Computation of rational invariants.

Algebraic moving frames & dimensional analysis

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Based on joint works with I. Kogan, or G. Labahn
and invaluable discussions with E. Mansfield & P. Olver

Computation of rational invariants

1. Construction of rational invariants : a general algorithm

2. Scalings and parameter reduction in biological models

AIDA : Algebraic Invariants and their Differential Algebras

Overview article : *Algebraic and Differential Invariants* (2012)

1999 Fels & Olver - *Moving coframes. Regularization and foundations.*

2010 Mansfield - *Invariant Calculus*

Differential Invariants :

2009 Differential invariants of a Lie group action: syzygies & generating set.

2007 Generating properties of Maurer-Cartan invariants

2007 Smooth and Algebraic Invariants of a Group Action. With Kogan.

2005 Differential Algebra for Derivations with Nontrivial Commutation

Maple libraries : *diffalg*, AIDA

Algorithms for Computing Invariants: The subject of today's talk

Rational action $\star$ of an affine algebraic group $\mathcal{G}$

Group action:

$$\begin{array}{rcl} \star: \ \mathcal{G} \times \mathbb{R}^n & \rightarrow & \mathbb{R}^n \text{ or } \mathbb{C}^n \\ (\lambda, z) & \mapsto & \lambda \star z \end{array} \qquad \text{s.t} \qquad \begin{array}{rcl} 1 \star z & = & z \\ (\lambda \cdot \mu) \star z & = & \lambda \star (\mu \star z) \end{array}$$

$\mathcal{G} \subset \mathbb{K}^l$ an algebraic variety $\qquad G \subset \mathbb{K}[\lambda_1, \ldots, \lambda_l]$ its ideal

Rational action of $\mathcal{G}$ on $\mathbb{K}^n$

$$\lambda \star z = \left(\frac{p_1(\lambda, z)}{q(\lambda, z)}, \ldots, \frac{p_n(\lambda, z)}{q(\lambda, z)} \right)$$

$$q, p_1, \ldots, p_n \in \mathbb{K}[\lambda_1, \ldots, \lambda_l, z_1, \ldots, z_n]$$

Orbit $\mathcal{O}_z$ of $z \in \mathbb{K}^n$: the image of $\mathcal{G}$ under $\lambda \mapsto \lambda \star z$

Examples of Algebraic Groups

$$\mathrm{SL}_n(\mathbb{K}) = \{A \in \mathbb{K}^{n\times n} \mid \det A = 1\}$$

$$\mathrm{O}_n(\mathbb{K}) = \{A \in \mathbb{K}^{n\times n} \mid A\,{}^tA = I_n\}$$

$$\mathrm{GL}_n(\mathbb{K}) = \{A \in \mathbb{K}^{n\times n} \mid \det A \neq 0\}$$

naturally act on $\mathbb{K}^n$

Examples of rational actions of SL_2

The action of $\mathrm{SL}_2(\mathbb{C})$ on *forms* $z_0x^2 + z_1xy + z_2y^2$ of degree 2

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \star \begin{pmatrix} z_0 \\ z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} a^2 & ac & c^2 \\ 2ab & ad+bc & 2cd \\ b^2 & bd & d^2 \end{pmatrix} \begin{pmatrix} z_0 \\ z_1 \\ z_2 \end{pmatrix}$$

Projective action of $\mathrm{SL}_2(\mathbb{R})$ on quadrapules of $\mathbb{R}$:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \star \begin{pmatrix} z_0 \\ z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} \frac{a\,z_0+b}{c\,z_0+d} \\ \frac{a\,z_1+b}{c\,z_1+d} \\ \frac{a\,z_2+b}{c\,z_2+d} \\ \frac{a\,z_3+b}{c\,z_3+d} \end{pmatrix}$$

Action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathrm{M}_{2\times 2}$:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \star \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}$$

Rational invariants

$$\star: \mathcal{G} \times \mathcal{Z} \rightarrow \mathcal{Z} \qquad \mathcal{O}_z = \{\lambda \star z \mid \lambda \in \mathcal{G}\}$$

Rational invariant: $f \in \mathbb{K}(z_1, \ldots, z_n)$ s.t. $f(\lambda \star z) = f(z)$, $\forall \lambda \in \mathcal{G}$

