On tangent geometry and generalized continuum with defects

Nguyen Van Hoi
Joint work with Prof. G. Casale, Prof. L. Le Marrec

Department of Mathematics, IRMAR
University of Rennes 1.

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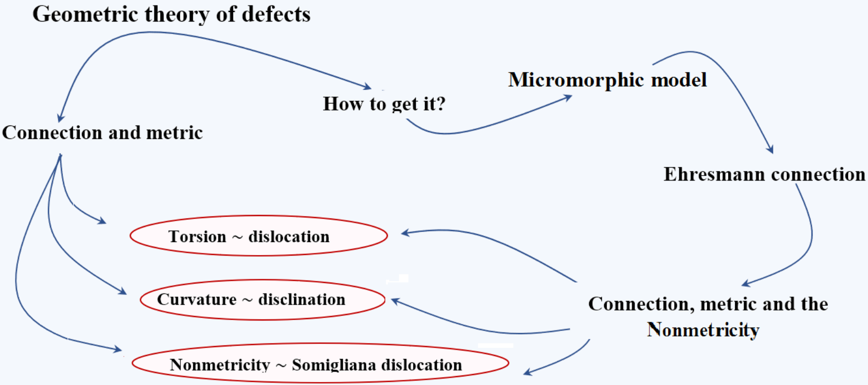
Abstract



Geometric theory of defects

How to get it?

Micromorphic model

Connection and metric

Ehresmann connection

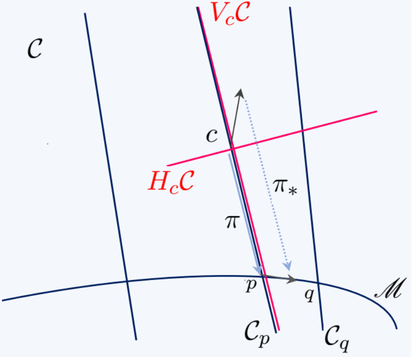
Torsion ~ dislocation

Curvature ~ disclination

Nonmetricity ~ Somigliana dislocation

Connection, metric and the Nonmetricity

Ehresmann connection

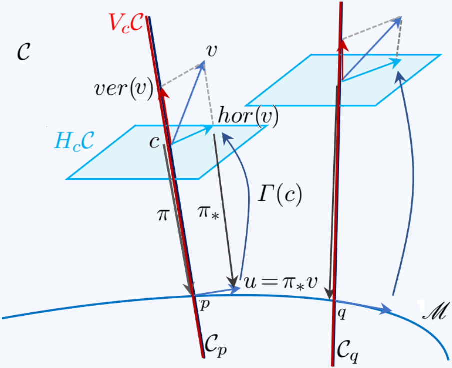


Ehresmann connection consists of a smooth assignment to each point $c \in \mathcal{C}$ of an n-dimensional horizontal subspace $H_c\mathcal{C} \subset T_c\mathcal{C}$ s.t

$$T_c\mathcal{C} = H_c\mathcal{C} \oplus V_c\mathcal{C} \quad \text{where} \quad V_c\mathcal{C} = \{V \in T_c\mathcal{C} : \pi_* V = 0\}$$

Throughout this presentation, we consider $\mathcal{C} = T\mathscr{M}$.

Operator lift $\mathit{\Gamma} : T\mathscr{M} \to T\mathcal{C}$



Let $u = u^a \partial/\partial x^a \in T_p\mathscr{M}$ and $c \in \mathcal{C}_p$, there is a unique horizontal lift $\mathit{\Gamma}(c)u$ at a point $c \in \mathcal{C}_p$, st $\pi_*(\mathit{\Gamma}(c)u) = u$:

$$\mathit{\Gamma}(c)u = u^a \frac{\partial}{\partial x^a} - \mathit{\Gamma}^i{}_a(c) u^a \frac{\partial}{\partial y^i}.$$

Horizontal decomposition

The horizontal and vertical spaces:

