

Homogeneous model of rigid surfaces in \mathbb{C}^2 and \mathbb{C}^3

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1 Dimension 2

1.1 Preparation

Consider \mathbb{C}^2 coordinated by z, w . Let $w = u + iv$. Consider a graphed surface $u = F(z, \bar{z})$ defined by an analytic function F . Consider the rigid transformation group

$$RT := \{(z' = f(z), w' = r w + h(z))\}$$

where f is a local biholomorphism and $r > 0$. The group acting on the set of surfaces by point-wise transformation.

Lemma 1.1. *The properties $F_{z,\bar{z}} \neq 0, F_{z^2,\bar{z}^2} \neq 0$ are invariant.*

Suppose $F_{z,\bar{z}} \neq 0, F_{z^2,\bar{z}^2} \neq 0$, then the surface with a fixed marked point p is, in a unique way, mapped to a unique normal form

$$u = z\bar{z} + \frac{1}{4}z^2\bar{z}^2 + \frac{Q}{2!3!}z^2\bar{z}^3 + \frac{Q}{2!3!}z^3\bar{z}^2 + \sum_{j,k \geq 2, j+k \geq 6} \frac{F_{j,k}}{j!k!}z^j\bar{z}^k$$

and p is mapped to the origin. Here Q is real. It represent an invariant of RT

$$Q := (F_{z,\bar{z}}F_{z^2,\bar{z}^2} - F_{z,\bar{z}^2}F_{z^2,\bar{z}})^{-3/2} \sqrt{E\bar{E}},$$

where

$$E := F_{z^3,\bar{z}^2}F_{z,\bar{z}}^2 - F_{z,\bar{z}}F_{z,\bar{z}^2}F_{z^3,\bar{z}} - 3F_{z,\bar{z}}F_{z^2,\bar{z}}F_{z^2,\bar{z}^2} + 3F_{z,\bar{z}^2}F_{z^2,\bar{z}}^2.$$

To get a homogeneous model we need to find 3 linearly independent vector fields tangent to the graph $\{u = F(z, \bar{z})\}$. We have $L_{triv} := \frac{\partial}{\partial v} = i \frac{\partial}{\partial w} - i \frac{\partial}{\partial \bar{w}}$ for free. It remains to find the other 2, induced by RT , in the form

$$L = f(z) \frac{\partial}{\partial z} + (h(z) + r w) \frac{\partial}{\partial w} + \text{conjugates}$$

satisfying $L(\frac{w+\bar{w}}{2} - F(z, \bar{z}))|_{\frac{w+\bar{w}}{2}=F(z, \bar{z})} = 0$. That is

$$\begin{aligned} f(z)F_z + \overline{f(z)}F_{\bar{z}} &= \frac{1}{2}h(z) + \frac{1}{2}\overline{h(z)} + r \frac{w+\bar{w}}{2} \Big|_{\frac{w+\bar{w}}{2}=F(z, \bar{z})} \\ &= \frac{1}{2}h(z) + \frac{1}{2}\overline{h(z)} + rF \end{aligned}$$

We compare the harmonic monomials $z^n + \bar{z}^n$ of both side.

$$f(0) \bar{z} + \overline{f(0)} z = \frac{1}{2} h(z) + \frac{1}{2} \overline{h(z)}.$$

Thus $h(z) = 2 \overline{f(0)} z + c_0$ is of degree at most 1.

$$\begin{aligned} L &= f(z) \frac{\partial}{\partial z} + \overline{f(z)} \frac{\partial}{\partial \bar{z}} + (c_0 + r w + 2 \overline{f(0)} z) \frac{\partial}{\partial w} + (\bar{c}_0 + r \bar{w} + 2 f(0) \bar{z}) \frac{\partial}{\partial \bar{w}} \\ &= f(z) \frac{\partial}{\partial z} + \overline{f(z)} \frac{\partial}{\partial \bar{z}} + (\operatorname{Re}(c_0) + r u + 2 \operatorname{Re}(\overline{f(0)} z)) \frac{\partial}{\partial u} + (\operatorname{Im}(c_0) + r v + 2 \operatorname{Im}(\overline{f(0)} z)) \frac{\partial}{\partial v}. \end{aligned}$$

From the tangency condition $L(u - F(z, \bar{z})) = 0$,

$$\begin{aligned} & f(z) \left(\bar{z} + \frac{1}{2} z \bar{z}^2 + \frac{Q}{3!} z \bar{z}^3 + \frac{Q}{4} z^2 \bar{z}^2 + \sum_{j,k \geq 2, j+k \geq 6} \frac{F_{j,k}}{(j-1)! k!} z^{j-1} \bar{z}^k \right) \\ &+ \overline{f(z)} \left(z + \frac{1}{2} z^2 \bar{z} + \frac{Q}{3!} z^3 \bar{z} + \frac{Q}{4} z^2 \bar{z}^2 + \sum_{j,k \geq 2, j+k \geq 6} \frac{F_{j,k}}{j! (k-1)!} z^j \bar{z}^{k-1} \right) \\ &= \operatorname{Re}(c_0) + \overline{f(0)} z + f(0) \bar{z} + r(z \bar{z} + \frac{1}{4} z^2 \bar{z}^2 + \frac{Q}{2! 3!} z^2 \bar{z}^3 + \frac{Q}{2! 3!} z^3 \bar{z}^2 + \sum_{j,k \geq 2, j+k \geq 6} \frac{F_{j,k}}{j! k!} z^j \bar{z}^k) \end{aligned}$$

Suppose $f(z) = \sum_{n=0}^{\infty} \frac{a_n}{n!} z^n$. We compare the coefficients

$$\begin{aligned} 1 : & \operatorname{Re}(c_0) = 0, \\ z \bar{z} : & r = a_1 + \bar{a}_1, \\ z^2 \bar{z} : & a_2 = -\bar{a}_0, \\ z^3 \bar{z} : & a_3 = -\bar{a}_0 Q, \\ z^2 \bar{z}^2 : & a_1 + \bar{a}_1 = -Q(a_0 + \bar{a}_0). \end{aligned}$$