Field of rational invariants: $\mathbb{K}(\mathcal{Z})^{\mathcal{G}}$ finitely generated

THM: $\mathbb{K}(\mathcal{Z})^{\mathcal{G}} = \mathbb{K}(r_1, \ldots, r_k) \Leftrightarrow \{r_1, \ldots, r_k\}$ separating

[Rosenlicht 56]

Separation property: For $z, z' \in \mathcal{Z} \setminus \mathcal{W}$

$$z' \in \mathcal{O}_z \quad \Leftrightarrow \quad r_1(z) = r_1(z'), \ldots, r_k(z) = r_k(z')$$

Section

Section of degree e :

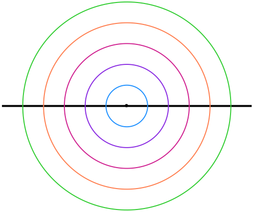
An irreducible variety $\mathcal{P}$ that intersects generic orbits in e points.

F.i. a generic affine space of complementary dimension to the orbit

Linear actions in the plane

$\mathcal{G} = \mathrm{SO}_2, \quad G = \left(\lambda^2 + \mu^2 - 1\right)$

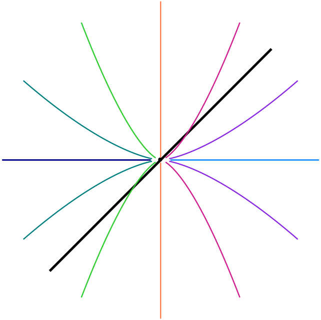
$\lambda \star \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \lambda & -\mu \\ \mu & \lambda \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$



$Q = \{Y, X^2 - (x^2 + y^2)\}$

$\mathcal{G} = \mathbb{K}^*, \quad G = (\lambda\,\mu - 1)$

$\lambda \star \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \lambda^2 z_1 \\ \lambda^3 z_2 \end{pmatrix}$



$$Q = \left\{ Y - X, X - \frac{x^3}{y^2} \right\}$$

Section & intersection ideal

Section of degree e :

An irreducible variety $\mathcal{P}$ that intersects generic orbits in e points.

F.i. a generic affine space of complementary dimension to the orbit

Intersection ideal:

$$I \subset \mathbb{K}(z_1, \dots, z_n)[Z_1, \dots, Z_n], \quad \dim_{\mathbb{K}} \mathbb{K}(z)[Z]/I = e$$

Under specialization $z_i \mapsto \bar{z}_i \in \mathbb{K}$ for $\bar{z} \in \mathcal{Z} \setminus \mathcal{W}$

$I_{\bar{z}} \subset \mathbb{K}[Z]$ is the ideal of $\mathcal{O}_{\bar{z}} \cap \mathcal{P}$

Prp: $I_{\lambda \star \bar{z}} = I_{\bar{z}}$

Intersection ideal as an elimination ideal

$$I = (G + (Z - \lambda \star z) + P) : q^\infty \cap \mathbb{K}(z)[Z]$$

Example :

$$G = (\lambda^2 + \mu^2 - 1), \quad (Z - \lambda \star z) = (X - \lambda x + \mu y,\ Y - \mu x - \lambda y), \quad P = (Y)$$

- $(Z - \lambda \star z) = (q(\lambda, z)\, Z_1 - p_1(\lambda, z), \ldots, q(\lambda, z)\, Z_n - p_n(\lambda, z))$ an ideal of $\mathbb{K}(z)[\lambda, Z]$ where

$$\lambda \star z = \left(\frac{p_1(\lambda, z)}{q(\lambda, z)}, \ldots, \frac{p_n(\lambda, z)}{q(\lambda, z)} \right)$$

- P a prime ideal in $\mathbb{K}[Z]$, $\mathcal{P} = \mathcal{V}(P)$ an irreducible variety of complementary dimension to the generic orbits

Invariants from the reduced Gröbner basis

$$I = (P + (Z - \lambda \star z) + G) : q^{\infty} \cap \mathbb{K}(z)[Z]$$

Q reduced Gröbner basis of I

$\{r_1, \ldots, r_k\}$ its coefficients

$I_{\lambda \star z} = I_z \;\Rightarrow\; r_i \subset \mathbb{K}(z)^{\mathcal{G}}$