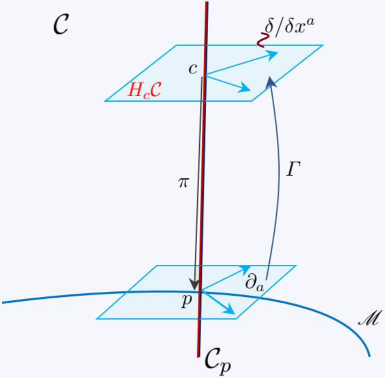
$$H_c\mathcal{C} = \mathrm{span}\ (\frac{\delta}{\delta x^a} := \frac{\partial}{\partial x^a} - \mathit{\Gamma}^i{}_a(c)\frac{\partial}{\partial y^i})$$

$$V_c\mathcal{C} = \mathsf{span}(\frac{\partial}{\partial y^i})$$

Their dual spaces

$$H^*\mathcal{C} = \mathsf{span}(dx^a)$$

$$V^*\mathcal{C} = \mathsf{span}(\delta y^i := dy^i + \mathit{\Gamma}^i{}_a dx^a)$$



Remark

The lift operator is independent on the chosen coordinate. Under a transformation of coordinated on $\mathscr{M}$, $\tilde{x}^b = \tilde{x}^a(x^b)$, *one has*

$$\frac{\partial \tilde{x}^b}{\partial x^a}\tilde{\Gamma}^k_{\ b} = \frac{\partial \tilde{x}^k}{\partial x^b}\Gamma^b_{\ a} - \frac{\partial^2 \tilde{x}^k}{\partial x^a \partial x^b} y^b.$$

Remark

- Holonomic basis: $\{\partial/\partial x^a, \partial/\partial y^i\}$ *and* $\{dx^a, dy^i\}$, *don't transform like a tensor.*
- Nonholonomic basis: $\{\delta/\delta x^a, \partial/\partial y^i\}$ *and* $\{dx^a, \delta y^i\}$ *transform like a tensor.*

Remark

The connection is linear if $\Gamma^i_{\ a}(x,v) = \Gamma^i_{\ aj} v^j$. *In the linear situation,* $\Gamma^i_{\ ab}$ *is defined an affine connection on* $\mathscr{M}$.

Ehresmann curvature and torsion on $T\mathscr{M}$

Γ^i_a Ehresmann connection on $\mathcal{C}$. Define $\Gamma^i{}_{aj} = \frac{\partial \Gamma^i{}_a}{\partial y^j}$

Ehresmann curvature

$$\mathfrak{R}^i{}_{jab} = \frac{\delta \Gamma^i{}_{aj}}{\delta x^b} - \frac{\delta \Gamma^i{}_{bj}}{\delta x^a} + \Gamma^k{}_{aj}\Gamma^i{}_{bk} - \Gamma^k{}_{bj}\Gamma^i{}_{ak}$$

Ehresmann torsion

$$\mathfrak{T}^i{}_{ab} = \Gamma^i{}_{ab} - \Gamma^i{}_{ba}$$

$\underline{\Gamma}^a{}_{bc}$ affine connection on $\mathscr{M}$.

Classical curvature

$$\mathsf{R}^a{}_{bcd} = \frac{\partial \Gamma^a{}_{cb}}{\partial x^d} - \frac{\partial \Gamma^a{}_{dj}}{\partial x^c} + \Gamma^k{}_{cb}\Gamma^a{}_{dk} - \Gamma^k{}_{db}\Gamma^a{}_{ck}$$

Classical torsion

$$\mathsf{T}^a{}_{bc} = \Gamma^a{}_{bc} - \Gamma^a{}_{cb}$$

If $\Gamma^i{}_a(x,v) = \underline{\Gamma}^i{}_{aj} v^j$, one has $\mathfrak{R}^i{}_{jab} = \mathsf{R}^i{}_{jab}$ and $\mathfrak{T}^i{}_{ab} = \mathsf{T}^i{}_{ab}$.

Sasaki metric

The nonholonomic bases $(\delta_a, \bar{\partial}_i)$ and $(dx^a, \delta y^i)$ are convenient local basis on $T\mathcal{C}$ and $T^*\mathcal{C}$, respectively.

