)\left(l^2 a_i^{-4}p_i^2 - 1\right), \\ x_{i+1} - x_i & = & ha_i^{-2}p_i. \end{array} \tag{74}$$

7.8.3. *Using a shifted discrete Hamiltonian system for the Eringen's nonlocal elastica.* A different version of discrete Hamiltonian systems has been introduced by C.D. Ahlbrandt, M. Bohner and J. Ridendour in [1]. In that case, the so called shifted Hamiltonian system is given by

$$\begin{array}{rcl} \Delta_+[p] & = & a(\sigma(x))\beta\sin(\sigma(x))\left(l^2 a^{-4}(\sigma(x))p^2 - 1\right), \\ \Delta_+[x] & = & a^{-2}(\sigma(x))p, \end{array} \tag{75}$$

where σs is the shift operator on the time-scale $\mathbb{T}$ defined by $\sigma(t_i) = t_{i+1}$ and as a consequence $\sigma(x_i) = x_{i+1}$.

We then obtain

$$\begin{array}{rcl} p_{i+1} - p_i & = & ha_{i+1}\beta\sin(x_{i+1})\left(l^2 a_{i+1}^{-4}p_i^2 - 1\right), \\ x_{i+1} - x_i & = & ha_{i+1}^{-2}p_i. \end{array} \tag{76}$$

As we can see, the first equation gives an explicit formula for p_{i+1} as long as x_{i+1} is known, which relies on the resolution of the second equation which is implicit. This semi-implicit scheme is very close to the discrete Eringen's nonlocal elastica introduced by N. Challamel and al. in [6].

7.8.4. *Simulations of the shifted and nonshifted discrete Hamiltonian.* In the following, we provide simulations of the shifted and non shifted discrete Hamiltonian for the Eringen's nonlocal elastica on the same figure. As we can see the difference between the two integrators is very small up to $l = 0.7$ and become ony significant for $l = 0.9$.

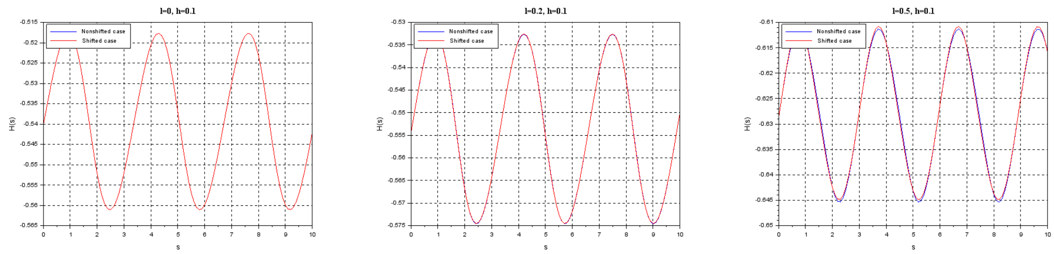


FIGURE 19. Simulations for $l = 0$, $l = 0.2$ and $l = 0.5$ - shifted and non-shifted Hamiltonian integrator - $h = 0.1$

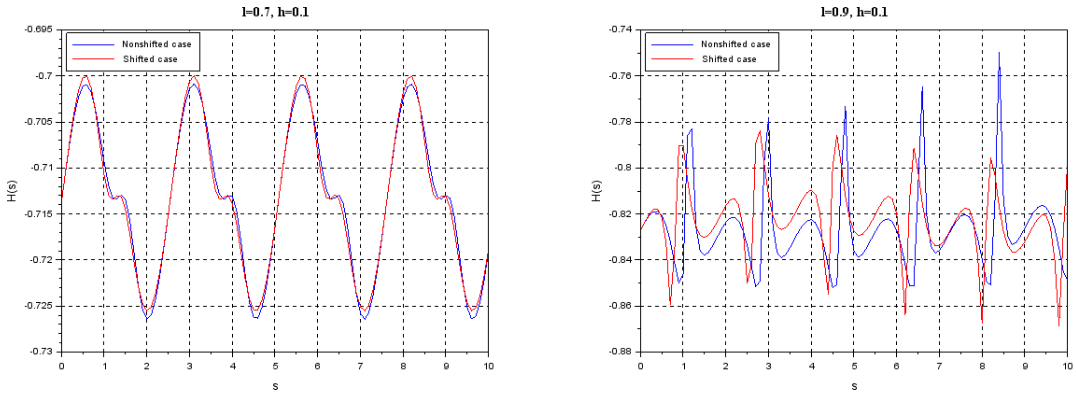


FIGURE 20. Simulations for $l = 0.7$ and $l = 0.9$ - shifted and non-shifted Hamiltonian integrator - $h = 0.1$

