Uncertainty principle in two fluid–mechanics

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Examples of multi-component heterogeneous media
Carbonated drinks, moist or dusty air, porous solids, ...

Respiratory droplets

Porous solids

Two-fluid model derivation methods

- Averaging methods
- Landau approach : conservation laws + general principles of non-equilibrium thermodynamics (applications to superfluid helium and multi-component mixtures of gases
- Variational approach + irreversible thermodynamics (demands physical intuition but gives a right mathematical structure of the model) (Sedov, Berdichevsky, Bedford & Drumheller, Geurst, Lhuillier, SG & Gouin, ...)

Hamilton's principle

Definitions

- T – kinetic energy
- U – potential energy
- $L = T - U$ – Lagrangian
- $a = \int_{t_0}^{t_1} \int_{\mathcal{D}(t)} L dD dt$ - Hamilton's action

Hamilton's principle

The governing equations are stationary 'points' of Hamilton's action (under certain constraints to be defined).

Constraints

They should be integrable in the reference configuration!

- Conservation of the density
- Conservation of the entropy
- ...

One-velocity model

- $T = \rho \dfrac{|\mathbf{v}|^2}{2}$
- $U = \alpha_1 \rho_1 \varepsilon_1 (\rho_1, \eta_1) + \alpha_2 \rho_2 \varepsilon_2 (\rho_1, \eta_1) =$
  $= \bar{\rho}_1 \varepsilon_1 \left(\dfrac{\bar{\rho}_1}{\alpha_1}, \eta_1 \right) + \bar{\rho}_2 \varepsilon_2 \left(\dfrac{\bar{\rho}_2}{\alpha_2}, \eta_2 \right)$
- $\rho = \bar{\rho}_1 + \bar{\rho}_2,\ \bar{\rho}_i = \alpha_i \rho_i,\ \alpha_1 + \alpha_2 = 1,\ i = 1, 2.$

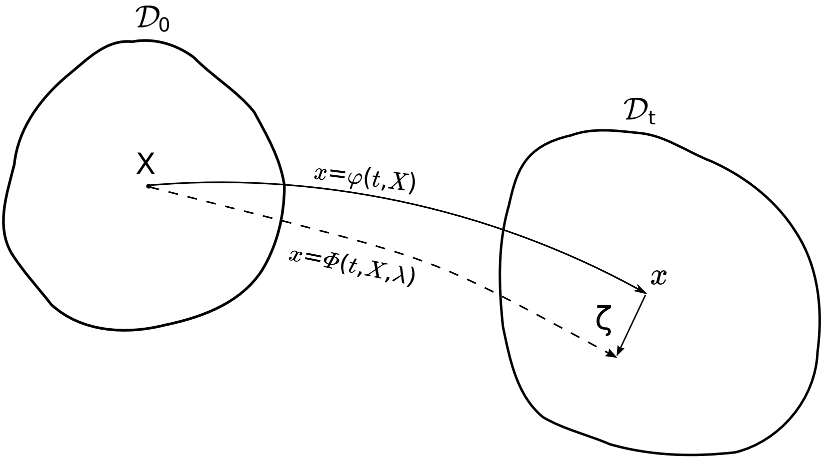
Constraints

:

$$\frac{\partial \bar{\rho}_i}{\partial t} + div\ (\bar{\rho}_i \mathbf{v}) = 0,$$

$$\frac{\partial (\bar{\rho}_i \eta_i)}{\partial t} + div\ (\bar{\rho}_i \eta_i \mathbf{v}) = 0,\ \ i = 1, 2.$$

Motion and virtual motion



One-velocity model

Berdichevski, V. L. (2009), SG (2011) (Eulerian variations)

$$\delta\bar{\rho}_i = -div(\bar{\rho}_i\,\zeta), \quad \zeta = \zeta(t,\mathbf{x}),$$

$$\delta\eta_i = -\nabla\eta_i\cdot\zeta,$$

$$\delta\mathbf{v} = \frac{D\zeta}{Dt} - \frac{\partial\mathbf{v}}{\partial\mathbf{x}}\zeta,$$

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla}.$$

One-velocity model

$$\delta a = \int_{t_0}^{t_1} \int_{\mathcal{D}} \mathbf{M} \cdot \zeta \, dD \, dt = 0.$$

