Nœther's theorem in fluid mechanics

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Goal of the talk

- Nœther's theorem
- Extension to non-variational problem
- Conservation laws in fluid mechanics

Conservation law

PDE: $E(y, u, \ldots, u_{(n)}) = 0$, Solution manifold $\mathcal{S}_E \subset \mathbb{R}^{n_y} \times \mathbb{R}^{n_u}$

Conservation law:

$$Div\ F = 0 \qquad \text{on} \quad \mathcal{S}_E$$

- Flux: $F = F(y, u, u_{(1)}, u_{(2)}, \ldots)$ is a $\mathbb{R}^{n_y}$-valued function
- Total divergence: Div $F = D_1 F^1 + D_2 F^2 + \cdots + D_{n_y} F^{n_y}$

Total derivative $D_i = \frac{\partial}{\partial y^i} + u^a_i \frac{\partial}{\partial u^a} + u^a_{ij} \frac{\partial}{\partial u^a_j} + \cdots$

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- If $y = t$ is the time then F is a scalar constant ($D_t F = 0$) on $\mathcal{S}_E$
- If $y = (t, x)$ and $F = (F^t, -F^x)$ then $D_t F^t = Div_x F^x$

(total density variation) $\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega F^t \, \mathrm{d}x = \int_{\partial\Omega} F^x \, \mathrm{d}x$ (flux through $\partial\Omega$)

- Better understanding of the dynamics of the system
  Analysis of integrability, existence of solution, stability, ...
  Development of robust numerical schemes

(First) Nœther's theorem

(Dirichlet) Variational problem

- $\mathcal{L} = \int_\Omega L(y, u, \dot{u}, \ldots, u_{(p)})\ \mathrm{d}y$
- $\delta\mathcal{L} = 0 \quad \Longrightarrow \quad$ Euler-Lagrange: $\quad EL(y, u, \dot{u}, \ldots, u_{(p)}) = 0$

$$EL_a := \frac{\delta L}{\delta u^a} = \frac{\partial L}{\partial u^a} - D_i\left(\frac{\partial L}{\partial u_i^a}\right) + \cdots$$

Nœther's theorem

Variational symmetry group $\mathcal{L} \Longleftrightarrow$ Conservation law of $\mathrm{EL} = 0$

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Nœther's theorem

Variational symmetry group $\mathcal{L} \Longleftrightarrow$ Conservation law of EL $= 0$

- (Local) one-parameter transformation group
  $$G = \{ \quad g_\epsilon : (y, u) \longmapsto (\; \widehat{y}(y, u, \epsilon) \; , \; \widehat{u}(y, u, \epsilon) \;) \quad \}$$
- G is a variational symmetry group of L if

$$\int_{\widehat{\Omega}} L(\widehat{y}, \widehat{u}, \ldots, \widehat{u}_{(p)}) \, \mathrm{d}\widehat{y} = \int_\Omega L(y, u, \ldots, u_{(p)}) \, \mathrm{d}y$$

Infinitesimal transformation

- Infinitesimal generator of G

(If $g_{\epsilon=0} = Id$)

$$X = \xi^i \frac{\partial}{\partial y^i} + \eta^a \frac{\partial}{\partial u^a} \qquad \text{with} \qquad \xi^i = \frac{\partial \widehat{y}^i}{\partial \epsilon}_{|\epsilon=0}, \qquad \eta^a = \frac{\partial \widehat{u}^a}{\partial \epsilon}_{|\epsilon=0}$$

Eg.

- $(y,u) \longmapsto (y+\epsilon, u)$, $\quad X = \frac{\partial}{\partial y}$, $\quad \mathrm{pr}\, X = X$
- $(y,u) \longmapsto (e^{\alpha\epsilon}\, y, e^{\beta\epsilon}\, u)$, $\quad X = \alpha y \frac{\partial}{\partial y} + \beta u \frac{\partial}{\partial u}$, $\quad \mathrm{pr}\, X = X + (\beta - \alpha) u_y \frac{\partial}{\partial u_y}$

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- Generalized infinitesimal transformation: $X = \xi^i \dfrac{\partial}{\partial y^i} + \eta^a \dfrac{\partial}{\partial u^a}$

with $\quad \xi^i = \xi^i(y, u, u_{(1)}, \dots),\ \eta^a = \eta^a(y, u, u_{(1)}, \dots)$