Note that here we only solve $\operatorname{Re}(a_1)$ not a_1 . Moreover for any $k \geq 4$

$$z^k \bar{z} : a_k = -\bar{a}_0 F_{k,2}.$$

Thus only a_0 , $\operatorname{Im}(a_1)$ and $\operatorname{Im}(c_0)$ remain unsolved.

$$\begin{aligned} z^3 \bar{z}^2 : & \frac{3Q^2 - 2F_{4,2}}{24} a_0 + \frac{3Q^2 - 2F_{3,3} + 6}{24} \bar{a}_0 - \frac{iQ}{12} \operatorname{Im}(a_1) = 0, \\ z^2 \bar{z}^3 : & \frac{3Q^2 - 2F_{3,3} + 6}{24} a_0 + \frac{3Q^2 - 2F_{2,4}}{24} \bar{a}_0 + \frac{iQ}{12} \operatorname{Im}(a_1) = 0, \end{aligned}$$

Adding them together

$$\frac{3Q^2 - F_{3,3} - F_{4,2} + 3}{12} a_0 + \frac{3Q^2 - F_{2,4} - F_{3,3} + 3}{12} \bar{a}_0 = 0$$

1.2 Case $Q \neq 0$

In this case $Im(a_1)$ can always be solved in terms of $Re(a_0)$ and $Im(a_0)$.

Theorem 1.2. *A necessary condition for a surface in its normal form to be homogeneous is $3Q^2 - F_{3,3} - F_{4,2} + 3 = 0$ and $3Q^2 - F_{2,4} - F_{3,3} + 3 = 0$.*

Proof. If $3Q^2 - F_{3,3} - F_{4,2} + 3 \neq 0$ or $3Q^2 - F_{2,4} - F_{3,3} + 3 \neq 0$ then a_0 takes value in a real line of \mathbb{C} . There are 2 \mathbb{R} -linearly independent choices of a_0 and $Im(c_0)$, which give at most 2 \mathbb{R} -linearly independent vector fields tangent to the graph. The graph cannot be homogeneous. \square

This necessary condition coincides with Olver's recurrence relation

$$\begin{aligned} 0 = \mathcal{D}_z Q &= \frac{F_{4,2}}{2} - \frac{3}{2} - \frac{3Q^2}{2} + \frac{F_{3,3}}{2} \\ 0 = \mathcal{D}_{\bar{z}} Q &= \frac{F_{3,3}}{2} - \frac{3}{2} - \frac{3Q^2}{2} + \frac{F_{2,4}}{2} \end{aligned}$$

Now all the other coefficients of L can be expressed in terms of $Re(a_0)$ and $Im(a_0)$:

$$\begin{aligned} Re(a_1) &= -Q Re(a_0), \\ Im(a_1) &= -\frac{3Q^2 - 2F_{3,3} + 6}{Q} Im(a_0), \\ a_2 &= -\bar{a}_0, \\ a_3 &= -Q \bar{a}_0, \\ a_k &= -F_{k,2} \bar{a}_0, \quad (k \geq 4) \\ Re(c_0) &= 0, \\ r &= -2Q Re(a_0). \end{aligned}$$

By taking $(a_0, c_0) = (1, 0), (i, 0), (0, i)$ we obtain 3 linearly independent vector fields

$$\begin{aligned} L_1 &:= \left(1 - Qz - \frac{z^2}{2} - \frac{Qz^3}{3!} - \sum_{k=4}^{\infty} \frac{F_{k,2}}{k!} z^k\right) \frac{\partial}{\partial z} \\ &+ \left(1 - Q\bar{z} - \frac{\bar{z}^2}{2} - \frac{Q\bar{z}^3}{3!} - \sum_{k=4}^{\infty} \frac{F_{2,k}}{k!} \bar{z}^k\right) \frac{\partial}{\partial \bar{z}} \\ &+ (-2Qu + z + \bar{z}) \frac{\partial}{\partial u} + (-2Qv - iz + i\bar{z}) \frac{\partial}{\partial v} \\ L_2 &:= i \left(1 - \frac{3Q^2 - 2F_{3,3} + 6}{Q} z + \frac{z^2}{2} + \frac{Qz^3}{3!} + \sum_{k=4}^{\infty} \frac{F_{k,2}}{k!} z^k\right) \frac{\partial}{\partial z} \\ &- i \left(1 - \frac{3Q^2 - 2F_{3,3} + 6}{Q} \bar{z} + \frac{\bar{z}^2}{2} + \frac{Q\bar{z}^3}{3!} + \sum_{k=4}^{\infty} \frac{F_{2,k}}{k!} \bar{z}^k\right) \frac{\partial}{\partial \bar{z}} \\ &+ (-iz + i\bar{z}) \frac{\partial}{\partial u} + (-z - \bar{z}) \frac{\partial}{\partial v} \\ L_3 &:= \frac{\partial}{\partial v} \end{aligned}$$

Theorem 1.3. *A necessary and sufficient condition for a surface in its normal form to be homogeneous is $\{L_1, L_2, L_3\}$ Frobenius integrable.*

Proof. If the 3 dimensional surface is homogeneous then $\{L_1, L_2, L_3\}$ generates a rank 3 sub-algebra of the Lie algebra of RT . Their brackets are linear combinations of L_1, L_2 and L_3 with constant coefficients. If $\{L_1, L_2, L_3\}$ is Frobenius integrable, then there exists a unique 3 dimensional surface M passing the origin, $\{L_1, L_2, L_3\}$ tangent to M and $span(L_1, L_2, L_3)$ translate points on M . So M is homogeneous. \square

In fact $F_{3,3}$ can also be solved in terms of Q . In the tangency conditions $L_1(u - F(z, \bar{z})) = 0$ we compare the coefficients

$$\begin{aligned} z^2 \bar{z}^4 : \quad & \frac{Q^3}{4} - \frac{Q F_{3,3}}{12} + \frac{11 Q}{24} - \frac{F_{2,5}}{48} - \frac{F_{3,4}}{48} = 0, \\ z^3 \bar{z}^3 : \quad & -\frac{F_{4,3}}{36} + \frac{Q F_{3,3}}{9} + \frac{Q}{6} - \frac{F_{3,4}}{36} = 0, \\ z^4 \bar{z}^2 : \quad & \frac{Q^3}{4} - \frac{Q F_{3,3}}{12} + \frac{11 Q}{24} - \frac{F_{4,3}}{48} - \frac{F_{5,2}}{48} = 0. \end{aligned}$$