It implies

$$\mathbf{M} = \rho \frac{D\mathbf{v}}{Dt} + \nabla p = \mathbf{0}, \quad p = \alpha_1 p_1 + \alpha_2 p_2.$$

Remark In the case of compressible phases the volume fraction equation should also be determined : the variation with respect to the volume fraction yields the pressure equality in the phases:

$$p_1 = p_2.$$

1. Mathematical consistency : the model is hyperbolic (the Cauchy problem is well posed at least in 1D case)
2. Physical consistency : the model admits the momentum and energy conservation laws (Noether theorem)

Two-velocity model

For simplicity, one supposes that one of the phases is incompressible (for example, $\rho_2 = \rho_{20} = const$) and barotropic (isentropic or isothermic motion).

Example of the internal energy function

:

$$U = \alpha_1 \, \rho_1 \, \varepsilon_1 \left(\rho_1\right) + \alpha_2 \, \rho_{20} \, \varepsilon_{int} \left(\alpha_2\right) =$$

$$= \bar{\rho}_1 \varepsilon_1 \left(\frac{\bar{\rho}_1}{1 - \dfrac{\bar{\rho}_2}{\rho_{20}}} \right) + \bar{\rho}_2 \varepsilon_{int} \left(\frac{\bar{\rho}_2}{\rho_{20}} \right) = U(\bar{\rho}_1, \bar{\rho}_2).$$

Reversible model

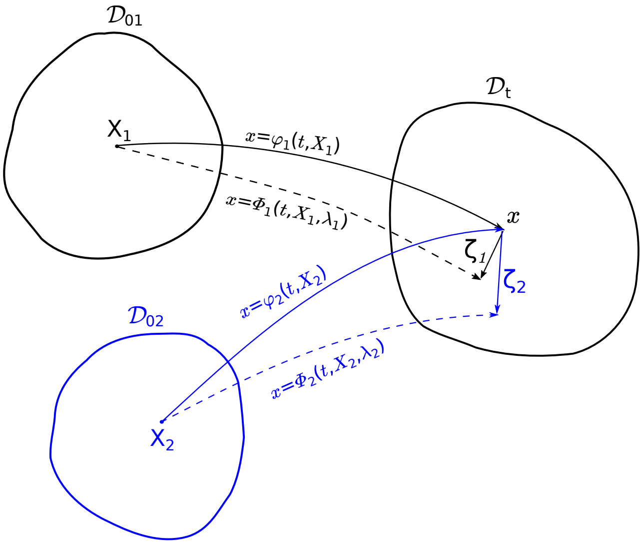
$$\delta a = 0, \quad a = \int_{t_1}^{t_2}\int_{\mathcal{D}} L \; d\mathbf{x}dt, \qquad L = \bar{\rho}_1 \frac{|\mathbf{v}_1|^2}{2} + \bar{\rho}_2 \frac{|\mathbf{v}_2|^2}{2} - U(\bar{\rho}_1, \bar{\rho}_2).$$

Constraints :

$$\frac{\partial \bar{\rho}_i}{\partial t} + div\ (\bar{\rho}_i \mathbf{v}_i) = 0, \quad i = 1, 2.$$

The mass equation for the incompressible phase '2' is the evolution equation for the volume fraction !

Motion and virtual motion



Reversible model

Eulerian variations :

$$\delta_i \bar{\rho}_i = -div(\bar{\rho}_i \boldsymbol{\zeta}_i),$$

$$\delta_i \mathbf{v}_i = \frac{D_i \boldsymbol{\zeta}_i}{Dt} - \frac{\partial \mathbf{v}_i}{\partial \mathbf{x}} \boldsymbol{\zeta}_i,$$

$$\frac{D_i}{Dt} = \frac{\partial}{\partial t} + \mathbf{v}_i \cdot \boldsymbol{\nabla}.$$

$$\delta_i a = \int_{t_0}^{t_1} \int_{\mathcal{D}} \mathbf{M}_i \cdot \boldsymbol{\zeta}_i \, dD \, dt = 0.$$

