

Differential invariants, recurrence formulas and homogeneous models

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Statement of the problem

General Problem: Cartan's equivalence problem

Given $M, N \subset \mathbb{R}^n$ two smooth submanifolds, G a local Lie group. Fix $p \in M$, $q \in N$ two points. Is there a diffeomorphism $\varphi \in G$ between two neighborhoods $U_p, V_q \subset \mathbb{R}^n$ of p, q such that $\varphi(p) = q$ and $\varphi(U_p \cap M) = V_q \cap N$?

Specific Problem: biholomorphic equivalence problem

Given $M, N \subset \mathbb{C}^n$ two **real analytic hypersurfaces**, G a local Lie group of **biholomorphisms**. Fix $p \in M$, $q \in N$ two points. Is there a biholomorphism $\varphi \in G$ between two neighborhoods $U_p, V_q \subset \mathbb{C}^n$ of p, q such that $\varphi(p) = q$ and $\varphi(U_p \cap M) = V_q \cap N$?

Background: Poincaré's observation

(Poincaré 1907)[3] Existence of biholomorphic invariants

Consider an arbitrary real analytic hypersurface $M^3 \subset \mathbb{C}^2$ with coordinates $(z, w) = (x + iy, u + iv)$ graphed by $u = \Phi(x, y, v)$, there are

$$N' = \frac{(n+1)(n+2)(n+3)}{6} - 1 \sim \frac{n^3}{6}$$

many Taylor coefficients up to order n . Consider a biholomorphism $(z', w') = (f(z, w), g(z, w))$ where f and g are holomorphic. Their expansion up to order n has

$$2\left(\frac{(n+1)(n+2)}{2} - 1\right) \sim n^2$$

arbitrary complex coefficients, so $\sim 2n^2$ real coefficients. Thus there are more hypersurfaces $\sim \frac{n^3}{6}$ than biholomorphisms $\sim 2n^2$.

So there are infinitely many inequivalent classes. In conclusion, there are **infinitely many** ($\sim \frac{n^3}{6}$ up to order n) biholomorphic differential invariants of $M^3 \subset \mathbb{C}^2$. Same argument holds for $M^5 \subset \mathbb{C}^3$.

Rigid equivalence problem. Definitions in \mathbb{C}^2 .

(Definition) Let \mathbb{C}^2 be coordinated by $(z, w) = (x + iy, u + iv)$. An analytic hypersurface $M^3 \subset \mathbb{C}_{z,w}^2$ is called **rigid** if it is a graph

$$u = \Phi(x, y) = F(z, \bar{z}),$$

independent of v .

(Definition) A local biholomorphism $(z', w') = (f(z, w), g(z, w))$ is called **rigid** if

$$f(z, w) = f(z), \quad g(z, w) = rw + h(z),$$

for some $r \in \mathbb{R}_{>0}$ and h holomorphic. We write G^{RT} for the group of rigid transformations.

(Proposition) The following properties of analytic hypersurfaces $M^3 \subset \mathbb{C}^2$ are invariant under rigid biholomorphisms:

- being rigid;
- being Levi-nondegenerate, i.e. $F_{z,\bar{z}} \neq 0$. If $F_{z,\bar{z}} \equiv 0$ then M^3 is flat;
- having $F_{z^2,\bar{z}^2} \neq 0$. If $F_{z^2,\bar{z}^2} \equiv 0$ then M^3 is equivalent to $u = z\bar{z}$.

(Problem) Determining the equivalence classes of rigid hypersurfaces with $F_{z,\bar{z}} \neq 0$ and $F_{z^2,\bar{z}^2} \neq 0$ under rigid biholomorphisms.

Equivalence classes and differential invariants

Let $M^3 \subset \mathbb{C}^2$ be a graph $\{u = F(z, \bar{z})\}$ passing the origin, F real-analytic. Locally M is parameterised by two independent coordinates z, \bar{z} .

Definition: Jet space

The jet space $J_{2,1}^n$ is a subspace of a complex vector space spanned by coordinates $z, \bar{z}, u, u_z, u_{\bar{z}}, u_{zz}, u_{z\bar{z}}, \dots, u_{\bar{z}\dots\bar{z}}$ (up to n^{th} derivatives) satisfying $u_{z^j\bar{z}^k} = \overline{u_{z^k\bar{z}^j}}$. It has real dimension $\sim n^2$.