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  with $\quad \xi^i = \xi^i(y, u, u_{(1)}, \dots),\ \eta^a = \eta^a(y, u, u_{(1)}, \dots)$

- Prolongation: $\quad \mathrm{pr}\,\mathrm{X} = \mathrm{X} + \eta^a_J \frac{\partial}{\partial u^a_J} \qquad$ with $\qquad \eta^a_J = D_J Q^a + \xi^j u^a_{J,j}$
- Characteristic: $(Q^1, \dots, Q^{n_u}) \qquad$ where $\qquad Q^a = \eta^a - \xi^j u^a_j$

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- Characteristic: $(Q^1, \dots, Q^{n_u}) \quad$ where $\quad Q^a = \eta^a - \xi^j u^a_j$

Nœther's identity

$$(\mathrm{pr}\,\mathrm{X}) \cdot L + L\, \mathrm{Div}\, \xi = Q^a\, \mathrm{EL}_a + \mathrm{Div}\, P$$

P results from integrations by parts $\qquad P^i = \xi^i L + Q^a \dfrac{\partial L}{\partial u^a_i} + \cdots$

Back to Nœther's theorem

G, connected, generated by X, is a variational symmetry group of L

$$\int_{\widehat{\Omega}} L(\widehat{y}, \widehat{u}, \ldots)\ \mathrm{d}\widehat{y} = \int_{\Omega} L(y, u, \ldots)\ \mathrm{d}y$$

$$\iff \qquad (\mathrm{pr}\,\mathrm{X}) \cdot L + L\,\mathrm{Div}\,\xi = 0$$

- A generalized (divergence) variational symmetry is a generalized infinitesimal transformation X verifying

$$(\mathrm{pr}\,\mathrm{X}) \cdot L + L\,\mathrm{Div}\,\xi = \mathrm{Div}\,B \qquad \text{for some } B$$

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Nœther's theorem

X is a generalized variational symmetry of L

$$\iff \qquad Q^a \, \mathrm{EL}_a + \operatorname{Div}(P - B) = 0 \qquad\qquad (\operatorname{pr} \mathrm{X}) \cdot L + L \operatorname{Div} \xi = Q^a \, \mathrm{EL}_a + \operatorname{Div} P$$

$$\iff \qquad \operatorname{Div}(P - B) = 0 \qquad \text{on} \qquad \mathcal{S}_{EL}$$

Kepler's problem

- Position of the planet: $u = u(t)$

$$L = \frac{1}{2}\|\dot{u}\|^2 - \frac{m}{\|u\|}, \qquad \text{EL} \equiv \ddot{u} + \frac{mu}{\|u\|^3} = 0$$

  - Time translation $X = \frac{\partial}{\partial t}$ $\longleftrightarrow$ energy: $m\|u\|^2/2$
  - Rotation $X = u^i \frac{\partial}{\partial u^j} - u^j \frac{\partial}{\partial u^i}$ $\longleftrightarrow$ angular momentum: $mu \times \dot{u}$
  - ??? $\longleftrightarrow$ Runge-Lenz vector: $m\dot{u} \times (u \times \dot{u}) - m\frac{u}{\|u\|}$

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- Nœther's identity $(\mathrm{pr}\,\mathrm{X}) \cdot L + L\,\mathrm{Div}\,\xi = Q^a\,\mathrm{EL}_a + \mathrm{Div}\,P$

$$\mathrm{X} = \Big(\dot{u} \otimes u - 2u \otimes \dot{u} + (u \cdot \dot{u})\,\mathrm{Id}\Big)\frac{\partial}{\partial u}$$

Conservation laws of a non-variational problem ?