It implies

$$\mathbf{M}_i = \mathbf{0}.$$

Reversible model

$$\frac{\partial \bar\rho_i}{\partial t} + div\ (\bar\rho_i \mathbf{v}_i) = 0,\ i = 1, 2,$$

$$\frac{\partial(\bar\rho_1 \mathbf{v}_1)}{\partial t} + \mathrm{div}\left(\bar\rho_1 \mathbf{v}_1 \mathbf{v}_1^T + \left(\bar\rho_1 \frac{\partial U}{\partial \bar\rho_1} - U\right)\mathbf{I}\right) = -\frac{\partial U}{\partial \bar\rho_2}\boldsymbol{\nabla}\bar\rho_2,$$

$$\frac{\partial(\bar\rho_2 \mathbf{v}_2)}{\partial t} + \mathrm{div}\left(\bar\rho_2 \mathbf{v}_2 \mathbf{v}_2^T + \bar\rho_2 \frac{\partial U}{\partial \bar\rho_2}\mathbf{I}\right) = \frac{\partial U}{\partial \bar\rho_2}\boldsymbol{\nabla}\bar\rho_2.$$

Reversible model

If the energy $U(\bar{\rho}_1, \bar{\rho}_2)$ is convex, then the reversible model is hyperbolic for small relative velocity $\mathbf{w} = \mathbf{v}_2 - \mathbf{v}_1$ (SG & Gouin).

A trick

New variables :

$$\bar{\rho} = \bar{\rho}_1 + \bar{\rho}_2, \ \bar{\rho}\mathbf{v} = \bar{\rho}_1\mathbf{v}_1 + \bar{\rho}_2\mathbf{v}_2.$$

Then

$$\bar{\rho}_1\frac{|\mathbf{v}_1|^2}{2} + \bar{\rho}_2\frac{|\mathbf{v}_2|^2}{2} = \bar{\rho}\frac{|\mathbf{v}|^2}{2} + \frac{\bar{\rho}\bar{\rho}_2}{2(\bar{\rho}-\bar{\rho}_2)}|\mathbf{v}_2 - \mathbf{v}|^2,$$

$$U(\bar{\rho}_1, \bar{\rho}_2) = U(\bar{\rho} - \bar{\rho}_2, \bar{\rho}_2).$$

A trick

Constraints:

$$\bar{\rho}_t + \operatorname{div}(\bar{\rho}\mathbf{v}) = 0,\ (\bar{\rho}_2)_t + \operatorname{div}(\bar{\rho}_2\mathbf{v}_2) = 0.$$

Two-fluid models

Variations defined by $\mathbf{v}_1,\ \mathbf{v}_2$ and $\mathbf{v},\ \mathbf{v}_2$.

$$\frac{\partial\bar\rho_i}{\partial t} + div\ (\bar\rho_i\mathbf{v}_i) = 0,\ i = 1, 2,$$

$$\frac{\partial(\bar\rho_1\mathbf{v}_1)}{\partial t} + \mathrm{div}\left(\bar\rho_1\mathbf{v}_1\mathbf{v}_1^T + \left(\bar\rho_1\frac{\partial U}{\partial\bar\rho_1} - U\right)\mathbf{I}\right) + \mathbf{F} = -\frac{\partial U}{\partial\bar\rho_2}\boldsymbol{\nabla}\bar\rho_2,$$

$$\frac{\partial(\bar\rho_2\mathbf{v}_2)}{\partial t} + \mathrm{div}\left(\bar\rho_2\mathbf{v}_2\mathbf{v}_2^T + \bar\rho_2\frac{\partial U}{\partial\bar\rho_2}\mathbf{I}\right) - \mathbf{F} = \frac{\partial U}{\partial\bar\rho_2}\boldsymbol{\nabla}\bar\rho_2,$$

with

$$\mathbf{F} = \bar\rho_1\bar\rho_2\frac{\boldsymbol{\omega}_1}{\bar\rho}\wedge(\mathbf{v}_2 - \mathbf{v}_1),\ \boldsymbol{\omega}_1 = \mathrm{curl}\,(\mathbf{v}_1)\,.$$

Two-fluid models

Variations defined by $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}$, $\mathbf{v}_2$.

