Caves Beneath a Waterfall

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This draft displays the technique of calculating differential invariants under infinite dimensional lie group action. First I introduce some notations.

1 Finite dimensional approximations

Definition 1.1. The rigid transformation group of $\mathbb{C}^{2+1}$ fixing the origin is denoted by

$$RT := \big\{(z,\zeta,w) \mapsto (z',\zeta',w') = \big(f(z,\zeta), g(z,\zeta), \rho\, w\big)\big\},$$

where $\rho \in \mathbb{R}^*$ and f, g are holomorphic functions near $0 \in \mathbb{C}^2$ with $f(0,0) = g(0,0) = 0$ and with invertible Jacobian

$$\begin{pmatrix} f_z & f_\zeta \\ g_z & g_\zeta \end{pmatrix}.$$

Multiplications and inversions are induced by compositions and inversions of transformations.

Proposition 1.2. (f,g) *defines a biholomorphism between neighborhoods of* $0 \in \mathbb{C}^2$ *if and only if the jacobian matrix is invertible at* 0.

Proof. Let us explain only the existence of a *formal* inverse. Expand the holomorphic functions f, g as

$$f(z,\zeta) = \sum_{n=1}^{\infty} \sum_{j=0}^{n} \tfrac{f_{j,n-j}}{j!\,(n-j)!}\, z^j\, \zeta^{n-j},$$

$$g(z,\zeta) = \sum_{n=1}^{\infty} \sum_{j=0}^{n} \tfrac{g_{j,n-j}}{j!\,(n-j)!}\, z^j\, \zeta^{n-j}.$$

Let us construct progressively the formal inverse, which will be expanded as

$$\tilde{f}(z,\zeta) = \sum_{n=1}^{\infty} \sum_{j=0}^{n} \tfrac{\tilde{f}_{j,n-j}}{j!\,(n-j)!}\, z^j\, \zeta^{n-j},$$

$$\tilde{g}(z,\zeta) = \sum_{n=1}^{\infty} \sum_{j=0}^{n} \tfrac{\tilde{g}_{j,n-j}}{j!\,(n-j)!}\, z^j\, \zeta^{n-j}.$$

Then

$$f\big(\tilde{f}(z,\zeta),\tilde{g}(z,\zeta)\big)\equiv z,$$
$$g\big(\tilde{f}(z,\zeta),\tilde{g}(z,\zeta)\big)\equiv \zeta.$$

At each degree we get a linear system. For example at degree 1 we have

$$\begin{pmatrix} f_{1,0} & f_{0,1} \\ g_{1,0} & g_{0,1} \end{pmatrix}\cdot\begin{pmatrix} \tilde{f}_{1,0} & \tilde{f}_{0,1} \\ \tilde{g}_{1,0} & \tilde{g}_{0,1} \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Here $\tilde{f}_{1,0},\tilde{f}_{0,1},\tilde{g}_{1,0},\tilde{g}_{0,1}$ can be uniquely solved thanks to the invertibility of the Jacobian of (f,g).

Suppose by induction, for some $\delta\in\mathbb{Z}_{\geqslant 1}$ that all the coefficients $\tilde{f}_{j,k}$ and $\tilde{g}_{j,k}$ with $j+k\leqslant\delta$ have been already solved as rational functions of $f_{l,n-l}$ and $g_{l,n-l}$ with $n\leqslant\delta$. Then for $j+k=\delta+1$, we expand $f(\tilde{f},\tilde{g})$ and $g(\tilde{f},\tilde{g})$ to degree $\delta+1$ and compare the coefficients of $z^j\zeta^{\delta+1-j}$:

$$\begin{aligned}
0&=\mathrm{Coef}_{z^j\,\zeta^{\delta+1-j}}\Big\{\sum_{n=1}^{\delta+1}\sum_{l=0}^{n}\tfrac{f_{l,n-k}}{l!\,(n-l)!}\,\big(\tilde{f}(z,\zeta)\big)^j\,\big(\tilde{g}(z,\zeta)\big)^{n-l}\Big\}\\
&=f_{1,0}\,\tilde{f}_{j,\delta+1-j}+f_{0,1}\,\tilde{g}_{j,\delta+1-j}+\mathrm{Coef}_{z^j\,\zeta^{\delta+1-j}}\Big\{\sum_{n=2}^{\delta+1}\sum_{l=0}^{n}\tfrac{f_{l,n-k}}{l!\,(n-l)!}\,\big(\tilde{f}(z,\zeta)\big)^j\,\big(\tilde{g}(z,\zeta)\big)^{n-l}\Big\},\\
0&=\mathrm{Coef}_{z^j\,\zeta^{\delta+1-j}}\Big\{\sum_{n=1}^{\delta+1}\sum_{l=0}^{n}\tfrac{g_{l,n-l}}{l!\,(n-l)!}\,\big(\tilde{f}(z,\zeta)\big)^j\,\big(\tilde{g}(z,\zeta)\big)^{n-l}\Big\}\\
&=g_{1,0}\,\tilde{f}_{j,\delta+1-j}+g_{0,1}\,\tilde{g}_{j,\delta+1-j}+\mathrm{Coef}_{z^j\,\zeta^{\delta+1-j}}\Big\{\sum_{n=2}^{\delta+1}\sum_{l=0}^{n}\tfrac{g_{l,n-l}}{l!\,(n-l)!}\,\big(\tilde{f}(z,\zeta)\big)^j\,\big(\tilde{g}(z,\zeta)\big)^{n-l}\Big\}.
\end{aligned}$$

i.e.