From multipliers

$$L = \operatorname{Div} \ell \text{ for some } \ell \iff \frac{\delta L}{\delta u} \equiv 0 \qquad \text{ie. } \operatorname{Im} \operatorname{Div} = \ker \frac{\delta}{\delta u}$$

$\mathcal{L}(u) = \int_\Omega L \, \mathrm{d}y, \qquad \delta \mathcal{L} = 0 \longrightarrow \dfrac{\delta L}{\delta u} = 0$

$L = \operatorname{Div} \ell \implies$ any $u(y)$ with the prescribed $u_{|\partial\Omega}$ is an optimum of $\mathcal{L} \implies \dfrac{\delta L}{\delta u} \equiv 0$

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$E[u] \equiv E(y, u, \ldots, u_{(p)}) = 0,$ Solution manifold $\mathcal{S}_E$

Multipliers of CoLas and their determining equation

$\operatorname{Div} P = 0$ on $\mathcal{S}_E$

$\iff$ $\operatorname{Div} P = \Lambda^k E_k$ for some $\Lambda(y, u, u_{(1)}, \ldots)$

(unique for Cauchy-Kovalevskaya PDE's)

$\iff$ $\dfrac{\delta(\Lambda \cdot E)}{\delta u} = 0$

Solve $\dfrac{\delta(\Lambda \cdot E)}{\delta u} = 0$ $\longrightarrow$ "All" local CoLa's (up to a prescribed order)

From a "Bilagrangian" (Ibragimov)

$E[u] \equiv E(y, u, \ldots, u_{(p)}) = 0,$ Solution manifold $\mathcal{S}_E$

- Adjoint variable: $v = (v^1, \ldots, v^{n_u})$
- "Bilagrangian": $L[y, u, \ldots, u_{(p)}, v] = v \cdot E$
- Euler-Lagrange eq.: $\begin{cases} \dfrac{\delta(v \cdot E)}{\delta v} \equiv E[u] = 0 \\ \\ \end{cases}$

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- "Bilagrangian": $L[y, u, \ldots, u_{(p)}, v] = v \cdot E$
- Euler-Lagrange eq.: $\begin{cases} \dfrac{\delta(v \cdot E)}{\delta v} \equiv E[u] = 0 \\ \dfrac{\delta(v \cdot E)}{\delta u} \equiv E^*[u, v] = 0 \end{cases}$ Shortcut: $\mathbb{E}[u, v] = 0$
- Nœther's theorem $\longrightarrow$ Local and non-local CoLa's in (u, v)
- Condition: $\dim E = \dim u = n_u$
- $E^*[u, \Lambda] = 0.$ $E^*[u, v]_{|\mathcal{S}_E} = 0.$

Examples in fluid mechanics

- Heat equation (Kolsrud, Ibragimov, Brandao)
  - ★ $E[u] \equiv u_t - \kappa u_{xx} = 0$
  - ★ $E^*[u,v] \equiv v_t + \kappa v_{xx} = 0$
  - ★ $L[u,v] = \frac{1}{2}\left(u_t v - u v_t\right) + \kappa u_x v_x$
- Burgers'equation (Kolsrud, Ibragimov, Brandao)
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  - ★ $L[u,v] = \frac{1}{2}\left(v u_t - u v_t\right) + \kappa u_x v_x + \frac{1}{3}(u v_x - v u_x)$

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- Navier-Stokes equation (Hamdouni, Razafindralandy)
  - ★ $E[u,p]: \quad \dfrac{\partial u}{\partial t} + (\nabla u)u + \dfrac{1}{\rho}\nabla p - \nu \Delta u = 0, \qquad \operatorname{div} u = 0$
  - ★ $E^*[u,p,v,q]: \dfrac{\partial v}{\partial t} + (\nabla v)u - ({}^{\top}\nabla u)v + \dfrac{1}{\rho}\nabla q + \nu \Delta v = 0, \qquad \operatorname{div} v = 0$
  - ★ $L = \dfrac{1}{2}\left(\dfrac{\mathrm{d}u}{\mathrm{d}t} \cdot v - u \cdot \dfrac{\mathrm{d}v}{\mathrm{d}t}\right) + \left(\dfrac{q}{\rho} - \dfrac{u \cdot v}{2}\right) \operatorname{div} u - \dfrac{p}{\rho} \operatorname{div} v + \nu \operatorname{tr}({}^{\top}\nabla u \, \nabla v)$

Dynamical symmetry

How to find variational symmetries ?