On $\mathcal{C}$, *Sasaki metric* or the fundamental tensor are introduced by

$$\mathrm{g}(x,y) = \bar{\mathrm{g}}_{ab}(x) dx^a \otimes dx^b + \tilde{\mathrm{g}}_{ij}(x,y) \delta y^i \otimes \delta y^j .$$

Material transformation

– Let $\mathscr{B}$ be a material three-dimensional differentiable manifold and (X^A, Y^I) be a coordinate system on $T\mathscr{B}$.
– $T\mathbb{E}^3$ is an Euclidean bundle with (x^a, y^i) denotes coordinate systems. The ambient metric is

$$\mathsf{g} = \mathsf{g}_{ab}(x) dx^a \otimes dx^b + \mathsf{g}_{ij}(x) \delta y^i \otimes \delta y^j .$$

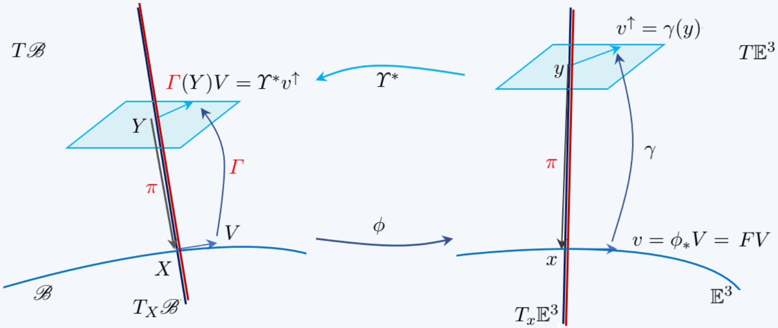
– A bundle mapping one-to-one $\mathscr{C}^1$, have inverses $\mathscr{C}^1$:

$$\Upsilon : T\mathscr{B} \to T\mathbb{E}^3, \qquad (X, V) \mapsto (\phi(X), \Psi(X, V))$$

– Total gradient

$$\mathcal{F} : TT\mathscr{B} \to TT\mathbb{E}^3 \qquad ((X, V), W) \mapsto (\Upsilon(X, V), D\Upsilon(X, V)W)$$
$$\mathcal{F} = \partial_C \phi^a \partial_a \otimes dX^C + \partial_C \Psi^i \bar{\partial}_i \otimes dX^C + \bar{\partial}_K \Psi^i \bar{\partial}_i \otimes dY^K .$$

Induced geometrical structure



An induced Ehresmann connection and a fundamental tensor are

$$\Gamma^I{}_A = \mathcal{F}^I_i \partial_A \Psi^i \qquad \bar{\mathtt{G}}_{AB} = F^a_A g_{ab} F^b_B \qquad \tilde{\mathtt{G}}_{IJ} = \mathcal{F}^i_I g_{ij} \mathcal{F}^j_J.$$

Remark

By the definition of the connection, for any closed curve σ *on* $\mathscr{B}$*, its horizontal-lift is defined by* $\sigma^{\uparrow} = \Upsilon^{-1}(\phi \circ \sigma)^{\uparrow}$*. Hence,* $\mathfrak{R} = 0$*.*

Remark

Assuming that the transformation is linear i.e. $\Psi^i(X, V) = \Psi^i_J(X) V^J$*. Hence, the connection is linear (i.e.* $\Gamma^K_{\ C} = \Gamma^K_{\ CJ}(X) V^J$*) with*

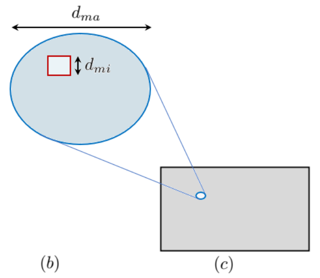
$$\Gamma^K_{\ CJ} = \Psi^K_i \partial_C \Psi^i_J \qquad\qquad \tilde{\mathtt{G}}_{IJ} = \Psi^i_I \mathrm{g}_{ij} \Psi^j_J .$$

Eventually, the couple $(\Gamma, \tilde{\mathtt{G}})$ *offers a natural Riemann-Cartan geometry on* $\mathscr{B}$ *with torsion, but vanishing curvature and* $\nabla \tilde{\mathtt{G}} = 0$*.*

An extended material transformation

A material transformation is then

$$\begin{array}{rcl} \tilde{\Upsilon}: \quad VT\mathscr{B} & \rightarrow & VT\mathbb{E}^3 \\ X & \mapsto & \phi(X) \\ {\color{blue}V} & {\color{blue}\mapsto} & {\color{blue}\Psi(X)V} \\ {\color{red}W} & {\color{red}\mapsto} & {\color{red}\Theta(X)W} \end{array}$$



The scale ratios $\ell_{ma} = d_{ma}/L$ and $\ell_{mi}^2 = d_{mi}/d_{ma}$ are defined for accounting for scale lengths.