And from $L_2(u - F(z, \bar{z})) = 0$:

$$\begin{aligned} z^2 \bar{z}^4 : \quad & i \left(-\frac{3 Q^3}{8} + \frac{3 Q F_{3,3}}{8} - \frac{11 Q}{12} + \frac{F_{2,5}}{48} - \frac{F_{3,4}}{48} - \frac{F_{3,3}^2}{12 Q} + \frac{F_{3,3}}{2 Q} - \frac{3}{4 Q} \right) = 0, \\ z^3 \bar{z}^3 : \quad & i \left(-\frac{F_{4,3}}{36} + \frac{F_{3,4}}{36} \right) = 0, \\ z^4 \bar{z}^2 : \quad & i \left(\frac{3 Q^3}{8} - \frac{3 Q F_{3,3}}{8} + \frac{11 Q}{12} + \frac{F_{4,3}}{48} - \frac{F_{5,2}}{48} + \frac{F_{3,3}^2}{12 Q} - \frac{F_{3,3}}{2 Q} + \frac{3}{4 Q} \right) = 0. \end{aligned}$$

To solve this system, first, we eliminate the 7th order coefficients $F_{2,5}, F_{3,4}, F_{4,3}, F_{5,2}$ to solve $F_{3,3}$.

$$2 F_{3,3}^2 - (5 Q^2 + 12) F_{3,3} + 3 Q^4 + 14 Q^2 + 18 = 0.$$

There are two roots for each $Q \in \mathbb{R}$:

$$F_{3,3} = \frac{5 Q^2}{4} + 3 + \epsilon \frac{Q \sqrt{Q^2 + 8}}{4}, \quad \epsilon = \pm 1.$$

Then we solve the 7th order coefficients

$$\begin{aligned} F_{2,5} = F_{5,2} &= 12 Q^3 - 6 Q F_{3,3} + 19 Q, \\ F_{3,4} = F_{4,3} &= 2 Q F_{3,3} + 3 Q. \end{aligned}$$

Such procedure can calculate all the coefficients $F_{j,k}$ of a homogeneous surface if it exists.

Denote by \mathfrak{g} the Lie algebra spanned by L_1, L_2, L_3 . To prove the existence without infinite calculation, we calculate the derived algebra $\mathfrak{g}^{(1)}$ by calculating the brackets among L_1, L_2, L_3 .

$$\begin{aligned} [L_1, L_2] &= (\dots) \frac{\partial}{\partial z} + (\dots) \frac{\partial}{\partial \bar{z}} + \frac{Q + \epsilon \sqrt{Q^2 + 8}}{2} (-i z + i \bar{z}) \frac{\partial}{\partial u} \\ &\quad + \frac{Q + \epsilon \sqrt{Q^2 + 8}}{2} (-z - \bar{z}) \frac{\partial}{\partial v} - 4 \frac{\partial}{\partial v} \\ [L_1, L_3] &= 2 Q L_3, \quad [L_2, L_3] = 0, \end{aligned}$$

From Frobenius integrability

$$\begin{aligned} [L_1, L_2] &= \frac{Q + \epsilon \sqrt{Q^2 + 8}}{2} L_2 - 4 L_3, \\ [L_1, L_3] &= 2 Q L_3, \quad [L_2, L_3] = 0, \end{aligned}$$

Thus $g^{(1)}$ has rank 2, generated by L_2, L_3 . Write $P := \frac{Q + \epsilon \sqrt{Q^2 + 8}}{2}$. The adjoint map

$$ad_{L_1} : g^{(1)} \longrightarrow g^{(1)}$$

is an isomorphism. Under the basis $\{L_2, L_3\}$ it has matrix form

$$ad_{L_1} = \begin{pmatrix} P & -4 \\ 0 & 2Q \end{pmatrix}.$$

Using Adam Bowers' classification of three-dimensional real Lie algebras, there are 2 cases.

1. If $P \neq 2Q$ then ad_{L_1} is diagonalizable with two real distinct eigenvalues P and $2Q$. Case $L(3, 2, l)$ with $l = P/2Q = \frac{Q + \epsilon \sqrt{Q^2 + 8}}{4Q} \in (-\infty, 0) \cup (1/2, 1) \cup (1, +\infty)$.
2. If $P = 2Q$ then there are 2 solutions $P = 2, Q = 1, \epsilon = 1$ and $P = -2, Q = -1, \epsilon = -1$. Case $L(3, 3)$.

1.2.1 Case $P \neq 2Q$

By taking $L'_2 = L_2 + \frac{4}{2Q-P} L_3$ and $L'_1 = \frac{1}{2Q} L_1$

$$[L'_1, L'_2] = l L'_2, \quad [L'_1, L_3] = L_3, \quad [L'_2, L_3] = 0,$$

Recall

$$\begin{aligned} L_1 &= (\dots) \frac{\partial}{\partial z} + (2z - 2Qw) \frac{\partial}{\partial w} + (\text{conjugates}), \\ L_2 &= (\dots) \frac{\partial}{\partial z} - 2iz \frac{\partial}{\partial w} + (\text{conjugates}), \\ L_3 &= i \frac{\partial}{\partial w} + (\text{conjugates}), \end{aligned}$$

so

$$\begin{aligned} L'_1 &= (\dots) \frac{\partial}{\partial z} + \left(\frac{z}{Q} - w\right) \frac{\partial}{\partial w} + (\text{conjugates}), \\ L'_2 &= (\dots) \frac{\partial}{\partial z} + \left(-2iz + \frac{4i}{2Q-P}\right) \frac{\partial}{\partial w} + (\text{conjugates}), \\ L_3 &= i \frac{\partial}{\partial w} + (\text{conjugates}). \end{aligned}$$

We want to simplify these expressions by a local rigid transformation.