X infinitesimal variational symmetry of $L \Longrightarrow$
X infinitesimal dynamical symmetry of $\mathrm{EL} = 0$

- Search among the infinitesimal symmetries of $\mathrm{EL} = 0$ (easier to find)

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Dynamical symmetry group of a PDE

- $E[u] \equiv E(y, u, \ldots, u_{(p)}) = 0$ **(1)**
- $G = \{ \quad g \ : \ (y, u) \longmapsto (\widehat{y}, \widehat{u}) \quad \}$ is a symmetry group of **(1)** if

  $E(\widehat{y}, \widehat{u}, \ldots \widehat{u}_{(p)}) = 0$ on $\mathcal{S}_E$

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How to find the dynamical symmetry groups of (1) ?

- If G, generated by X, is a dynamical symmetry group of **(1)** then (conditions)

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How to find the dynamical symmetry groups of (1) ?

- If G, generated by X, is a dynamical symmetry group of **(1)** then (conditions)

  $(\mathrm{pr}\,\mathrm{X}) \cdot E = 0 \qquad \text{on} \qquad \mathcal{S}_E$
- Solve for (standard or generalized) X. Deduce G

$$\frac{\mathrm{d}\widehat{y}}{\mathrm{d}\epsilon} = \xi(\widehat{y}, \widehat{u}), \qquad \frac{\mathrm{d}\widehat{u}}{\mathrm{d}\epsilon} = \eta(\widehat{y}, \widehat{u}), \qquad \widehat{y}(\epsilon = 0) = y, \quad \widehat{u}(\epsilon = 0) = u$$

Generalized infinitesimal dynamical symmetry: $(\mathrm{pr}\,\mathrm{X}) \cdot E = 0 \quad \text{on} \quad \mathcal{S}_E$

Dynamical symmetry groups of Navier-Stokes's equation

$$\frac{\partial u}{\partial t} + \operatorname{div}(u \otimes u) + \tfrac{1}{\rho}\nabla p - \nu \Delta u = 0, \qquad \operatorname{div} u = 0$$

- Time translation: $(t, x, u, p) \mapsto (t+\epsilon, x, u, p)$
- Pressure translation: $(t, x, u, p+\zeta(t))$
- Rotation: (t, Rx, Ru, p)
- Generalized galilean transformation: $\big(t,\ x+\alpha(t),\ u+\dot{\alpha}(t),\ p-\rho x \cdot \ddot{\alpha}(t)\big)$
- Scale transformation: $(a^2 t,\ ax,\ a^{-1}u,\ a^{-2}p)$
- Equivalence (scale) transformation: $(t,\ ax,\ au,\ a^2 p,\ a^2 \nu)$

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Other symmetries:

- Reflections $(t,\ \Lambda x,\ \Lambda u,\ p)$
  $\Lambda = diag(\pm 1, \pm 1, \pm 1)$
- 2D material indifference $\left(t,\ R(t)x,\ R(t)u,\ p + -3\omega\psi + \frac{1}{3}\omega^2 \|x\|^2\right)$
  $R(t)$ plane rotation with angle ωt, ψ stream function

Anisothermal case

$$\frac{\partial u}{\partial t} + \operatorname{div}(u \otimes u) + \tfrac{1}{\rho}\nabla p - \nu \Delta u + \beta g \theta \mathrm{e}_3 = 0, \qquad \frac{\partial \theta}{\partial t} + \operatorname{div}(u\theta) - \kappa \Delta \theta = 0, \qquad \operatorname{div} u = 0$$

- Time translation: $(t+\epsilon, x, u, p, \theta)$
- Space translation: $(t, x + x_0, u, p, \theta)$
- Pressure translation pression : $(t, x, u, p+\zeta(t), \theta)$
- Pressure-temperature translation: $(t, x, u, p+\epsilon\beta x_3, \theta + \epsilon/\rho)$
- Generalized Galilean transformation: $\big(t, x+\alpha(t), u+\dot{\alpha}(t), p+\rho\, x \cdot \ddot{\alpha}(t), \theta\big)$
- Horizontal rotation: $(t, \mathrm{R}x, \mathrm{R}u, p, \theta)$
- Scale transformation: $(\mathrm{e}^{2\epsilon} t, \mathrm{e}^{\epsilon} x, \mathrm{e}^{-\epsilon} u, \mathrm{e}^{-2\epsilon} p, \mathrm{e}^{-3\epsilon}\theta)$
- Equivalence (scale) transformation: $(t, \mathrm{e}^{\epsilon} x, \mathrm{e}^{\epsilon} u, \mathrm{e}^{2\epsilon} p, \mathrm{e}^{\epsilon}\theta, \mathrm{e}^{2\epsilon}\nu, \mathrm{e}^{2\epsilon}\kappa)$