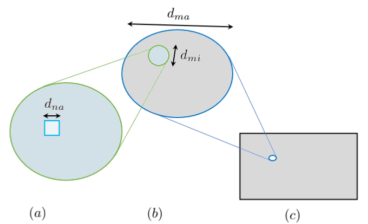
An extended material transformation

A material transformation is then

$$\begin{array}{rccl} \tilde{\Upsilon}: & VT\mathscr{B} & \to & VT\mathbb{E}^3 \\ & X & \mapsto & \phi(X) \\ & V & \mapsto & \Psi(X)V \\ & W & \mapsto & \Theta(X)W \end{array}$$

The main idea is to define

$$\begin{array}{rccl} \Upsilon: & TT\mathscr{B} & \to & TT\mathbb{E}^3 \\ s.t & \Upsilon_{|VT\mathscr{B}} & = & \tilde{\Upsilon} \end{array}$$



The scale ratios $\ell_{ma} = d_{ma}/L$, $\ell^2_{mi} = d_{mi}/d_{ma}$ and $\ell^2_{na} = d_{na}/d_{mi}$ are defined for accounting for scale lengths.

Linear contribution

A typical example:

$$\begin{array}{rccc} \Upsilon: & TT\mathscr{B} & \to & TT\mathbb{E}^3 \\ & (X,Y,W) & \mapsto & (\phi(X), \varPsi(X)Y, \varOmega(X,Y)W) \end{array}$$

with $\det F > 0$; $\det \varPsi > 0$ and $\det \varTheta > 0$.

$\varOmega$ being in the form:

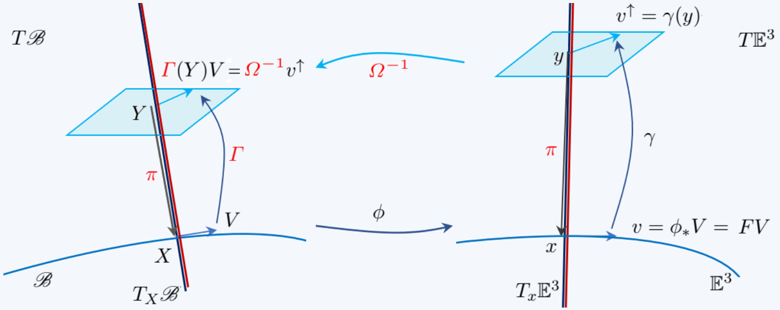
$$\varOmega = F^a_A \partial_a \otimes dX^A + \varOmega^i_A \bar{\partial}_i \otimes dX^A + \varTheta^i_K(X) \bar{\partial}_i \otimes dY^K$$

with $\varOmega^i_A = ((1-\ell^2)\partial_A \varPsi^i_J + \ell^2 \partial_A \varTheta^i_J) Y^J$ with $0 < \ell^2 \leq 1$.

It satisfies

$$\Upsilon_{|VT\mathscr{B}} = \tilde{\Upsilon}.$$

Linear contribution



Straightforward computations give the connection Γ is linear:

$$\Gamma^K_{AJ} = \Theta^K_i((1-\ell^2)\partial_A\Psi^i_J + \ell^2\partial_A\Theta^i_J).$$

This linear connection Γ may have torsion $\mathfrak{T}$ and curvature $\mathfrak{R}$.