The graph M is uniquely lifted to a surface $M^{(n)}$ on $J_{2,1}^n$, which is a 2-dim graph determined by equations $u = F(z, \bar{z})$, $u_z = F_z(z, \bar{z})$, $u_{\bar{z}} = F_{\bar{z}}(z, \bar{z})$, $u_{zz} = F_{zz}(z, \bar{z})$, etc. The conjugacy condition is satisfied since $\bar{F} = F$.

Equivalence classes and differential invariants

Definition: Lifted action on jet spaces

Any rigid transformation (close to the identity) $g \in G^{RT}$ defines a point-wise action on the set of graphed surfaces $M^3 \subset \mathbb{C}^2$, hence deduces a lifted action $g^{(n)}$ on the jet space $J_{2,1}^n$.

$$\begin{array}{ccc} M^{(n)} & \xrightarrow{g^{(n)}} & (g(M))^{(n)} \\ \downarrow & & \downarrow \\ M & \xrightarrow{g} & g(M), \end{array}$$

If g sends $\{u = F(z, \bar{z})\}$ to $\{u' = G(z', \bar{z}')\}$ pointwisely, then $g^{(n)}$ sends $u_z, u_{\bar{z}}, \dots$ to $u'_{z'}, u'_{\bar{z}'}, \dots$ analytically (with some complicated expressions).

Equivalence classes and differential invariants

Definition: Differential invariants

A **differential invariant of order n** is a analytic function $I(z, \bar{z}, u, u_z, \dots)$ on some domain of $J_{2,1}^n$ which is invariant under lifted actions, i.e. $g^{(n)*}I = I$ for all $g \in G^{RT}$ close to id .

- A necessary condition of equivalence: $I_M(p) = I_N(q)$.
- Orbits of $g^{(n)}$ actions \Leftrightarrow level sets of invariants $\{I = c\}$
- Equivalence classes are parameterised by values of invariants I .

Definition: Relative invariants

We say I is a **relative invariant** if we only have $g^{(n)*}I = h(g) \cdot I$ for some $h(g) \neq 0$. In other words $I = 0$ is an invariant property.

Example: $F_{z,\bar{z}}$ and F_{z^2,\bar{z}^2} are relative invariants.

A relative invariant R often brings "branches" in classification: $R \equiv 0$ and $R \neq 0$ in a small domain (Lie's technique). We avoid the mixed case when treating the problem on a generic point.

Rigid equivalence problem. Results in \mathbb{C}^2

(Chen-Foo-Merker-Ta 2019) Differential invariants in \mathbb{C}^2

Rigid hypersurfaces $M^3 \subset \mathbb{C}^2$ graphed by $u = F(z, \bar{z})$ with $F_{z,\bar{z}} \neq 0 \neq F_{z^2,\bar{z}^2}$ has a rigid invariant of order 5

$$Q = \frac{|F_{z^3,\bar{z}^2} F_{z,\bar{z}}^2 - F_{z,\bar{z}} F_{z,\bar{z}^2} F_{z^3,\bar{z}} - 3F_{z,\bar{z}} F_{z^2,\bar{z}} F_{z^2,\bar{z}^2} + 3F_{z,\bar{z}^2} F_{z^2,\bar{z}}^2|}{(F_{z,\bar{z}} F_{z^2,\bar{z}^2} - F_{z,\bar{z}^2} F_{z^2,\bar{z}})^{3/2}}.$$

When $Q \equiv 0$ the hypersurface is equivalent as $u = -2 \log(1 - \frac{1}{2} z \bar{z})$, not homogeneous. When $Q \neq 0$, the normal form is locally rigid-equivalent to a **unique normal form**

$$u = z \bar{z} + \frac{1}{4} z^2 \bar{z}^2 + \frac{Q}{12} z^2 \bar{z}^3 + \frac{Q}{12} z^3 \bar{z}^2 + \sum_{j,k \geq 2, j+k \geq 6} \frac{l_{j,k}}{j!k!} z^j \bar{z}^k$$

where all coefficients $Q \in \mathbb{R}$, $l_{j,k} \in \mathbb{C}$ are rigid differential invariants. Two rigid hypersurfaces are rigid-equivalent iff they have the same normal form.