7.8.5. *Comparison with the Challamel's integrator.* We can compare the previous result with the one obtained using the Challamel's integrator by comparing for each integrator the behavior of the first integral $I(x) = H(x, a\dot{x}) = H(x, p)$. As we will see, the Challamel's integrator diverge when l increases.

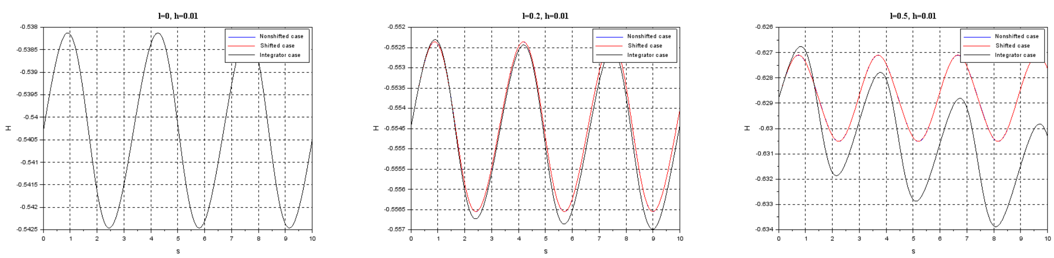


FIGURE 21. Simulations for $l = 0$, $l = 0.2$ and $l = 0.5$ -comparison of the value of the first integral for the shifted , non-shifted Hamiltonian and Challamel's integrator - $h = 0.01$

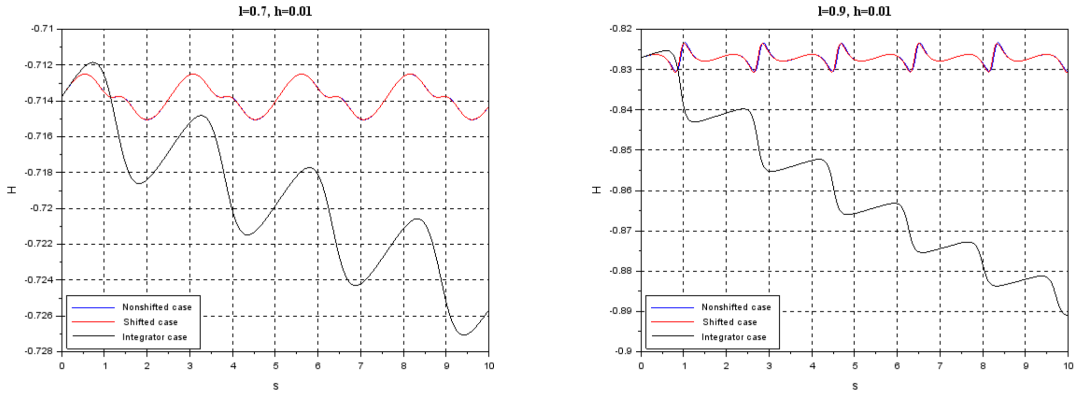


FIGURE 22. Simulations for $l = 0.7$ and $l = 0.9$ - comparison of the value of the first integral for the shifted, non-shifted Hamiltonian and Challamel's integrator - $h = 0.01$