- The linear connection $\mathit{\Gamma}$:

$$\mathit{\Gamma}^K_{AJ} = \mathit{\Theta}^K_i((1-\ell^2)\partial_A \mathit{\Psi}^i_J + \ell^2 \partial_A \mathit{\Theta}^i_J).$$

- The induced metric is obtained by

$$\tilde{\mathtt{G}}_{IJ} = \mathit{\Theta}^i_I \mathtt{g}_{ij} \mathit{\Theta}^j_J.$$

- The couple $(\mathit{\Gamma}, \tilde{\mathtt{G}})$ may be non-RC geometry (i.e. $\nabla\tilde{\mathtt{G}} \neq 0$). The connection has both torsion and curvature.

+ If $\frac{|dY|}{|dX|} \to 1$ then $\ell^2 \to 1$ and hence $\mathfrak{R} \to 0$ and $\nabla\tilde{\mathtt{G}} \to 0$.
+ If $\mathit{\Theta} = \mathit{\Psi}$, one has $\mathfrak{R} = 0$ and $\nabla\tilde{\mathtt{G}} = 0$.
+ If $\mathit{\Theta} = \mathit{\Psi} = D\phi$, it yields that $\mathfrak{T} = 0$; $\mathfrak{R} = 0$ and $\nabla\tilde{\mathtt{G}} = 0$.

Torsion yes. Curvature no. Nonmetricity no.

$$\Psi = Id \quad \Theta = \mathbb{I} + \cos X^1 \partial_1 \otimes \partial_2$$

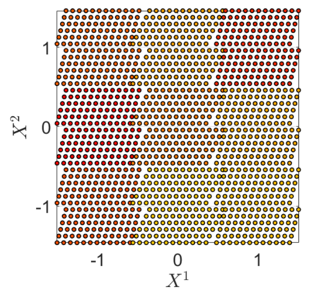


Figure: $\ell_{ma} = 1$, $\ell_{na}^2 = \ell_{mi}^2 = 0.25$; $\theta(X^1) = 0.5 \cos X^1$.

Torsion no. Curvature no. Nonmetricity yes.

$$\Psi = \mathbb{I} + \psi(X^1)\bar{\partial}_2 \otimes \bar{\partial}_1 \qquad \Theta = \mathbb{I} + \theta(X^1)\bar{\partial}_2 \otimes \bar{\partial}_1$$

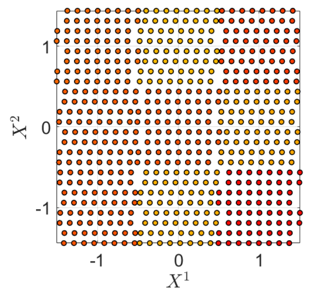


Figure: $\ell_{ma} = 1$, $\ell_{na}^2 = \ell_{mi}^2 = 0.25$, $\varepsilon = 0$ $\psi = 0.5\cos X^1$ and $\theta = 0.5\sin X^1$.

Torsion yes. Curvature no. Nonmetricity yes.

$$\varPsi = \mathbb{I} + \psi(X^2)\partial_1 \otimes \partial_2 \qquad \varTheta = \mathbb{I} + \theta(X^1)\bar{\partial}_1 \otimes \bar{\partial}_2$$

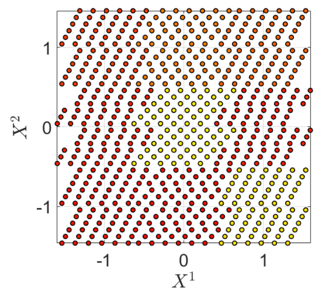


Figure: $\psi(X^2) = \cos X^2$ and $\theta(X^1) = \cos X^1$; $\ell_{ma} = 1, \ell_{na}^2 = 0.5$, $\ell_{mi}^2 = 0.25$.

Torsion no. Curvature yes. Nonmetricity yes.

$$\varPsi = \mathbb{I} + \psi(X^1)\partial_2 \otimes \partial_1 + \varepsilon(X^2)\partial_2 \otimes \partial_2 \qquad \varTheta = \mathbb{I} + \theta(X^1)\bar{\partial}_2 \otimes \bar{\partial}_1$$

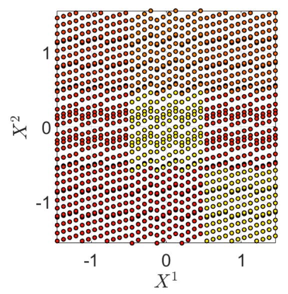


Figure: $\varepsilon(X^2) = 0.5\cos X^2$, $\psi(X^1) = 0.5\cos X^1$ and $\theta(X^1) = 0.1\cos X^1$. $\ell_{ma} = 1$, $\ell_{mi}^2 = 0.1111$, $\ell_{na}^2 = 0.0625$.