$$\begin{pmatrix} f_{1,0} & f_{0,1} \\ g_{1,0} & g_{0,1} \end{pmatrix}\cdot\begin{pmatrix} \tilde{f}_{j,\delta+1-j} \\ \tilde{g}_{j,\delta+1-j} \end{pmatrix}+\begin{pmatrix} \mathscr{R}_1 \\ \mathscr{R}_2 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

where $\mathscr{R}_1$ and $\mathscr{R}_2$ are polynomials of $f_{l,n-l},g_{l,n-l}$ with $n\leqslant\delta+1$ and $\tilde{f}_{p,q},\tilde{g}_{p,q}$ with $p+q\leqslant\delta$. By inductive assumption $\tilde{f}_{p,q},\tilde{g}_{p,q}$ are rational functions of $f_{l,n-l},g_{l,n-l}$ with $n\leqslant\delta$. So $\mathscr{R}_1$ and $\mathscr{R}_2$ are rational functions of $f_{l,n-l},g_{l,n-l}$ with $n\leqslant\delta+1$. We can solve $\tilde{f}_{j,\delta+1-j}$ and $\tilde{g}_{j,\delta+1-j}$ as rational functions of $f_{l,n-l},g_{l,n-l}$ with $n\leqslant\delta+1$. □

Definition 1.3. The space of all Levi-rank 1 and 2 non-degenerate CR graphed hypersurfaces passing by the origin in $\mathbb{C}^3$ is denoted by

$$\mathscr{H}:=\big\{u:=Re(w)=F(z,\zeta,\overline{z},\overline{\zeta})\big\}$$

where

- (real-valued analytic) F is an analytic and real-valued function in a neigborhood of $(0,0)\in\mathbb{C}^2$;
- (passing by the origin) $F(0,0,0,0)=0$;
- (no harmonic monomials) $\partial_z^a\partial_\zeta^b F(0,0,0,0)=0$, for any $a,b\geqslant 0$.

- (Levi-rank 1) the matrix
$$\begin{pmatrix} F_{z\,\overline{z}} & F_{z\,\overline{\zeta}} \\ F_{\zeta\,\overline{z}} & F_{\zeta\,\overline{\zeta}} \end{pmatrix}$$
has rank 1 everywhere;
- (2-non degenerate) the matrix
$$\begin{pmatrix} F_{z\,\overline{z}} & F_{z\,\overline{\zeta}} \\ F_{z\,z\,\overline{z}} & F_{z\,z\,\overline{\zeta}} \end{pmatrix}$$
is invertible at the origin.

There is a natural action of the group RT on the space $\mathscr{H}$: a graphed hypersurface $u = Re(w) = F(z, \zeta, \overline{z}, \overline{\zeta})$ is transformed into another hypersurface $u' = Re(w') = F'(z', \zeta', \overline{z'}, \overline{\zeta'})$. The expression of F' is obtained by solving the fundamental equation

$$F'\big(f(z,\zeta), g(z,\zeta), \overline{f(z,\zeta)}, \overline{g(z,\zeta)}\big) = \rho\, F(z, \zeta, \overline{z}, \overline{\zeta}).$$

Indeed $F'(z, \zeta, \overline{z}, \overline{\zeta}) = \rho\, F\big(\tilde{f}(z,\zeta), \tilde{g}(z,\zeta), \overline{\tilde{f}(z,\zeta)}, \overline{\tilde{g}(z,\zeta)}\big)$ where $(\tilde{f}, \tilde{g})$ is the inverse of (f, g). The inverse transformation brings convenience to obtain the explicit action.

Both the group RT and the space $\mathscr{H}$ are infinite-dimensional in the sense that they admet infinitely many linearly independent parameters.

For RT, any transformation is defined by $\rho \in \mathbb{R}^*$ and two holomorphic functions f, g with expansions

$$f(z,\zeta) = \sum_{n=1}^{\infty} \sum_{j=0}^{n} \tfrac{f_{j,n-j}}{j!\,(n-j)!}\, z^j\, \zeta^{n-j},$$
$$g(z,\zeta) = \sum_{n=1}^{\infty} \sum_{j=0}^{n} \tfrac{g_{j,n-j}}{j!\,(n-j)!}\, z^j\, \zeta^{n-j}.$$

where $f_{j,k}, g_{j,k} \in \mathbb{C}$, $f_{1,0}\, g_{0,1} - f_{0,1}\, g_{1,0} \neq 0$. The group RT is hence parametrized by $f_{j,k}, g_{j,k}$ and ρ.

