Quantized numbers: examples

Golden Ratio

The "Golden Ratio"

$$\varphi = \frac{1+\sqrt{5}}{2} = [1, 1, 1, 1, 1, \ldots] \, .$$

is the simplest irrational number. The equation : $\varphi^2 = \varphi + 1$.

The convergents of this continued fraction :

$$\varphi_n = \underbrace{[1, 1, \ldots, 1]}_{n} = \frac{F_{n+1}}{F_n},$$

where F_n is the n^{th} Fibonacci number.

The q-deformations :

$$
\begin{aligned}
[\varphi_6]_q &= \frac{1+2q+3q^2+3q^3+3q^4+q^5}{1+2q+2q^2+2q^3+q^4}, \\
[\varphi_8]_q &= \frac{1+3q+5q^2+7q^3+7q^4+6q^5+4q^6+q^7}{1+3q+4q^2+5q^3+4q^4+3q^5+q^6}, \\
[\varphi_9]_q &= \frac{1+4q+7q^2+10q^3+11q^4+10q^5+7q^6+4q^7+q^8}{1+4q+6q^2+7q^3+7q^4+5q^5+3q^6+q^7}.
\end{aligned}
$$

The coefficients : A123245 of OEIS and its mirror A079487.

The stabilization phenomenon

The Taylor series of the convergents :

$$
\begin{aligned}
[\varphi_6]_q &= 1 + q^2 - q^3 + 2q^4 - 3q^5 + 3q^6 - 3q^7 + 4q^8 \\
&\quad -5q^9 + 5q^{10} - 5q^{11} + 6q^{12} \cdots
\end{aligned}
$$

$$
\begin{aligned}
[\varphi_8]_q &= 1 + q^2 - q^3 + 2q^4 - 4q^5 + 8q^6 - 16q^7 + 30q^8 \\
&\quad -55q^9 + 103q^{10} - 195q^{11} + 368q^{12} \cdots
\end{aligned}
$$

$$
\begin{aligned}
[\varphi_9]_q &= 1 + q^2 - q^3 + 2q^4 - 4q^5 + 8q^6 - 17q^7 + 37q^8 \\
&\quad -82q^9 + 184q^{10} - 414q^{11} + 932q^{12} \cdots
\end{aligned}
$$

The coefficients stabilize !

q-deformed Golden Ratio

The q-deformation $[\varphi]_q$ is given by the series

$$\begin{aligned}[\varphi]_q &= 1 + q^2 - q^3 + 2q^4 - 4q^5 + 8q^6 - 17q^7 + 37q^8 \\ &\quad -82q^9 + 185q^{10} - 423q^{11} + 978q^{12} - 2283q^{13} \\ &\quad +5373q^{14} - 12735q^{15} + 30372q^{16} - 72832q^{17} \\ &\quad +175502q^{18} - 424748q^{19} + 1032004q^{20} \cdots\end{aligned}$$

The coefficients coincide with the sequence A004148 of OEIS called the *generalized Catalan numbers*.

The series $[\varphi]_q$ satisfies the equation

$$q\,[\varphi]_q^2 = \left(q^2 + q - 1\right)[\varphi]_q + 1.$$

This is the q-analogue of $\varphi^2 = \varphi + 1$.

Generating function :

$$[\varphi]_q = \frac{q^2 + q - 1 + \sqrt{q^4 + 2q^3 - q^2 + 2q + 1}}{2q}$$

$$= \frac{q^2 + q - 1 + \sqrt{(q^2 + 3q + 1)(q^2 - q + 1)}}{2q}.$$

The continued fraction

$[\varphi]_q$ can be written as infinite continued fraction :

$$[\varphi]_q = 1 + \cfrac{q^2}{q + \cfrac{1}{1 + \cfrac{q^2}{q + \cfrac{1}{\ddots}}}} = 1 + \cfrac{1}{q^{-1} + \cfrac{1}{q^2 + \cfrac{1}{q^{-3} + \cfrac{1}{\ddots}}}}$$

NB : The celebrated Rogers-Ramanujan continued fraction

$$R(q) = 1 + \cfrac{1}{1 + \cfrac{q}{1 + \cfrac{q^2}{1 + \cfrac{q^3}{\ddots}}}}$$

The "quantum π"

$$
\begin{aligned}
[\pi]_q = \; & 1 + q + q^2 + q^{10} - q^{12} - q^{13} + q^{15} + q^{16} \\
& - q^{20} - 2q^{21} - q^{22} + 2q^{23} + 4q^{24} + q^{25} \\
& - 4q^{27} - 4q^{28} - 2q^{29} + q^{30} + 5q^{31} + 8q^{32} + 3q^{33} \\
& - 3q^{34} - 10q^{35} - 12q^{36} - 5q^{37} + 8q^{38} + 19q^{39} + 20q^{40} \\
& + 2q^{41} - 18q^{42} - 32q^{43} - 25q^{44} + 31q^{46} + 51q^{47} \\
& + 45q^{48} - 7q^{49} - 65q^{50} - 94q^{51} - 57q^{52} + 35q^{53} \cdots
\end{aligned}
$$

The "quantum e"

Euler's number $e = [2, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, \ldots]$

$$
\begin{aligned}
[e]_q \quad = \quad & 1 + q + q^3 - q^5 + 2q^6 - 3q^7 + 3q^8 - q^9 \\
& -3q^{10} + 9q^{11} - 17q^{12} + 25q^{13} - 29q^{14} + 23q^{15} + 2q^{16} \\
& -54q^{17} + 134q^{18} - 232q^{19} \\
& +320q^{20} - 347q^{21} + 243q^{22} + 71q^{23} \\
& -660q^{24} + 1531q^{25} - 2575q^{26} \\
& +3504q^{27} - 3804q^{28} + 2747q^{29} + 488q^{30} \cdots
\end{aligned}
$$

Observations :

- the coefficients of q^{2+7k} are smaller !
- the signs $+, +$ appear with periodicity 7 !

More examples

The "silver ratio" $\sqrt{2} = [1, 2, 2, 2, \ldots]$:

$$\left[\sqrt{2}\right]_q = \frac{q^3 - 1 + \sqrt{(q^4 + q^3 + 4q^2 + q + 1)(q^2 - q + 1)}}{2q^2}$$

satisfies $q^2 \left[\sqrt{2}\right]_q^2 - \left(q^3 - 1\right) \left[\sqrt{2}\right]_q = q^2 + 1.$

Square root of 5 :

$$\left[\sqrt{5}\right]_q = \frac{q^5 + q^3 - q^2 - 1 + \sqrt{\begin{array}{l}(q^8 + q^7 + 2q^6 + 3q^5 + 6q^4 \\ +3q^3 + 2q^2 + q + 1)(q^2 - q + 1)\end{array}}}{2q^3},$$

Compare with the golden ratio !

Radius of convergence

(Work in progress, Leclere-MG-Ovsienko-Veselov)

$$R_\varphi = \frac{3-\sqrt{5}}{2},$$

$$R_{\sqrt{2}} = \frac{1+\sqrt{2}-\sqrt{2\sqrt{2}-1}}{2},$$

$$R_{a_3} = \frac{1+\sqrt{13}-\sqrt{2\left(\sqrt{13}-1\right)}}{4},$$

where $a_3 = \frac{9+\sqrt{221}}{14}$ the "bronze ratio".

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