☞ 8-dimensional Lie group and 4 infinite dimensional groups

Importance of dynamical symmetry groups

- Encode important physical properties
  - Fundamental principles in physics (invariance galiléenne, d'échelle, ...)
  - Self-similar solutions (vortex, chocs, ...)
  - In turbulence: wall laws, scaling laws, ...
    - Linear law: $U_1 = C_1 x_2 + C_3, \quad \Theta = C_2 x_2 + C_4$
    - Logarithmic law : $U_1 = C_1 \ln(x_2 + b) + C_3, \quad \Theta = C_2 [x_2 + b]^{-1} + C_4$
    - Exponential law: $U_1 = C_1 \exp(C x_2) + C_3, \quad \Theta = C_2 \exp(2 C x_2) + C_4$
    - Power law: $U_1 = C_1 (x_2 + b)^a + C_3, \quad \Theta = C_2 (x_2 + b)^{2a-1} + C_4$
- Construction of turbulence model (Hamdouni, Razafindralandy)
- Design of robust numerical schemes (Hamdouni, Chhay)

Infinitesimal standard symmetries of Navier-Stokes + adjoint equations

$$\begin{cases} \dfrac{\partial u}{\partial t} + (\nabla u)u + \dfrac{1}{\rho}\nabla p - \nu \Delta u = 0, & \operatorname{div} u = 0 \\ \dfrac{\partial v}{\partial t} + (\nabla v)u - ({}^{\top}\nabla u)v + \dfrac{1}{\rho}\nabla q + \nu \Delta v = 0, & \operatorname{div} v = 0 \end{cases}$$

- $\dfrac{\partial}{\partial t}, \qquad \pi(t)\dfrac{\partial}{\partial p}, \qquad \varphi(t)\dfrac{\partial}{\partial q}$
- $w^i \dfrac{\partial}{\partial v^i} + \rho\left(w^i u^i - w^i_t x^i\right)\dfrac{\partial}{\partial q},$ $\qquad i = 1,2,3$
- $x^j \dfrac{\partial}{\partial v^i} - x^i \dfrac{\partial}{\partial v^j} + \rho(x^j u^i - x^i u^j)\dfrac{\partial}{\partial q}$ $\qquad (i,j) \in \{(1,2),(2,3),(3,1)\}$
- $x^j \dfrac{\partial}{\partial x^i} - x^i \dfrac{\partial}{\partial x^j} + u^j \dfrac{\partial}{\partial u^i} - u^i \dfrac{\partial}{\partial u^j} + v^j \dfrac{\partial}{\partial v^i} - v^i \dfrac{\partial}{\partial v^j},$ $\qquad (i,j) \in \{(1,2),(2,3),(3,1)\}$
- $z^i \dfrac{\partial}{\partial x^i} + z^i_t \dfrac{\partial}{\partial u^i} - \rho x^i z^i_{tt} \dfrac{\partial}{\partial p},$ $\qquad i = 1,2,3$
- $2t\dfrac{\partial}{\partial t} + x^k \dfrac{\partial}{\partial x^k} - u^k \dfrac{\partial}{\partial u^k} - 2p\dfrac{\partial}{\partial p} - q\dfrac{\partial}{\partial q}$ $\qquad$ (sum over k)
- $v^k \dfrac{\partial}{\partial v^k} + q\dfrac{\partial}{\partial q}.$ $\qquad$ (sum over k)

$\pi(t)$, $\varphi(t)$, $w^i(t)$ and $z^i(t)$ are arbitrary functions of t

Local dynamical symmetry group of Navier-Stokes + adjoint equations

$$\begin{cases} \dfrac{\partial u}{\partial t} + (\nabla u)u + \dfrac{1}{\rho}\nabla p - \nu\Delta u = 0, & \text{div } u = 0 \\ \dfrac{\partial v}{\partial t} + (\nabla v)u - ({}^{\top}\nabla u)v + \dfrac{1}{\rho}\nabla q + \nu\Delta v = 0, & \text{div } v = 0 \end{cases}$$