$$\frac{\partial \bar\rho_i}{\partial t} + div\ (\bar\rho_i \mathbf{v}_i) = 0,\ i = 1, 2,$$

$$\frac{\partial(\bar\rho_1 \mathbf{v}_1)}{\partial t} + \mathrm{div}\left(\bar\rho_1 \mathbf{v}_1 \mathbf{v}_1^T + \left(\bar\rho_1 \frac{\partial U}{\partial \bar\rho_1} - U\right)\mathbf{I}\right) + \mathbf{F} = -\frac{\partial U}{\partial \bar\rho_2} \boldsymbol{\nabla} \bar\rho_2,$$

$$\frac{\partial(\bar\rho_2 \mathbf{v}_2)}{\partial t} + \mathrm{div}\left(\bar\rho_2 \mathbf{v}_2 \mathbf{v}_2^T + \bar\rho_2 \frac{\partial U}{\partial \bar\rho_2}\mathbf{I}\right) - \mathbf{F} = \frac{\partial U}{\partial \bar\rho_2} \boldsymbol{\nabla} \bar\rho_2,$$

$$\frac{\partial \boldsymbol{\omega}_1}{\partial t} + \mathrm{div}\left(\mathbf{v}\boldsymbol{\omega}_1^T - \boldsymbol{\omega}_1 \mathbf{v}^T\right) = 0,$$

with

$$\mathbf{F} = \bar\rho_1 \bar\rho_2 \frac{\boldsymbol{\omega}_1}{\bar\rho} \wedge (\mathbf{v}_2 - \mathbf{v}_1),\ \boldsymbol{\omega}_1 = \mathrm{curl}\,(\mathbf{v}_1).$$

Non-conservative equation for the vorticity

$$\frac{\partial}{\partial t}\left(\frac{\omega_1}{\bar{\rho}}\right) + \frac{\partial}{\partial \mathbf{x}}\left(\frac{\omega_1}{\bar{\rho}}\right)\mathbf{v} - \frac{\partial \mathbf{v}}{\partial \mathbf{x}}\left(\frac{\omega_1}{\bar{\rho}}\right) = 0.$$

Conservation laws

The same conservation laws of momentum and energy!
Equations are “almost” identical!

Uncertainty

Equations are not uniquely defined : two types of governing equations can be obtained.

What is a right model ??? There is no mathematical solution to this question !!!

It looks like the modeling of the Euler equations : we should indicate *a priori* the number of contact surfaces before to solve the equations. And it is a personal choice!

Nature of the ‘extra-force’

Recall on gyroscopic forces in classical mechanics

$$\mathbf{A}\ddot{\mathbf{x}} + \mathbf{B}\mathbf{x} + \mathbf{C}\dot{\mathbf{x}} = 0, \quad \mathbf{A} = \mathbf{A}^{\mathbf{T}} > 0, \quad \mathbf{B} = \mathbf{B}^{\mathbf{T}}, \quad \mathbf{C} = -\mathbf{C}^{\mathbf{T}}.$$

Energy equation

$$H = \frac{\dot{\mathbf{x}}^T \mathbf{A}\dot{\mathbf{x}}}{2} + \frac{\mathbf{x}^T \mathbf{B}\mathbf{x}}{2} = const.$$

Variational principle

$$L = \int_{t_0}^{t_1} \left(\frac{\dot{\mathbf{x}}^T \mathbf{A}\dot{\mathbf{x}}}{2} - \frac{\mathbf{x}^T \mathbf{B}\mathbf{x}}{2} + \frac{\dot{\mathbf{x}}^T \mathbf{C}\mathbf{x}}{2} \right) dt.$$

Consequences

- Hyperbolicity condition is the same for both systems (convexity of $U(\bar{\rho}_1, \bar{\rho}_2)$).
- Both systems admit the same total momentum and total energy (**F** is a gyroscopic force).
- But the energy distribution for each component is completely different.

Exact solutions

$$\mathbf{v}_1 = (u_1(z), v_1(z), 0)^T, \ \mathbf{v}_2 = (u_2(z), v_2(z), 0)^T, \ \bar{\rho}_1 = \bar{\rho}_1(z), \ \bar{\rho}_2 = \bar{\rho}_2(z).$$

$$\boldsymbol{\omega}_1 \wedge (\mathbf{v}_2 - \mathbf{v}_1) = H(z)\mathbf{k},$$

with

$$H(z) = -\frac{dv_1}{dz}(v_2 - v_1) - \frac{du_1}{dz}(u_2 - u_1).$$

The momentum equation implies :

$$\bar{\rho}_1 \frac{\partial U}{\partial \bar{\rho}_1} + \bar{\rho}_2 \frac{\partial U}{\partial \bar{\rho}_2} - U = const. \tag{1}$$

It is complemented by the differential equation :

$$\frac{d}{dz}\left(\frac{\partial U}{\partial \bar{\rho}_1}\right) + \frac{\bar{\rho}_2}{\bar{\rho}} H(z) = 0. \tag{2}$$

They can be integrated.