Theorem 1.4. For a general vector field

$$L := A(z) \frac{\partial}{\partial z} + (B(z) + C w) \frac{\partial}{\partial w} + (\text{conjugates}),$$

after a rigid transformation

$$\begin{cases} \tilde{z} = f(z), \\ \tilde{w} = g(z) + \lambda w, \end{cases} \quad \begin{cases} z = f^{-1}(\tilde{z}), \\ w = -\lambda^{-1} g \circ f^{-1}(\tilde{z}) + \lambda^{-1} \tilde{w}. \end{cases}$$

in its new coordinates

$$L = \tilde{A}(\tilde{z}) \frac{\partial}{\partial \tilde{z}} + (\tilde{B}(\tilde{z}) + \tilde{C} \tilde{w}) \frac{\partial}{\partial \tilde{w}} + (\text{conjugates}).$$

one has

$$\tilde{A} = (A f') \circ f^{-1}, \quad \tilde{B} = (A g' + B \lambda - C g) \circ f^{-1}, \quad \tilde{C} = C.$$

Proof. To calculate $\tilde{A}, \tilde{B}, \tilde{C}$ we try

$$L(\tilde{z}) = \tilde{A}(\tilde{z}) = L(f(z)) = A(z) f'(z) = A \circ f^{-1}(\tilde{z}) f' \circ f^{-1}(\tilde{z}),$$

and

$$\begin{aligned} L(\tilde{w}) &= \tilde{B}(\tilde{z}) + \tilde{C} \tilde{w} \\ &= L(g(z) + \lambda w) = A(z) g'(z) + (B(z) + C w) \lambda \\ &= A \circ f^{-1}(\tilde{z}) g' \circ f^{-1}(\tilde{z}) + B \circ f^{-1}(\tilde{z}) \lambda - C g \circ f^{-1}(\tilde{z}) + C \tilde{w}, \end{aligned}$$

and we conclude. □

First rigid transformation Dropping primes, for

$$\begin{aligned} L_1 &= A_1(z) \frac{\partial}{\partial z} + \left(\frac{z}{Q} - w\right) \frac{\partial}{\partial w} + (\text{conjugates}), \\ L_2 &= A_2(z) \frac{\partial}{\partial z} + \left(-2iz + \frac{4i}{2Q - P}\right) \frac{\partial}{\partial w} + (\text{conjugates}), \\ L_3 &= i \frac{\partial}{\partial w} + (\text{conjugates}), \\ [L_1, L_2] &= l L_2, \quad [L_1, L_3] = L_3, \quad [L_2, L_3] = 0, \end{aligned}$$

with $A_1(0) \neq 0, A_2(z) \neq 0$. Consider a rigid transformation

$$\begin{cases} \tilde{z} = f(z), \\ \tilde{w} = g(z) + w, \end{cases}$$

where $f(z)$ be a solution of $A_1(z) f'(z) = 1$ and $g(z)$ be a solution of $A_1(z) g'(z) + \frac{z}{Q} + g(z) = 0$. Such two solutions locally exists since $A_1(0) \neq 0$. Then under the new coordinates

$$\begin{aligned} L_1 &= \frac{\partial}{\partial \tilde{z}} - \tilde{w} \frac{\partial}{\partial \tilde{w}} + (\text{conjugates}), \\ L_2 &= \tilde{A}_2(\tilde{z}) \frac{\partial}{\partial \tilde{z}} + \tilde{B}_2(\tilde{z}) \frac{\partial}{\partial \tilde{w}} + (\text{conjugates}), \\ L_3 &= i \frac{\partial}{\partial \tilde{w}} + (\text{conjugates}), \end{aligned}$$

with $\tilde{A}_2(0) \neq 0$. We drop the tildes and calculate the Lie brackets

$$\begin{aligned} [L_1, L_3] &= L_3, & [L_2, L_3] &= 0, \\ [L_1, L_2] &= A_2'(z) \frac{\partial}{\partial z} + (B_2(z) + B_2'(z)) \frac{\partial}{\partial w} + \text{conjugates}, \end{aligned}$$

and compare with $[L_1, L_2] = l L_2$. We get

$$\begin{aligned} A_2'(z) &= l A_2(z) \\ B_2'(z) + B_2(z) &= l B_2(z) \end{aligned}$$

Solving these ODEs

$$\begin{aligned} A_2(z) &= C_1 e^{lz} \\ B_2(z) &= C_2 e^{(l-1)z} \end{aligned}$$

where $C_1 = A_2(0) = r e^{i\theta} \neq 0$. We get the Lie algebra

$$\begin{aligned} L_1 &= \frac{\partial}{\partial z} - w \frac{\partial}{\partial w} + (\text{conjugates}), \\ L_2 &= r e^{lz+i\theta} \frac{\partial}{\partial z} + C_2 e^{(l-1)z} \frac{\partial}{\partial w} + (\text{conjugates}), \\ L_3 &= i \frac{\partial}{\partial w} + (\text{conjugates}). \end{aligned}$$

We may assume $|C_1| = r = 1$ after scaling L_2 .

Second rigid transformation We apply another rigid transformation $\tilde{z} = z$, $\tilde{w} = g(z) + w$. Under new coordinates

$$L_1 = \frac{\partial}{\partial \tilde{z}} + (g'(z) + g(z) - \tilde{w}) \frac{\partial}{\partial \tilde{w}} + (\text{conjugates}),$$

We let $g(z) = C e^{-z}$ to preserve the simple expression of L'_1 . Then

$$\begin{aligned} L_2 &= e^{l\tilde{z}+i\theta} \frac{\partial}{\partial \tilde{z}} + (-e^{i\theta} C + C_2) e^{(l-1)\tilde{z}} \frac{\partial}{\partial \tilde{w}} + (\text{conjugates}), \\ L_3 &= i \frac{\partial}{\partial \tilde{w}} + (\text{conjugates}). \end{aligned}$$

Take $C = e^{-i\theta} C_2$ and drop tildes we get nice expressions

$$\begin{aligned} L_1 &= \frac{\partial}{\partial z} - w \frac{\partial}{\partial w} + (\text{conjugates}) = \frac{\partial}{\partial x} - u \frac{\partial}{\partial u}, \\ L_2 &= e^{lz+i\theta} \frac{\partial}{\partial z} + (\text{conjugates}) = e^{lx} \cos(l y + \theta) \frac{\partial}{\partial x} + e^{lx} \sin(l y + \theta) \frac{\partial}{\partial y} \\ L_3 &= i \frac{\partial}{\partial w} + (\text{conjugates}) = \frac{\partial}{\partial v}. \end{aligned}$$

where $z = x + iy$ and $w = u + iv$. We may assume the surface is a graph $u = F(x, y, v)$.