- Time translation: $(t, x, u, p, v, q) \longmapsto (t+\epsilon, x, u, p, v, q)$
- Pressure translation: $(t, x, u, p+\pi(t), v, q)$
- Adjoint-pressure translation: $(t, x, u, p, v, q+\varphi(t))$
- 1st (v, q) translation : $(t, x, u, p, v+w, q+\rho w \cdot u - \rho w_t \cdot x)$
  where $w(t)$ is an arbitrary function of t
- 2nd (v, q) translation sur (v, q): $(t, x, u, p, v+\omega \times x, q+\rho x \cdot \omega \times u)$
  where ω is a vectorial parameter
- Constant rotation matrix R: (t, Rx, Ru, p, Rv, q)
- Generalized Galilean transformation: $(t, x+z, u+z_t, p+\rho z_{tt} \cdot x, v, q)$
  where $z(t)$ is an arbitrary function of t
- 1st scale transformation: $(e^{2b}t, e^{b}x, e^{-b}u, e^{-2b}p, v, e^{-b}q)$
- 2nd scale transformation: $(t, x, u, p, e^{c}v, e^{c}q)$

Computation of conservation laws

- Search among (combinaisons of) these symmetries
  - variational symmetries
  - ie. verifying $(\mathrm{pr}\,\mathrm{X}) \cdot L + L\,\mathrm{Div}\,\xi = \mathrm{Div}\,B$
- Conservation laws: $F = P - B$ avec $P^i = \xi^i L + (\eta^a - \xi^j u_j^a) \dfrac{\partial L}{\partial u_i^a}$

Some conservation laws

- Generator: $\mathrm{V}_{ij} = x^j \frac{\partial}{\partial v^i} - x^i \frac{\partial}{\partial v^j} + \rho(x^j u^i - x^i u^j) \frac{\partial}{\partial q}$ $\qquad (i,j) \in \{(1,2),(2,3),(3,1)\}$

- Translation : $(t, x, u, p, v, q) \longmapsto (t, x, u, p, v + \omega \times x, q + \rho x \cdot \omega \times u)$

Some conservation laws

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- Translation : $(t,x,u,p,v,q) \longmapsto (t,x,u,p,v+\omega \times x, q+\rho x \cdot \omega \times u)$

▶ $(i,j) = (1,2)$

- Divergence symmetry: $\mathrm{pr}\,\mathrm{V}_{12} \cdot L + L\,\mathrm{Div}\,\xi = \mathrm{Div}\,B$ où

$$B = \tfrac{1}{2}(x^2u^1 - x^1u^2)\begin{pmatrix}1\\u^1\\u^2\\u^3\end{pmatrix} - \nu\begin{pmatrix}0\\u^2\\-u^1\\0\end{pmatrix}$$

- Flux: $P = \frac{1}{2}(x^1u^2 - x^2u^1)\begin{pmatrix}1\\u^1\\u^2\\u^3\end{pmatrix} - \frac{p}{\rho}\begin{pmatrix}0\\x^2\\-x^1\\0\end{pmatrix} + \nu\begin{pmatrix}0\\x^2u^1_1 - x^1u^2_1\\x^2u^1_2 - x^1u^2_2\\x^2u^1_3 - x^1u^2_3\end{pmatrix}$

- Local conservation law: $\mathrm{Div}(P - B) = 0$

▶ $(i,j) = (2,3)$ et $(i,j) = (3,1)$: ...

- Generator $\mathrm{R}_{ij} = x^j \frac{\partial}{\partial x^i} - x^i \frac{\partial}{\partial x^j} + u^j \frac{\partial}{\partial u^i} - u^i \frac{\partial}{\partial u^j} + v^j \frac{\partial}{\partial v^i} - v^i \frac{\partial}{\partial v^j}$, $(i, j) \in \{(1, 2), (2, 3), (3, 1)\}$
- Rotation $(t, x, u, p, v, q) \longmapsto (t, Rx, Ru, p, Rv, q)$

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- Rotation $(t, x, u, p, v, q) \longmapsto (t, Rx, Ru, p, Rv, q)$

▶ $(i,j) = (1,2)$

- Variational symmetry: $\mathrm{pr}\,\mathrm{R}_{12} \cdot L + L\,\mathrm{Div}\,\xi = 0$