Exact solutions

Let us consider the Legendre transform $U^\star(R_1, R_2)$ of U. Here $R_i = \dfrac{\partial U}{\partial \bar{\rho}_i}$, $i = 1, 2$. In particular, (1) implies :

$$U^\star(R_1, R_2) = const, \tag{3}$$

$$R_1 - R_2 + \int^z H(s)ds = const. \tag{4}$$

The relations give a very simple algorithm to find the solution R_1 and R_2 : it is sufficient then to find the intersection of the closed convex curve defined by (3) with the strait line defined by (4) at any z. Once R_i are found as functions of z, the densities $\bar{\rho}_i$ can be found. When the "lift" force is absent, such a solution is constant.

Exact solutions

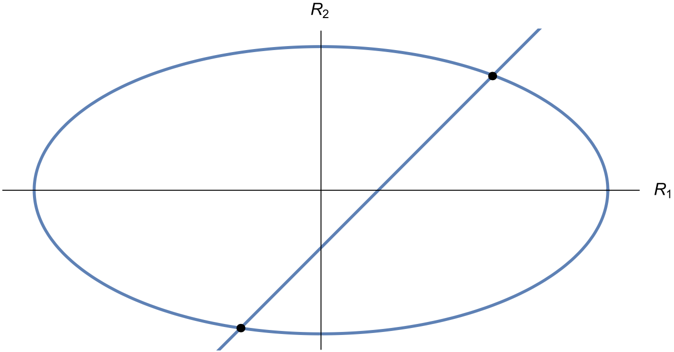


Figure: For a given velocity field, the variables R_1 and R_2 that are conjugate to the densities $\bar{\rho}_1$ and $\bar{\rho}_2$ can be determined as the intersection of the level set of the Legendre transform $U^\star(R_1, R_2)$ of the volume energy $U(\bar{\rho}_1, \bar{\rho}_2)$, and a strait line.

External forces and friction

One can add external conservative and friction forces :

$$\frac{\partial \bar{\rho}_1}{\partial t} + div\ (\bar{\rho}_1 \mathbf{v}_1) = 0, \tag{5}$$

$$\frac{\partial \bar{\rho}_2}{\partial t} + div\ (\bar{\rho}_2 \mathbf{v}_2) = 0,$$

$$\frac{\partial (\bar{\rho}_1 \mathbf{v}_1)}{\partial t} + \mathrm{div}\left(\bar{\rho}_1 \mathbf{v}_1 \mathbf{v}_1^T\right) + \boldsymbol{\nabla}\left(\bar{\rho}_1 \frac{\partial U}{\partial \bar{\rho}_1} - U\right) + \mathbf{F} = -\frac{\partial U}{\partial \bar{\rho}_2} \boldsymbol{\nabla} \bar{\rho}_2 + \mu(\mathbf{v}_2 - \mathbf{v}_1),$$

$$\frac{\partial (\bar{\rho}_2 \mathbf{v}_2)}{\partial t} + \mathrm{div}\left(\bar{\rho}_2 \mathbf{v}_2 \mathbf{v}_2^T\right) + \boldsymbol{\nabla}\left(\bar{\rho}_2 \frac{\partial U}{\partial \bar{\rho}_2}\right) - \mathbf{F} = \frac{\partial U}{\partial \bar{\rho}_2} \boldsymbol{\nabla} \bar{\rho}_2 - \mu(\mathbf{v}_2 - \mathbf{v}_1),$$

$$\frac{\partial \boldsymbol{\omega}_1}{\partial t} + \mathrm{div}\left(\mathbf{v} \boldsymbol{\omega}_1^T - \boldsymbol{\omega}_1 \mathbf{v}^T\right) = \mathrm{curl}\left(\frac{\mu}{\bar{\rho}_1}(\mathbf{v}_2 - \mathbf{v}_1)\right).$$

Here μ is a positive parameter. The friction force is responsible for the vorticity creation due to the relative motion of components.

Hyperbolicity and stability

- Equations are hyperbolic for small relative velocities.
- The gyroscopic forces can stabilize the governing equations and prevent the solution to leave the hyperbolicity region (must be studied....)

Conclusion

I know that I know nothing.
Socrate.