From $L_3(u - F(x, y, v)) = 0$ we know $F_v = 0$ so $u = F(x, y)$.

From $L_1(u - F(x, y)) = 0$ we know $-F - F_x = 0$ so $u = F(y) e^{-x}$.

Now calculate $L_2(u - F(y) e^{-x}) = 0$ we get

$$-\cos(l y + \theta) F(y) + \sin(l y + \theta) F'(y) = 0$$

so $F(y) = C_0 \sin(l y + \theta)^{1/l}$ for some $C_0 \in \mathbb{R}$. The graphed surface $u = C_0 e^{-x} \sin(l y + y_0)^{1/l}$ is rigidly equivalent to $u = e^{-x} \sin(l y)^{1/l}$.

For the surface $u = F(z, \bar{z}) := e^{\frac{-z-\bar{z}}{2}} \sin\left(\frac{-iz+i\bar{z}}{2}\right)^{1/l}$, even though it is not defined in a neighbourhood of the origin for general l , one can still calculate its invariant Q by using

$$Q^2 = (F_{z,\bar{z}} F_{z^2,\bar{z}^2} - F_{z,\bar{z}^2} F_{z^2,\bar{z}})^{-3} E \bar{E},$$

$$E = F_{z^3,\bar{z}^2} F_{z,\bar{z}}^2 - F_{z,\bar{z}} F_{z,\bar{z}^2} F_{z^3,\bar{z}} - 3 F_{z,\bar{z}} F_{z^2,\bar{z}} F_{z^2,\bar{z}^2} + 3 F_{z,\bar{z}^2} F_{z^2,\bar{z}}.$$

It turns out that $Q^2 = \frac{1}{(2l-1)l}$ is a constant. So the surface is homogeneous.

Note that even though the real Lie algebras $L(2, 3, l)$ and $L(2, 3, 1/l)$ are isomorphic, they are not rigidly equivalent since switching z and w is not a rigid transformation.

1.2.2 Case $P = 2Q$

Recall

$$L_1 = (\dots) \frac{\partial}{\partial z} + (2z - 2Qw) \frac{\partial}{\partial w} + (\text{conjugates}),$$

$$L_2 = (\dots) \frac{\partial}{\partial z} - 2iz \frac{\partial}{\partial w} + (\text{conjugates}),$$

$$L_3 = i \frac{\partial}{\partial w} + (\text{conjugates}),$$

When $P = 2, Q = 1, \epsilon = 1$,

$$[L_1, L_2] = 2L_2 - 4L_3, \quad [L_1, L_3] = 2L_3, \quad [L_2, L_3] = 0.$$

By taking $L'_1 = \frac{1}{2} L_1$ and $L'_3 = -2L_3$

$$[L'_1, L_2] = L_2 + L'_3, \quad [L'_1, L'_3] = L'_3, \quad [L_2, L'_3] = 0,$$

When $P = -2, Q = -1, \epsilon = -1$,

$$[L_1, L_2] = -2L_2 - 4L_3, \quad [L_1, L_3] = -2L_3, \quad [L_2, L_3] = 0.$$

By taking $L'_1 = -\frac{1}{2} L_1$ and $L'_3 = 2L_3$

$$[L'_1, L_2] = L_2 + L'_3, \quad [L'_1, L'_3] = L'_3, \quad [L_2, L'_3] = 0,$$

Dropping primes, it suffices to treat the case

$$L_1 = A_1(z) \frac{\partial}{\partial z} + (B_1(z) - w) \frac{\partial}{\partial w} + (\text{conjugates}),$$

$$L_2 = A_2(z) \frac{\partial}{\partial z} + B_2(z) \frac{\partial}{\partial w} + (\text{conjugates}),$$

$$L_3 = -2\epsilon i \frac{\partial}{\partial w} + (\text{conjugates}),$$

$$[L_1, L_2] = L_2 + L_3, \quad [L_1, L_3] = L_3, \quad [L_2, L_3] = 0.$$

First rigid transformation Consider a rigid transformation

$$\begin{cases} \tilde{z} = f(z), \\ \tilde{w} = g(z) + w, \end{cases}$$

where $f(z)$ be a solution of $A_1(z) f'(z) = 1$ and $g(z)$ be a solution of $A_1(z) g'(z) + B_1(z) + g(z) = 0$. Such two solutions locally exists since $A_1(0) \neq 0$. Then under the new coordinates

$$\begin{aligned} L_1 &= \frac{\partial}{\partial \tilde{z}} - \tilde{w} \frac{\partial}{\partial \tilde{w}} + (\text{conjugates}), \\ L_2 &= \tilde{A}_2(\tilde{z}) \frac{\partial}{\partial \tilde{z}} + \tilde{B}_2(\tilde{z}) \frac{\partial}{\partial \tilde{w}} + (\text{conjugates}), \\ L_3 &= -2\epsilon i \frac{\partial}{\partial \tilde{w}} + (\text{conjugates}), \end{aligned}$$

with $\tilde{A}_2(0) \neq 0$. We drop the tildes and calculate the Lie brackets

$$\begin{aligned} [L_1, L_3] &= L_3, \quad [L_2, L_3] = 0, \\ [L_1, L_2] &= A_2'(z) \frac{\partial}{\partial z} + (B_2(z) + B_2'(z)) \frac{\partial}{\partial w} + \text{conjugates}, \end{aligned}$$

and compare with $[L_1, L_2] = L_2 + L_3$. We get

$$\begin{aligned} A_2'(z) &= A_2(z) \\ B_2(z) + B_2'(z) &= B_2(z) - 2\epsilon i \end{aligned}$$