Rigid equivalence problem. Results in \mathbb{C}^2

(Chen-Foo-Merker-Ta 2019) [1] Differential invariants in \mathbb{C}^2

Moreover, when $Q \neq 0$

- 1 the invariants Q and $I_{j,k}$ can be explicitly expressed as analytic functions of $F_{j,k}$;
- 2 all invariants $I_{j,k}$ can be generated by Q , $I_{3,3}$ and their differentials.
- 3 Rigid homogeneous models exist for any $Q \in \mathbb{R}^*$. The surface graphed by

$$u = e^{\frac{-z-\bar{z}}{2}} \sin\left(\frac{-iz + i\bar{z}}{2}\right)^{\frac{1}{l}}$$

where $l = \frac{1}{2} \pm \sqrt{\frac{1}{2Q^2} + \frac{1}{4}}$, is homogeneous having constant invariant $Q = \frac{1}{\sqrt{(2l-1)l}}$. The three tangent vectors are ∂_v ,

$$L_1 = \frac{\partial}{\partial z} - w \frac{\partial}{\partial w} + (\text{conjugates}) = \frac{\partial}{\partial x} - u \frac{\partial}{\partial u} \quad \text{and}$$

$$L_2 = e^{lz+i\theta} \frac{\partial}{\partial z} + (\text{conjugates}) = e^{lx} \cos(l y + \theta) \frac{\partial}{\partial x} + e^{lx} \sin(l y + \theta) \frac{\partial}{\partial y}.$$



Prenormalization and Power-series invariants

(Step 1) Start from an arbitrary rigid hypersurface $M^3 \subset \mathbb{C}^2$ passing the origin

$$u = \frac{w + \bar{w}}{2} = F(z, \bar{z}) = F_{1,0}z + F_{0,1}\bar{z} + \frac{F_{2,0}}{2}z^2 + F_{1,1}z\bar{z} + \frac{F_{0,2}}{2}\bar{z}^2 + O(3),$$

after a transformation $w' = w + h(z)$

$$u' = F(z, \bar{z}) - \frac{h(z) + \overline{h(z)}}{2}$$

we may assume (we normalize) all $F_{j,0} = F_{0,j} = 0$.

(Step 2) Now from

$$\begin{aligned} u &= F_{1,1}z\bar{z} + \frac{F_{2,1}}{2}z^2\bar{z} + \frac{F_{1,2}}{2}z\bar{z}^2 + O(4) \\ &= F_{1,1} \underbrace{\left(z + \frac{F_{2,1}}{2F_{1,1}}z^2 + O_z(3) \right)}_{z'} \underbrace{\left(\bar{z} + \frac{F_{1,2}}{2F_{1,1}}\bar{z}^2 + O_{\bar{z}}(3) \right)}_{\bar{z}'} + \frac{F_{2,2}}{4}z^2\bar{z}^2 + O(5), \end{aligned}$$

after a transformation $z' = z + \frac{F_{2,1}}{2F_{1,1}}z^2 + O_z(3)$ we normalize all $F_{j,1} = F_{1,j} = 0$ except $F_{1,1}$.

Normalization and Power-series invariants

(Step 3) The stabilizer of

$$u = F_{1,1}z\bar{z} + \frac{F_{2,2}}{4}z^2\bar{z}^2 + \frac{F_{3,2}}{12}z^3\bar{z}^2 + \frac{F_{2,3}}{12}z^2\bar{z}^3 + O(6),$$

is now finite (3) dimensional: $(z', w') = (re^{i\theta}z, \lambda w)$ with $re^{i\theta} \in \mathbb{C}^*$ and $b > 0$.

Using the 2 dimensional dilations $(z', w') = (rz, \lambda w)$ we normalize $F_{1,1} = F_{2,2} = 1$.