- Non-local conservation law with

$$P = \begin{pmatrix} u^2 v^1 - u^1 v^2 + \frac{1}{2}\left(v \cdot \mathrm{R}_{12}^{(0)} u - u \cdot \mathrm{R}_{12}^{(0)} v\right) \\ x^2 L + u^2 \dfrac{\partial L}{\partial u_1^1} - u^1 \dfrac{\partial L}{\partial u_1^2} + \mathrm{R}_{12}^{(0)} u^k \dfrac{\partial L}{\partial u_1^k} + v^2 \dfrac{\partial L}{\partial v_1^1} - v^1 \dfrac{\partial L}{\partial v_1^2} + \mathrm{R}_{12}^{(0)} v^k \dfrac{\partial L}{\partial v_1^k} \\ -x^1 L + u^2 \dfrac{\partial L}{\partial u_2^1} - u^1 \dfrac{\partial L}{\partial u_2^2} + \mathrm{R}_{12}^{(0)} u^k \dfrac{\partial L}{\partial u_2^k} + v^2 \dfrac{\partial L}{\partial v_2^1} - v^1 \dfrac{\partial L}{\partial v_2^2} + \mathrm{R}_{12}^{(0)} v^k \dfrac{\partial L}{\partial v_2^k} \\ u^2 \dfrac{\partial L}{\partial u_3^1} - u^1 \dfrac{\partial L}{\partial u_3^2} + \mathrm{R}_{12}^{(0)} u^k \dfrac{\partial L}{\partial u_3^k} + v^2 \dfrac{\partial L}{\partial v_3^1} - v^1 \dfrac{\partial L}{\partial v_3^2} + \mathrm{R}_{12}^{(0)} v^k \dfrac{\partial L}{\partial v_3^k} \end{pmatrix}$$

where $\mathrm{R}_{ij}^{(0)} = x^j \dfrac{\partial}{\partial x^i} - x^i \dfrac{\partial}{\partial x^j}$ is the base part of R_{ij}

▶ $(i,j) = (2,3)$ et $(i,j) = (3,1)$: ...

- Generator $S_1 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - u\frac{\partial}{\partial u} - 2p\frac{\partial}{\partial p} - q\frac{\partial}{\partial q}$

- 1st scale transformation: $(e^{2b}t, e^{b}x, e^{-b}u, e^{-2b}p, v, e^{-b}q)$

$\mathrm{pr}\, S_1 \cdot L + L\, \mathrm{Div}\, \xi = 2L \longrightarrow$ not a generalized variationnal symmetry

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$\mathrm{pr}\, S_1 \cdot L + L\, \mathrm{Div}\, \xi = 2L \longrightarrow$ not a generalized variationnal symmetry

- Générateur $S_2 = v\frac{\partial}{\partial v} + q\frac{\partial}{\partial q}$

- 2ère transformation d'échelle: $(t, x, u, p, e^{c}v, e^{c}q)$

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$\mathrm{pr}\, S_2 \cdot L + L\,\mathrm{Div}\,\xi = L \longrightarrow$ not a generalized variationnal symmetry

But: $S = S_1 - 2S_2$
$\mathrm{pr}\, S \cdot L + L\,\mathrm{Div}\,\xi = 0 \longrightarrow$ variational symmetry

- Generator $$\mathrm{S} = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - u\frac{\partial}{\partial u} - 2p\frac{\partial}{\partial p} - q\frac{\partial}{\partial q} - 2v\frac{\partial}{\partial v} - 2q\frac{\partial}{\partial q}$$

- Scale transformation: $(\mathrm{e}^{2b}t, \mathrm{e}^{b}x, \mathrm{e}^{-b}u, \mathrm{e}^{-2b}p, \mathrm{e}^{-2b}v, \mathrm{e}^{-3b}q)$

- Variational symmetry: $\operatorname{pr}\mathrm{S}\cdot L + L\,\mathrm{Div}\,\xi = 0$