Solving these ODEs

$$\begin{aligned} A_2(z) &= C_1 e^z \\ B_2(z) &= -2\epsilon i z + C_2 \end{aligned}$$

where $C_1 = A_2(0) = r e^{i\theta} \neq 0$. We get the Lie algebra

$$\begin{aligned} L_1 &= \frac{\partial}{\partial z} - w \frac{\partial}{\partial w} + (\text{conjugates}), \\ L_2 &= r e^{z+i\theta} \frac{\partial}{\partial z} + (-2\epsilon i z + C_2) \frac{\partial}{\partial w} + (\text{conjugates}), \\ L_3 &= -2\epsilon i \frac{\partial}{\partial w} + (\text{conjugates}). \end{aligned}$$

Second rigid transformation We apply another rigid transformation $\tilde{z} = z + i\theta$, $\tilde{w} = g(z) + w$. Under new coordinates

$$L_1 = \frac{\partial}{\partial \tilde{z}} + (g'(z) + g(z) - \tilde{w}) \frac{\partial}{\partial \tilde{w}} + (\text{conjugates}),$$

We let $g(z) = C e^{-z}$ to preserve the simple expression of L_1 . Then

$$\begin{aligned} L_2 &= r e^{\tilde{z}} \frac{\partial}{\partial \tilde{z}} + (-2\epsilon i \tilde{z} - 2\epsilon\theta - r e^{i\theta} C + C_2) \frac{\partial}{\partial \tilde{w}} + (\text{conjugates}), \\ L_3 &= -2\epsilon i \frac{\partial}{\partial \tilde{w}} + (\text{conjugates}). \end{aligned}$$

Take $C = r^{-1} e^{-i\theta} (C_2 - 2\epsilon\theta)$ and drop tildes we get nice expressions

$$\begin{aligned} L_1 &= \frac{\partial}{\partial z} - w \frac{\partial}{\partial w} + (\text{conjugates}) = \frac{\partial}{\partial x} - u \frac{\partial}{\partial u}, \\ L_2 &= r e^z \frac{\partial}{\partial z} - 2\epsilon i z \frac{\partial}{\partial w} (\text{conjugates}) \\ &= r e^x \cos(y) \frac{\partial}{\partial x} + r e^x \sin(y) \frac{\partial}{\partial y} + 2\epsilon y \frac{\partial}{\partial u} - 2\epsilon x \frac{\partial}{\partial v}, \\ L_3 &= -2\epsilon i \frac{\partial}{\partial w} + (\text{conjugates}) = -2\epsilon \frac{\partial}{\partial v}. \end{aligned}$$

where $z = x + iy$ and $w = u + iv$. We may assume the surface is a graph $u = F(x, y, v)$.

From $L_3(u - F(x, y, v)) = 0$ we know $F_v = 0$ so $u = F(x, y)$.

From $L_1(u - F(x, y)) = 0$ we know $-F - F_x = 0$ so $u = F(y) e^{-x}$.

Now calculate $L_2(u - F(y) e^{-x}) = 0$ we get

$$-r \cos(y) F(y) + r \sin(y) F'(y) - 2\epsilon y = 0$$

so

$$F(y) = \frac{\epsilon}{r} (-2y \cos(y) + 2 \sin(y) \ln(\sin(y))) + C_0 \sin(y)$$

for some $C_0 \in \mathbb{R}$. The surface graphed by

$$u = e^{-x} \frac{\epsilon}{r} (-2y \cos(y) + 2 \sin(y) \ln(\sin(y))) + C_0 e^{-x} \sin(y),$$

after a rigid transformation $w \longrightarrow \frac{1}{r}(w - i C_0 e^{-z})$, is equivalent to

$$u = e^{-x} \epsilon (-2y \cos(y) + 2 \sin(y) \ln(\sin(y)))$$

Its invariant Q^2 is constant 1. It is homogeneous.

1.3 Case $Q = 0$

Then E is constant 0 which implies

$$F_{z^3 \bar{z}^2} \equiv \frac{F_{z, \bar{z}} F_{z \bar{z}^2} F_{z^3 \bar{z}} + 3 F_{z, \bar{z}} F_{z^2 \bar{z}} F_{z^2 \bar{z}^2} - 3 F_{z, \bar{z}^2} F_{z^2 \bar{z}}^2}{F_{z, \bar{z}}^2}$$

holds everywhere on the surface.

For simplicity we write as

$$D_{3,2} \equiv \frac{D_{1,1} D_{1,2} D_{3,1} + 3 D_{1,1} D_{2,1} D_{2,2} - 3 D_{1,2} D_{2,1}^2}{D_{1,1}^2}$$

a careful reader shall not be confused among the functions $D_{j,k}(z, w)$ and the constants $F_{j,k}$ introduced before. When valued at the origin $D_{j,k}(0, 0) = F_{j,k}$.

Lemma 1.5. *One has $F_{j,k} = 0$ whenever $j \neq k$.*

Proof. We do an induction on $j + k =: d$. It is true for $d \leq 5$. Suppose $F_{j,k} = 0$ for all $j \neq k$, $j + k \leq d$. Clearly $F_{0,d+1} = F_{1,d} = \frac{F_{d,1}}{d} = F_{d+1,0} = 0$. It suffice to calculate $F_{j,k}$ with $j, k \geq 2$, $j + k = d + 1$ and $j \neq k$. Since $F_{j,k} = \frac{F_{k,j}}{k}$ we only treat the case $D_{j,k}$ with $j > k$. Thus $j > \frac{d+1}{2} = 3$ and $k \geq 2$.