When $F_{3,2} \neq 0$, using the rotation $z' = e^{i\theta}z$ we normalize $F_{3,2}$ as a positive real number. Thus we get the unique normal form

$$u = z\bar{z} + \frac{1}{4}z^2\bar{z}^2 + \frac{Q}{12}z^2\bar{z}^3 + \frac{Q}{12}z^3\bar{z}^2 + \sum_{j,k \geq 2, j+k \geq 6} \frac{l_{j,k}}{j!k!} z^j \bar{z}^k$$

where all Taylor coefficients are uniquely determined by its rigid-equivalence class, hence are invariants.

Normalization and Power-series invariants

The explicit expression

$$Q = \frac{|F_{3,2}F_{1,1}^2 - F_{1,1}F_{1,2}F_{3,1} - 3F_{1,1}F_{2,1}F_{2,2} + 3F_{1,2}F_{2,1}^2|}{(F_{1,1}F_{2,2} - F_{1,2}F_{2,1})^{3/2}}$$

can be obtained by tracing the relation among Taylor coefficients before and after each step of normalization.

When $Q \equiv 0$, the PDE

$$0 \equiv F_{z^3, \bar{z}^2} F_{z, \bar{z}}^2 - F_{z, \bar{z}} F_{z, \bar{z}^2} F_{z^3, \bar{z}} - 3F_{z, \bar{z}} F_{z^2, \bar{z}} F_{z^2, \bar{z}^2} + 3F_{z, \bar{z}^2} F_{z^2, \bar{z}}^2$$

implies that $F_{j,k} = 0$ for all $j \neq k$. Thus $u = F(z, \bar{z}) = H(z\bar{z})$ is a function of the product $z\bar{z}$. Such surfaces turn out to be equivalent to $u = -2 \log(1 - \frac{1}{2}z\bar{z})$, not homogeneous.

From now on we assume $Q \neq 0$.

Olver's recurrence formulas: Invariant differential operators

(Definition) An **invariant differential operator** is a linear combination of the total differential operators

$$\mathcal{D} = A D_z + B D_{\bar{z}}$$

where A, B are analytic functions on certain jet space $J_{2,1}^N$, which (\mathcal{D}) sends an invariant of order n to an invariant of order $n + 1$.

There are two invariant differential operators \mathcal{D}_z and $\mathcal{D}_{\bar{z}}$, depending on the choice of the normal form. They generate relations among invariants of different orders, via Olver's recurrence formulas.

Olver's recurrence formulas: Lifted vector fields and invariantizations

Take generators of the Lie algebra of rigid transformations

$$v_1, v_2, \dots = \partial_w + \partial_{\bar{w}}, z\partial_w + \bar{z}\partial_{\bar{w}}, z^2\partial_w + \bar{z}^2\partial_{\bar{w}}, \dots, \\ \partial_z, \partial_{\bar{z}}, z\partial_z, \bar{z}\partial_{\bar{z}}, z^2\partial_z, \bar{z}^2\partial_{\bar{z}} \dots$$

They can be lifted as vector fields on the jet space $J_{2,1}^n$

$$\text{pr}v_\kappa = v_\kappa + \sum_{j,k} v_{j,k,\kappa} \partial_{u_{j,k}}$$

Write $\text{inv}(v_{j,k,\kappa}) \in \mathbb{C}$ as their **invariantizations**, i.e. evaluate them on our normal section

$$z = 0, \bar{z} = 0, u_{1,1} = 1 = u_{2,2}, u_{j,0} = u_{0,j} = u_{j+2,1} = u_{1,j+2} = 0, j = 0, 1, 2, \dots$$

Then

$$\mathcal{D}_z l_{j,k} = l_{j+1,k} + \sum_{\kappa} R_\kappa^1 \text{inv}(v_{j,k,\kappa})$$

$$\mathcal{D}_{\bar{z}} l_{j,k} = l_{j,k+1} + \sum_{\kappa} R_\kappa^2 \text{inv}(v_{j,k,\kappa})$$

for some **Maurer-Cartan invariants** R_κ^1, R_κ^2 .