Thm

$$\mathbb{K}(z)^{\mathcal{G}} = \mathbb{K}(r_1, \ldots, r_k)$$

Pf: Rewriting $\frac{p}{q} \in \mathbb{K}(z)^{\mathcal{G}}$

$y_1, \ldots, y_k$ a new indeterminates

$Q_y := Q(r_i \leftarrow y_i)$

$p(Z) \longrightarrow^*_{Q_y} \sum_\alpha a_\alpha(y)\, Z^\alpha$

$q(Z) \longrightarrow^*_{Q_y} \sum_\alpha b_\alpha(y)\, Z^\alpha$

$$\frac{p(z)}{q(z)} = \frac{a_\alpha(r)}{b_\alpha(r)}$$

[H. Kogan JSC 2007]

Rational sections of SL_2

- The action of $SL_2(\mathbb{C})$ on *forms* $z_0x^2 + z_1xy + z_2y^2$ of degree 2

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \star \begin{pmatrix} z_0 \\ z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} a^2 & ac & c^2 \\ 2ab & ad+bc & 2cd \\ b^2 & bd & d^2 \end{pmatrix} \begin{pmatrix} z_0 \\ z_1 \\ z_2 \end{pmatrix}$$

$$I = \left(\underbrace{Z_0 - 1,\ Z_1}_{P}, Z_2 + \frac{1}{4}\left(z_1^2 - 4\,z_0 z_2\right) \right)$$

- Projective action of $SL_2(\mathbb{R})$ on quadrapules of $\mathbb{R}$:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \star \begin{pmatrix} z_0 & z_1 & z_2 & z_3 \end{pmatrix} = \begin{pmatrix} \frac{a\,z_0+b}{c\,z_0+d} & \frac{a\,z_1+b}{c\,z_1+d} & \frac{a\,z_2+b}{c\,z_2+d} & \frac{a\,z_3+b}{c\,z_3+d} \end{pmatrix}$$

$$I = \left(\underbrace{Z_0^{-1}, Z_1, Z_2 - 1}_{P}, Z_3 - \frac{(z_0-z_2)(z_1-z_3)}{(z_0-z_3)(z_1-z_2)} \right)$$

Matrix similarity

Action: $\mathbb{K} = \mathbb{Q}, \mathbb{R}, \mathbb{C}$

$$\begin{array}{ccc} \mathrm{SL}_n(\mathbb{K}) \times \mathrm{M}_n(\mathbb{K}) & \rightarrow & \mathrm{M}_n(\mathbb{K}) \\ (P, M) & \mapsto & P^{-1}\, M\, P \end{array}$$

Invariants: The coefficients of the characteristic polynomial

$$\chi_0, \ldots, \chi_{n-1} \; : \mathrm{M}_n(\mathbb{K}) \rightarrow \mathbb{K}$$

Section: Companion matrices

$$\begin{pmatrix} . & . & . & \chi_0 \\ 1 & . & . & \chi_1 \\ . & \ddots & . & \vdots \\ . & . & 1 & \chi_{n-1} \end{pmatrix}$$

are normal forms for matrices M s.t. discr $\chi(M) \neq 0$

Algebraic invariants & replacement property

- $\mathcal{P}$ a section of degree $e = 1$

  $I = (\, Z_1 - g_1(z), \ldots, Z_n - g_n(z))$

$$r(z_1, \ldots, z_n) = r(g_1, \ldots, g_n) \qquad \forall r \in \mathbb{K}(z)^{\mathcal{G}}$$

- $\mathcal{P}$ a section of degree $e > 1$

  $I^G = I \cap \mathbb{K}(z)^{\mathcal{G}}[Z] = (Q)$ has e distinct $\overline{\mathbb{K}(z)}^G$-zeros

  $\xi = (\xi_1, \ldots, \xi_n)$ a $\overline{\mathbb{K}(z)}^G$-zero of I^G.

$$\forall r \in \mathbb{K}(z)^{\mathcal{G}}, r(z_1, \ldots, z_n) = r(\xi_1, \ldots, \xi_n)$$

$(\xi_1, \ldots, \xi_n)$ are the *normalized invariants* in the F&O'99 moving frame

[Hubert Kogan FoCM 07]