- $F = (F^0, F^1, F^2, F^3)^\top$ with

$$F^0 = \tfrac{1}{2}(u\cdot v) + t(u\cdot v_t - u_t\cdot v) + x\cdot\left(\nabla\tfrac{u\cdot v}{2} - (\nabla u)v\right)$$

$$F^i = x^i L - U\cdot\left[\overline{q}\,\mathrm{e}_i + \tfrac{u^i v}{2} + \nu v_i\right] - V\cdot\left[-\tfrac{p}{\rho}\,\mathrm{e}_i - \tfrac{u^i u}{2} + \nu u_i\right], \quad i = 1,2,3,$$

where $U = u + 2tu_t + (\nabla u)x, \qquad V = 2v + 2tv_t + (\nabla v)x$

- Generateur $Z_i = z^i \frac{\partial}{\partial x^i} + z^i_t \frac{\partial}{\partial u^i} - \rho x^i \, z^i_{tt} \frac{\partial}{\partial p}, \qquad i = 1, 2, 3$
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- Generalized Galilean transformation: $(t, x+z, u+z_t, p+\rho z_{tt}\cdot x, v, q)$

▶ $i = 1$

- Divergence symmetry: $\quad \mathrm{pr}\, Z_1 \cdot L + L\, \mathrm{Div}\, \xi = \mathrm{Div}\, B \quad$ où

$$B = -\tfrac{1}{2} z^1_t v^1 \begin{pmatrix} 1 \\ u^1 \\ u^2 \\ u^3 \end{pmatrix}$$

- Case z^1 constant

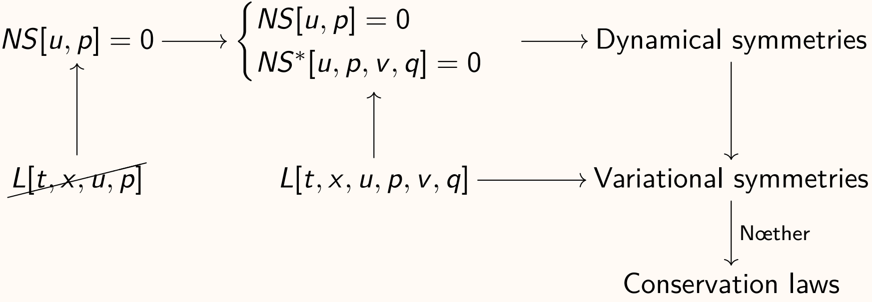
- Flux

$$P = \begin{pmatrix} \frac{1}{2}\left(v_1 \cdot u - u_1 \cdot v\right), \\ L + \frac{1}{2} u^1 \left(v_1 \cdot u - u_1 \cdot v\right) - 2\nu u_1 \cdot v_1 - u^1_1 \overline{q} + v^1_1 \overline{p}, \\ \frac{1}{2} u^2 \left(v_1 \cdot u - u_1 \cdot v\right) - \nu\left(u_1 \cdot v_2 + u_2 \cdot v_1\right) - u^2_1 \overline{q} + v^2_1 \overline{p}, \\ \frac{1}{2} u^3 \left(v_1 \cdot u - u_1 \cdot v\right) - \nu\left(u_1 \cdot v_3 + u_3 \cdot v_1\right) - u^3_1 \overline{q} + v^3_1 \overline{p}. \end{pmatrix} \qquad \begin{aligned} \overline{p} &= p + \rho \tfrac{u\cdot u}{2} \\ \overline{q} &= q - \rho \tfrac{u \cdot v}{2} \end{aligned}$$

▶ $i = 2, 3$: ... $\qquad$ ▶ z non constant ...

Conclusion

- Outline of the approach



- Interpretation of these conservation laws

  Integral form

- Non exhaustive
  - Other combinations of dynamical symmetries
  - Higher-order conservation laws: Bäcklund

Conclusion

- Inviscid flow $\nu = 0$
  - Euler'equations derive from a Lagrangian in a Euler-Poincaré sense
    - Euler's eq: $\dfrac{\partial u}{\partial t} + (\nabla u)u + \dfrac{1}{\rho}\nabla p = 0, \qquad \operatorname{div} u = 0$
    - Lagrangian: $L = \frac{1}{2}\|v\|^2$
  - Nœther's theorem in Euler-Poincaré sense
  - Compare with the previous conservation laws with $\nu = 0$

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- Inviscid flow $\nu = 0$
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  - ★ Compare with the previous conservation laws with $\nu = 0$
- Discrete Nœther's theorem
  - ★ Work in progress (Palafox, Cresson, Hamdouni)
  - ★ Numerical scheme preserving conservation laws