Olver's recurrence formulas: Maurer-Cartan invariants

The Maurer-Cartan invariants R_{κ}^1 , R_{κ}^2 can be solved from the **phantom invariants**, i.e. the constants we normalize:

$$l_{j,0} = l_{0,j} = l_{j+2,1} = l_{1,j+2} = 0, \quad j = 0, 1, 2, \dots$$
$$l_{1,1} = 1 = l_{2,2}, \quad l_{3,2} = l_{2,3} = Q$$

They give a finite linear system after truncation up to finite order, like

$$0 = \mathcal{D}_z l_{j,t} = l_{j+1,t} + \sum_{\kappa} R_{\kappa}^1 \text{inv}(v_{j,t,\kappa}), \quad t = 0, 1$$

$$0 = \mathcal{D}_{\bar{z}} l_{t,k} = l_{t,k+1} + \sum_{\kappa} R_{\kappa}^2 \text{inv}(v_{t,k,\kappa}), \quad t = 0, 1$$

$$0 = \mathcal{D}_z l_{2,2} = \underbrace{l_{3,2}}_Q + \sum_{\kappa} R_{\kappa}^1 \text{inv}(v_{j,1,\kappa}),$$

$$0 = \mathcal{D}_{\bar{z}} l_{2,2} = \underbrace{l_{2,3}}_Q + \sum_{\kappa} R_{\kappa}^2 \text{inv}(v_{1,k,\kappa}),$$

$$0 = \mathcal{D}_z (l_{3,2} - l_{2,3}) = l_{4,2} - l_{3,3} + \sum_{\kappa} R_{\kappa}^1 \text{inv}(v_{3,2,\kappa} - v_{2,3,\kappa}),$$

$$0 = \mathcal{D}_{\bar{z}} (l_{3,2} - l_{2,3}) = l_{3,3} - l_{2,4} + \sum_{\kappa} R_{\kappa}^2 \text{inv}(v_{3,2,\kappa} - v_{2,3,\kappa}).$$

Olver's recurrence formulas: Lower-orders

Once R_{κ}^1 , R_{κ}^2 are solved, the recurrence formulas on non-phantom invariants give

$$\mathcal{D}_z Q = \frac{l_{4,2}}{2} - \frac{3}{2} - \frac{3Q^2}{2} + \frac{l_{3,3}}{2}, \quad \mathcal{D}_{\bar{z}} Q = \frac{l_{3,3}}{2} - \frac{3}{2} - \frac{3Q^2}{2} + \frac{l_{2,4}}{2}.$$

(Fels-Olver 1999) [2, Thm 13.3] The generating system of all invariants

A generating system of differential invariants consists of

- (a) all nonphantom differential invariants coming from the nonnormalized zeroth order lifted invariants (not exist in our case)
- (b) all nonphantom differential invariants of the form $l_{j+1,k}$ and $l_{j,k+1}$ where $l_{j,k}$ is a phantom differential invariant.

In our case, the generating system consists of Q , $l_{3,3}$, $l_{2,4}$ and $l_{4,2}$. But the later two can be solved from the two recurrence formulas above

$$l_{2,4} = 2\mathcal{D}_{\bar{z}} Q + 3 + \frac{3Q^2}{2} - l_{3,3}, \quad l_{4,2} = 2\mathcal{D}_z Q + 3 + \frac{3Q^2}{2} - l_{3,3}.$$

Homogeneous models

A rigid hypersurface $M^3 \subset \mathbb{C}^2$ graphed by $u = F(z, \bar{z})$ passing the origin is **homogeneous** iff there exists three tangent vector fields from the Lie algebra of G^{RT} linearly independent at the origin.

Trivial vector field: $\partial_v = \frac{\partial_w - \partial_{\bar{w}}}{2i}$. It remains to find two more. Take

$L := f(z)\partial_z + (h(z) + rw)\partial_w + (\text{conjugates})$ an arbitrary vector field from the Lie algebra. It is tangent to the graph $u = \frac{w+\bar{w}}{2} = F(z, \bar{z})$ iff

$$f(z)F_z(z, \bar{z}) + \overline{f(z)}F_{\bar{z}} = \frac{1}{2}h(z) + \frac{1}{2}\overline{h(z)} + rF(z, \bar{z})$$

Assume $f(z) = a_0 + a_1z + \frac{a_2}{2}z^2 + \dots$, we solve the infinite system of (Taylor coefficients) = 0.