So far

- An algorithm to compute a generating set of rational invariants
- A way to rewrite any other invariants in terms of these
- An algebraic construction of the *normalized invariants*

. . . All relying on a choice of section

Sample of relevant problems

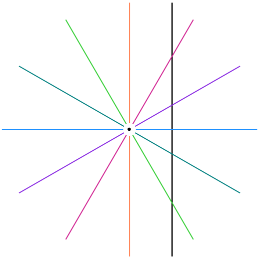
Finding sections of the lowest degree

More efficient algorithms for specific classes of group actions

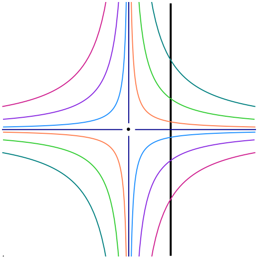
Section of degree 1 for scalings [Hubert Labahn FoCM 13]

$A = \begin{bmatrix} a & b \end{bmatrix}$

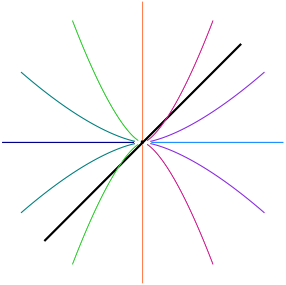
$$\begin{array}{rccc} \star : & \mathbb{K}^* \times \mathbb{K}^2 & \rightarrow & \mathbb{K}^2 \\ & (\lambda, (x, y)) & \mapsto & (\lambda^a x, \lambda^b y) \end{array}$$



$A = \begin{bmatrix} 1 & 1 \end{bmatrix}$



$A = \begin{bmatrix} 1 & -1 \end{bmatrix}$



$A = \begin{bmatrix} 2 & 3 \end{bmatrix}$

Dimensional analysis [Hubert Labahn FoCM 13]

Prey-predator model [Murray, Mathematical Biology (2002)]

$$\begin{cases} \dot{n} &= \left(\left(1-\frac{n}{k_1}\right) r - k_2 \frac{p}{n+e}\right) n, \\ \dot{p} &= s\left(1-h\frac{p}{n}\right) p. \end{cases}$$

$$\begin{cases} \dot{\mathfrak{n}} &= \left(1-\frac{\mathfrak{n}}{\mathfrak{k}} - \mathfrak{h}\frac{\mathfrak{p}}{\mathfrak{n}+1}\right) \mathfrak{n}, \\ \dot{\mathfrak{p}} &= \mathfrak{s}\left(1-\frac{\mathfrak{p}}{\mathfrak{n}}\right) \mathfrak{p}. \end{cases}$$

r, s, e, h, k_1, k_2 parameters.

$\mathfrak{s}, \mathfrak{h}, \mathfrak{k}$ parameters

$$\begin{array}{lllllllll} t & \to & \lambda^{-1}\, t, & r & \to & \lambda\, r, & h & \to & \nu\, h, \\ n & \to & \mu\, n, & s & \to & \lambda\, s, & k_1 & \to & \mu\, k_1, \\ p & \to & \mu\nu^{-1}\, p, & e & \to & \mu\, e, & k_2 & \to & \lambda\nu\, k_2 \end{array}$$

$$\mathfrak{t} = r\,t,\; \mathfrak{n} = \frac{n}{e},\; \mathfrak{p} = \frac{h\,p}{e},\; \mathfrak{s} = \frac{s}{r},\; \mathfrak{h} = \frac{k_2}{rh}, \mathfrak{k} = \frac{k_1}{e}.$$

THANKS!

- Rational Invariants. Construction and Rewriting.
  H. & Kogan, J. of Symbolic Coputation (2007)

- Smooth and Algebraic Invariants. Local and Global Constructions
  H. & Kogan, Foundations of Computational Mathematics (2007)

- Linear actions of $(\mathbb{K}^*)^m$ and parameter reduction.
  H. & Labahn, Foundations of Computational Mathematics (2013)

- Linear actions of finite abelian groups and solving polynomial systems:
  H. & Labahn, Mathematics of Computation (2016)

- $O(3)$ on $\mathbb{K}[x, y, z]_{2d}$ and neuroimaging
  Görlach, H. & Papadopoulo, Foundations of Computational Math. (2019)