The function $D_{j,k}$ can be obtained by differentiating $D_{3,2}$. It has the form

$$D_{j,k} = \sum_{a=1}^N c_a D_{1,1}^{-p(a)} \prod_{b=1}^{M(a)} D_{j_{a,b}, k_{a,b}}$$

for certain positive integers $N, M(a), p(a), j_{a,b}, k_{a,b}$ and certain constant c_a . This expression is unique up to permutations. We call each summand a monomial. For each monomial

$$S(a) := c_a D_{1,1}^{-p(a)} \prod_{b=1}^{M(a)} D_{j_{a,b}, k_{a,b}}$$

we define its numerical indices

$$\begin{aligned} \text{order}(S(a)) &:= \max\{j_{a,b} + k_{a,b} | b = 1, \dots, M(a)\}, \\ \text{difference}(S(a)) &:= \sum_{b=1}^{M(a)} j_{a,b} - k_{a,b}. \end{aligned}$$

For example in the expression of

$$D_{3,2} = D_{1,1}^{-1} D_{1,2} D_{3,1} + 3 D_{1,1}^{-1} D_{2,1} D_{2,2} - 3 D_{1,1}^{-2} D_{1,2} D_{2,1}^2$$

	$S(1)$	$S(2)$	$S(3)$
order	4	4	3
difference	1	1	1

and in the expression of

$$\begin{aligned} D_{4,2} &= -7 D_{1,1}^{-2} D_{2,1} D_{1,2} D_{3,1} - 6 D_{1,1}^{-2} D_{2,1}^2 D_{2,2} + 6 D_{1,1}^{-3} D_{1,2} D_{2,1}^3 \\ &\quad + 4 D_{1,1}^{-1} D_{2,2} D_{3,1} + 3 D_{1,1}^{-1} D_{3,2} D_{2,1} + D_{1,1}^{-1} D_{4,1} D_{1,2} \end{aligned}$$

	$S(1)$	$S(2)$	$S(3)$	$S(4)$	$S(5)$	$S(6)$
order	4	4	3	4	5	5
difference	2	2	2	2	2	2

Taking total differential D_z raises the order by at most 1 and the difference by 1. Taking $D_{\bar{z}}$ raises the order by at most 1 and the difference by -1. Thus

$$\text{order}(S(a)) \leq j + k - 1 = d, \quad \text{difference}(S(a)) = j - k \neq 0,$$

for all monomials. Thus $S(a)$ contains at least one factor $D_{j_{a,b}, k_{a,b}}$ with $j_{a,b} + k_{a,b} \leq d$ and $j_{a,b} \neq k_{a,b}$. By inductive assumption such $D_{j_{a,b}, k_{a,b}}$ is 0 when valued at origin. Hence $S(a) = 0$ when valued at the origin, so does $D_{j,k}$. So $F_{j,k} = 0$ when $j + k = d + 1$. \square

Now our graph function becomes

$$u = F(z, \bar{z}) = z\bar{z} + \frac{1}{4}z^2\bar{z}^2 + \sum_{j \geq 3}^{\infty} \frac{F_{j,j}}{j!j!} z^j \bar{z}^j = G(z\bar{z}).$$

We apply the vector field

$$L = f(z) \frac{\partial}{\partial z} + \overline{f(z)} \frac{\partial}{\partial \bar{z}} + (c_{0r} + r u) \frac{\partial}{\partial u} + (\dots) \frac{\partial}{\partial v}$$

We get

$$f(z) \bar{z} G'(z\bar{z}) + \overline{f(z)} z G'(z\bar{z}) = c_{0r} + r G(z\bar{z})$$

So $f(z) \bar{z} + \overline{f(z)} z$ should depend only on $z\bar{z}$. We conclude that $f(z) = a_1 z$ and

$$(a_1 + \bar{a}_1) z \bar{z} G'(z\bar{z}) = c_{0r} + r G(z\bar{z})$$

It amounts to solve an ODE

$$A x y'(x) = B + C y(x),$$

whose solution is $y(x) = -\frac{B}{C} + C_0 x^{\frac{C}{A}}$ for $C_0 \in \mathbb{R}$. In our case we get

$$G(z\bar{z}) = -\frac{c_{0r}}{r} + C_0 (z\bar{z})^{\frac{r}{a_1 + \bar{a}_1}}.$$

The surface graphed by $u = G(z\bar{z})$ is equivalent to the graph of

$$u = (z\bar{z})^\gamma \quad (\gamma \in \mathbb{R}),$$

which admits 2 linearly independent vector fields:

$$L_1 := i t z \frac{\partial}{\partial z} - i t \bar{z} \frac{\partial}{\partial \bar{z}} \quad L_2 := z \frac{\partial}{\partial z} + \bar{z} \frac{\partial}{\partial \bar{z}} + 2 \gamma u \frac{\partial}{\partial u} + 2 \gamma v \frac{\partial}{\partial v}.$$

In particular this surface is analytic at the origin if and only if $\gamma \in \mathbb{Z}_{\geq 0}$. This surface does not satisfy the condition $F_{z,\bar{z}} \neq 0$ and $F_{z^2,\bar{z}^2} \neq 0$. So there is no homogeneous model under in this branch.

1.4 Surface with vanishing Q

In fact when $Q \equiv 0$, i.e. $E \equiv 0$, the surface, by Lemma 1.5, is graphed by $u = G(z\bar{z})$. We calculate E and we will get an ODE for G of order 5:

$$\begin{aligned} & G''(z\bar{z})^2 z^4 \bar{z}^5 G''''(z\bar{z}) - 2 G''(z\bar{z})^2 z^3 \bar{z}^4 G''''(z\bar{z}) + 6 G''(z\bar{z})^2 z^2 \bar{z}^3 G''''(z\bar{z}) \\ & + G'(z\bar{z})^2 G''''(z\bar{z}) z^2 \bar{z}^3 + 6 G'(z\bar{z})^2 G''''(z\bar{z}) z \bar{z}^2 + 3 G''(z\bar{z}) z^3 \bar{z}^4 G''''(z\bar{z})^2 \\ & - 15 G'(z\bar{z}) G''''(z\bar{z})^2 z^2 \bar{z}^3 + 2 G''(z\bar{z}) z^3 \bar{z}^4 G'(z\bar{z}) G''''(z\bar{z}) + 4 G''(z\bar{z}) z^2 \bar{z}^3 G'(z\bar{z}) G''''(z\bar{z}) \\ & - 24 G''(z\bar{z}) z \bar{z}^2 G'(z\bar{z}) G''''(z\bar{z}) - 4 G''(z\bar{z}) z^4 \bar{z}^5 G''''(z\bar{z}) G''''(z\bar{z}) \\ & - 4 G'(z\bar{z}) G''''(z\bar{z}) z^3 \bar{z}^4 G''''(z\bar{z}) + 6 G'(z\bar{z})^2 G''''(z\bar{z}) \bar{z} + 12 G''(z\bar{z})^3 z \bar{z}^2 \\ & - 12 G'(z\bar{z}) G''''(z\bar{z})^2 \bar{z} + 3 G''''(z\bar{z})^3 z^4 \bar{z}^5 = 0 \end{aligned}$$

After dividing by \bar{z} we obtain a real ODE of order 5.

$$\begin{aligned} & 3 G'''(x)^3 x^4 - 4 G'''(x) G''(x) G''''(x) x^4 + G''(x)^2 G''''(x) x^4 - 4 G'(x) G'''(x) G''''(x) x^3 \\ & + 2 G'(x) G''(x) G''''(x) x^3 + 3 G'''(x)^2 G''(x) x^3 - 2 G''(x)^2 G''''(x) x^3 + G'(x)^2 G''''(x) x^2 \\ & - 15 G'(x) G'''(x)^2 x^2 + 4 G'(x) G''(x) G''''(x) x^2 + 6 G'''(x) G''(x)^2 x^2 + 6 G'(x)^2 G''''(x) x \\ & - 24 G'(x) G'''(x) G''(x) x + 12 G''(x)^3 x + 6 G'(x)^2 G''''(x) - 12 G'(x) G''(x)^2 = 0 \end{aligned}$$

From the initial value $G(0) = 0, G'(0) = 1, G''(0) = 1/2$ one can solve the Taylor coefficients of $G(x)$ inductively. We assume $G(x) = \sum_{n=1}^{\infty} G_n x^n$ with $G_1 = 1$ and $G_2 = 1/4$. We calculate E and from the vanishing of all coefficients of $z^j \bar{z}^k$ we obtain

$$\begin{aligned} G_3 &= 1/12, & G_4 &= 1/32, & G_5 &= 1/80, & G_6 &= 1/192, \\ G_7 &= 1/448, & G_8 &= 1/1024, & G_9 &= 1/2304, & G_{10} &= 1/5120, \\ G_{11} &= 1/11264, & G_{12} &= 1/24576, & G_{13} &= 1/53248, & G_{14} &= 1/114688. \end{aligned}$$

In fact $G(x) = -2 \log(1 - x/2)$ is a solution with those initial values and satisfying $E \equiv 0$. It is the unique solution since any other solution should have the same Taylor expansion as $-2 \log(1 - x/2)$. We conclude that the surface is graphed by

$$u = -2 \log\left(1 - \frac{1}{2} z \bar{z}\right)$$

It is not homogeneous by previous calculation with L .

2 Dimension 3

2.1 Preparation

Consider \mathbb{C}^3 coordinated by z, ζ, w . Let $w = u + iv$. Consider a graphed surface $u = F(z, \bar{z}, \zeta, \bar{\zeta})$ defined by an analytic function F . Consider the rigid transformation group

$$RT := \{(z' = f(z, \zeta), \zeta' = g(z, \zeta), w' = r w + h(z, \zeta))\}$$

where (f, g) defines a local biholomorphism and $r > 0$. The group acting on the set of surfaces by point-wise transformation. We only consider Levi-rank 1 and 2-non-degenerate surfaces in this paper.

Theorem 2.1. *There are 3 relevant invariants I_0, V_0 and Q_0 of order 5, dividing all surfaces into 3 equivalent types.*

- If $I_0 \neq 0$ then such surface is equivalent to the normal form

$$\begin{aligned} u = F(z, \zeta, \bar{z}, \bar{\zeta}) &= z \bar{z} + \frac{1}{2} z^2 \bar{\zeta} + \frac{1}{2} \zeta \bar{z}^2 \\ &+ \frac{1}{6} Q z \zeta \bar{z}^3 + \frac{1}{6} Q z^3 \bar{z} \bar{\zeta} + \frac{1}{24} V \zeta \bar{z}^4 + \frac{1}{24} \bar{V} z^4 \bar{\zeta} \\ &+ \frac{1}{12} \zeta^2 \bar{z}^3 + \frac{1}{12} z^3 \bar{\zeta}^2 \\ &+ \zeta \bar{\zeta} (\dots) + O(6), \end{aligned}$$

- If $I_0 = 0$, $V_0 \neq 0$ then such surface is equivalent to the normal form

$$\begin{aligned}
u = F(z, \zeta, \bar{z}, \bar{\zeta}) &= z \bar{z} + \frac{1}{2} z^2 \bar{\zeta} + \frac{1}{2} \zeta \bar{z}^2 \\
&+ \frac{1}{6} z \zeta \bar{z}^3 + \frac{1}{6} z^3 \bar{z} \bar{\zeta} + \frac{1}{24} V \zeta \bar{z}^4 + \frac{1}{24} \bar{V} z^4 \bar{\zeta} \\
&+ \frac{1}{12} I \zeta^2 \bar{z}^3 + \frac{1}{12} \bar{I} z^3 \bar{\zeta}^2 \\
&+ \zeta \bar{\zeta} (\dots) + O(6),
\end{aligned}$$

- If $I_0 = V_0 = 0$ then such surface is equivalent to the Gaussier-Merker model

$$u = F(z, \zeta, \bar{z}, \bar{\zeta}) = \frac{z \bar{z} + \frac{1}{2} z^2 \bar{\zeta} + \frac{1}{2} \bar{z}^2 \zeta}{1 - \zeta \bar{\zeta}}$$

which is homogeneous. It admits 5 vector fields