Homogeneous models, solving from the normal form

We may assume $u = F(z, \bar{z})$ is already normalized. Then $h(z) = 2\overline{f(0)}z + c_0$ for any $c_0 \in \mathbb{C}$. From the tangency condition $L(u - F(z, \bar{z})) = 0$,

$$\begin{aligned} & f(z) (\bar{z} + \frac{1}{2}z\bar{z}^2 + \frac{Q}{3!}z\bar{z}^3 + \frac{Q}{4}z^2\bar{z}^2 + \sum_{j,k \geq 2, j+k \geq 6} \frac{F_{j,k}}{(j-1)!k!} z^{j-1}\bar{z}^k) \\ + & \overline{f(\bar{z})} (z + \frac{1}{2}z^2\bar{z} + \frac{Q}{3!}z^3\bar{z} + \frac{Q}{4}z^2\bar{z}^2 + \sum_{j,k \geq 2, j+k \geq 6} \frac{F_{j,k}}{j!(k-1)!} z^j\bar{z}^{k-1}) \\ = & \operatorname{Re}(c_0) + \overline{f(0)}z + f(0)\bar{z} + r(z\bar{z} + \frac{1}{4}z^2\bar{z}^2 + \frac{Q}{2!3!}z^2\bar{z}^3 + \frac{Q}{2!3!}z^3\bar{z}^2 + \sum_{j,k \geq 2, j+k \geq 6} \frac{F_{j,k}}{j!k!} z^j\bar{z}^k) \end{aligned}$$

The lower order coefficients give

$$\begin{aligned} \mathbf{1} : & \quad \operatorname{Re}(c_0) = 0, \\ z\bar{z} : & \quad r = a_1 + \overline{a_1}, \\ z^2\bar{z} : & \quad a_2 = -\overline{a_0}, \\ z^3\bar{z} : & \quad a_3 = -\overline{a_0}Q, \\ z^2\bar{z}^2 : & \quad a_1 + \overline{a_1} = -Q(a_0 + \overline{a_0}), \\ z^k\bar{z} : & \quad a_k = -\overline{a_0}F_{k,2}, \quad k \geq 4 \end{aligned}$$

Thus only a_0 , $\operatorname{Im}(a_1)$ and $\operatorname{Im}(c_0)$ remain unsolved.



Homogeneous models and recurrence formulas

Look at

$$z^3 \bar{z}^2 : \frac{3Q^2 - 2F_{4,2}}{24} a_0 + \frac{3Q^2 - 2F_{3,3} + 6}{24} \bar{a}_0 - \frac{iQ}{12} \operatorname{Im}(a_1) = 0,$$

$$z^2 \bar{z}^3 : \frac{3Q^2 - 2F_{3,3} + 6}{24} a_0 + \frac{3Q^2 - 2F_{2,4}}{24} \bar{a}_0 + \frac{iQ}{12} \operatorname{Im}(a_1) = 0.$$

Adding them together

$$\frac{3Q^2 - F_{3,3} - F_{4,2} + 3}{12} a_0 + \frac{3Q^2 - F_{2,4} - F_{3,3} + 3}{12} \bar{a}_0 = 0.$$

Thus a **necessary** condition for a surface in its normal form to be homogeneous is $3Q^2 - F_{3,3} - F_{4,2} + 3 = 0$ and $3Q^2 - F_{2,4} - F_{3,3} + 3 = 0$. It **coincides** with Olver's recurrence relation

$$0 = \mathcal{D}_z Q = \frac{F_{4,2}}{2} - \frac{3}{2} - \frac{3Q^2}{2} + \frac{F_{3,3}}{2},$$

$$0 = \mathcal{D}_{\bar{z}} Q = \frac{F_{3,3}}{2} - \frac{3}{2} - \frac{3Q^2}{2} + \frac{F_{2,4}}{2},$$

because homogeneous models must have constant invariant Q , i.e.

$0 = \mathcal{D}_z Q = \mathcal{D}_{\bar{z}} Q$. Following this path one solves the model

$$u = e^{\frac{-z-\bar{z}}{2}} \sin\left(\frac{-iz+i\bar{z}}{2}\right)^{\frac{1}{l}} \text{ where } l = \frac{1}{2} \pm \sqrt{\frac{1}{2Q^2} + \frac{1}{4}}.$$



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