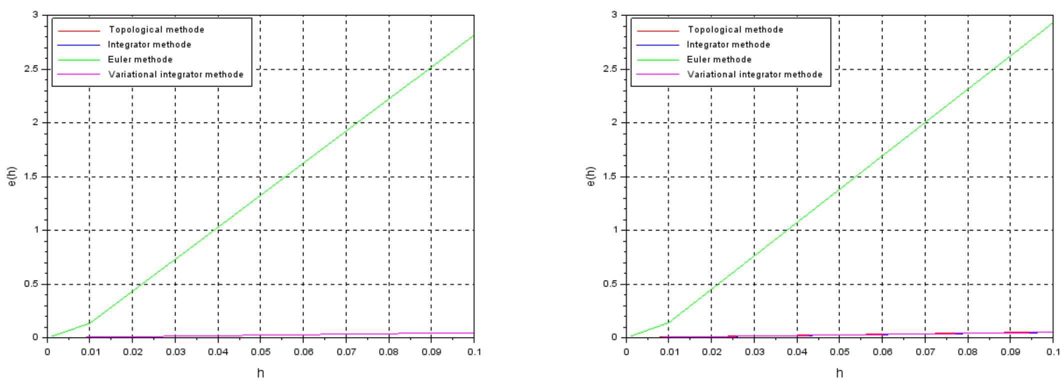
7.9. **Numerical study of the convergence of each scheme.** For each numerical scheme, we fix l and h, and we denote by $\star(t)$ the resulting discrete solution. We compute the error term as

$$e_\star(h) = \max_{t\in\mathbb{T}} \mid x(t) - \star(t) \mid, \tag{77}$$

where x is taken as a reference solution computed for $h = 10^{-5}$.

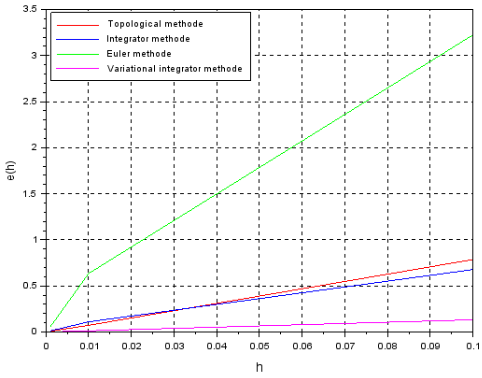
In the following, we provide a comparison for $l = 0$, $l = 0.2$, $l = 0.7$ and $l = 0.9$ between the Euler (in green), the topological (in red), the Challamel's integrator (in blue) and the variational integrator (in magenta) for value of $h = 10^{-3}$, $h = 10^{-2}$ and $h = 10^{-1}$.

For $l = 0$ and $l = 0.2$, we have:



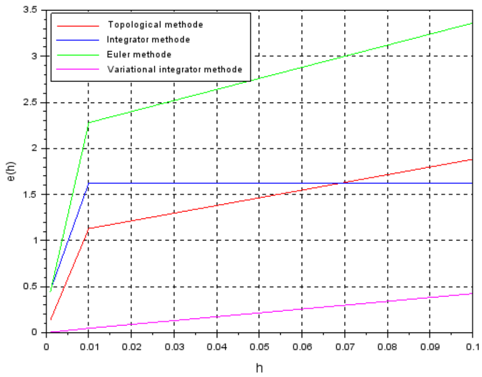
The Euler scheme is always the less good integrator but the topological and the Challamel's one behaves more or less equally and no significant difference is observable.

For $l = 0.7$ we obtain:



As we can see the topological integrator is better than the two other when h is below 0.1 and the variational integrator gives always a better result.

For greater values of l we have an increasing instability of the Euler and Challamel's integrator as can be seen in the following for $l = 0.9$:



As we can see, the variational integrator is very well adapted to the study of the Eringen's nonlocal elastica.

8. Conclusion and perspectives

The previous results only give partial answers to the problems raised in [4]. As already quoted in the Introduction, this article focus and the discrete and continuous Eringen's nonlocal elastica from the point of view of their algebro-geometric structures and how they are preserved from the continuous to the discrete case. However, in order to do applications in the mechanical context, we have to take into account the boundary conditions. This will be the subject of a forthcoming paper. The explicit expression of the solutions coming from the Lagrangian and Hamiltonian structure of the modified Eringen's nonlocal elastica will be of importance for this purpose. In that context, the asymptotic behavior of the solutions as well as the bifurcation diagram will be investigated.

9. Supplementary material

All the numerical scheme have been implemented using Scilab and can be downloaded from the web page of J. Cresson and K. Hariz Belgacem.

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