Torsion yes. Curvature yes. Nonmetricity yes.

$$\mathit{\Psi} = \mathbb{I} + \psi(X^1, X^2)\partial_1 \otimes \partial_2 + \psi(X^1, X^2)\partial_2 \otimes \partial_2$$
$$\mathit{\Theta} = \mathbb{I} + \theta(X^1)\bar{\partial}_1 \otimes \bar{\partial}_2$$

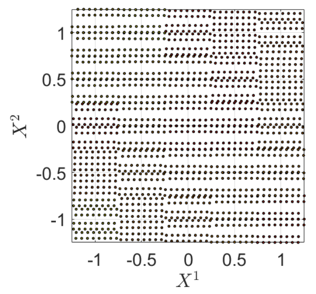
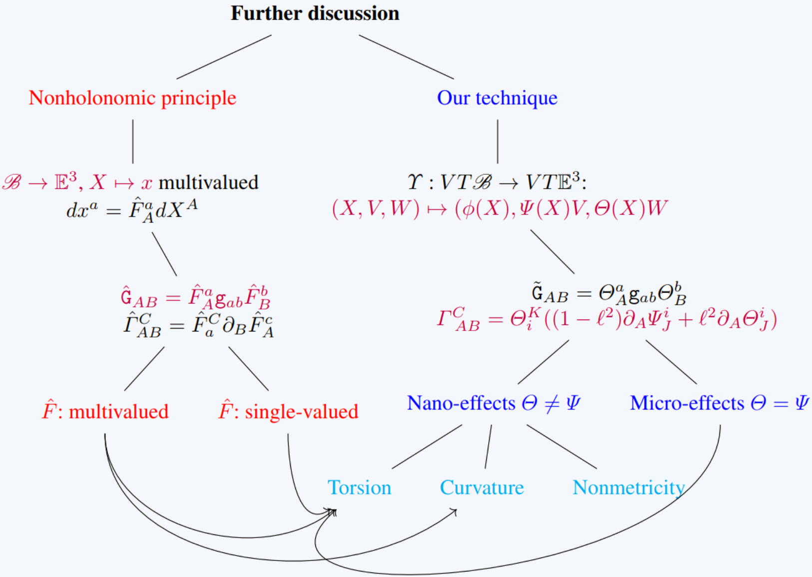


Figure: $\ell_{ma} = 0.25$, $\ell_{na}^2 = 0.25$; $\ell_{mi}^2 = 0.25$; $\theta(X^1) = 0.5\cos X^1$ and $\varepsilon(X^2) = \psi(X^1) = 0.5\cos(X^1 + X^2)$.



Further discussion

Nonholonomic principle

$\mathscr{B} \to \mathbb{E}^3$, $X \mapsto x$ multivalued
$dx^a = \hat{F}^a_A dX^A$

$\hat{\mathtt{G}}_{AB} = \hat{F}^a_A \mathtt{g}_{ab} \hat{F}^b_B$
$\hat{\Gamma}^C_{AB} = \hat{F}^C_a \partial_B \hat{F}^c_A$

$\hat{F}$: multivalued

$\hat{F}$: single-valued

Our technique

$\Upsilon : VT\mathscr{B} \to VT\mathbb{E}^3$:
$(X, V, W) \mapsto (\phi(X), \Psi(X)V, \Theta(X)W)$

$\tilde{\mathtt{G}}_{AB} = \Theta^a_A \mathtt{g}_{ab} \Theta^b_B$
$\Gamma^C_{AB} = \Theta^K_i((1-\ell^2)\partial_A \Psi^i_J + \ell^2 \partial_A \Theta^i_J)$

Nano-effects $\Theta \neq \Psi$

Micro-effects $\Theta = \Psi$

Torsion

Curvature

Nonmetricity

THANK FOR YOUR ATTENTION :)