For $\mathscr{H}$, any graphed hypersurface admets an expansion

$$u = F(z, \zeta, \overline{z}, \overline{\zeta}) = \sum_{n=2}^{\infty} \sum_{a+b+c+d=n} \tfrac{F_{a,b,c,d}}{a!\,b!\,c!\,d!}\, z^a\, \zeta^b\, \overline{z}^c\, \overline{\zeta}^d,$$

where $F_{a,b,c,d} \in \mathbb{C}$, $F_{c,d,a,b} = \overline{F_{a,b,c,d}}$, $F_{a,b,0,0} = 0$ and conditions of constant Levi-rank 1 and of 2-non degeneracy are satisfied. The space is hence parametrized by $F_{a,b,c,d}$.

But these infinite-dimensional objects have finite dimensional approximations. They can be truncated by degrees in expansions. Then they can be viewed as inverse or projective limits of those finite-dimensional truncations.

Definition 1.4. The δ^{th} residue group Res_δ is the subgroup of RT with

$$f(z,\zeta) = z + O(\delta), \qquad g(z,\zeta) = \zeta + O(\delta), \qquad \rho = 1.$$

Proposition 1.5. *The group Res_δ is a normal subgroup of RT.* □

Definition 1.6. The δ^{th} approximation group RT_δ is the quotient group $RT/Res_{\delta+1}$. Each element has a representative

$$f(z,\zeta) = \sum_{n=1}^{\delta}\sum_{j=0}^{n} \tfrac{f_{j,n-j}}{j!\,(n-j)!}\, z^j\, \zeta^{n-j},$$
$$g(z,\zeta) = \sum_{n=1}^{\delta}\sum_{j=0}^{n} \tfrac{g_{j,n-j}}{j!\,(n-j)!}\, z^j\, \zeta^{n-j}.$$

The group RT_δ is a finite dimensional Lie group parameterized by ρ and $f_{j,n-j}, g_{j,n-j}$ with $n \leqslant \delta$.

δ	1	2	3	4	5	6	7	δ
$\dim_{\mathbb{R}} RT_\delta$	9	21	37	57	81	109	141	$2\,\delta^2 + 6\,\delta + 1$

Its multiplication and inversion are obtained by dropping terms of degree $\geqslant \delta + 1$ in the multiplication and inversion of RT.

Proposition 1.7. *For any $\delta, \delta' \in \mathbb{Z}_+$ with $\delta > \delta'$ there is a projection $RT_\delta \longrightarrow RT_{\delta'}$ induced by the injection $Res_\delta \longrightarrow Res_{\delta'}$. For any $\delta, \delta', \delta'' \in \mathbb{Z}_+$ with $\delta > \delta' > \delta''$ The following diagram commutes.*

$$\begin{array}{ccc} RT_\delta & \longrightarrow & RT_{\delta'} \\ & \searrow & \downarrow \\ & & RT_{\delta''}. \end{array}$$

These projections define a projective system $\{RT_\delta\}_{\delta\in\mathbb{Z}_+}$. Projections $\pi_\delta : RT \longrightarrow RT_\delta$ are compatible with this system. By the universal property of the projective limit, there is a morphism

$$RT \longrightarrow \varprojlim_{\delta} RT_\delta$$

which is indeed an inclusion whose image consists of all convergent power series.

Definition 1.8. For any $\delta \geqslant 2$, the δ^{th} approximation of $\mathscr{H}$ is a manifold

$$\mathscr{H}_\delta := \big\{u := F(z,\zeta,\overline{z},\overline{\zeta}) = \sum_{n=2}^{\delta} \sum_{a+b+c+d=n} \tfrac{F_{a,b,c,d}}{a!\,b!\,c!\,d!}\, z^a\, \zeta^b\, \overline{z}^c\, \overline{\zeta}^d\big\},$$

where

- (real-valued) $F_{a,b,c,d} = \overline{F_{c,d,a,b}}$ for any $a, b, c, d \geqslant 0$;
- (passing by the origin) $F_{0,0,0,0} = 0$;
- (no harmonic monomials) $F_{a,b,0,0} = F_{0,0,c,d} = 0$ for any $a, b, c, d \geqslant 0$.
- (2-non-degenerate) the matrix
$$\begin{pmatrix} F_{1,0,1,0} & F_{1,0,0,1} \\ F_{2,0,1,0} & F_{2,0,0,1} \end{pmatrix}$$
is invertible.

- (Levi-rank 1 until degree δ) $F_{1,0,1,0}$, $F_{1,0,0,1} = \overline{F_{0,1,1,0}}$ and $F_{0,1,0,1}$ are not simultaneously 0. The complex Hessian of $F(z,\zeta,\overline{z},\overline{\zeta})$ vanishes up to order $\delta-2$, i.e. $F_{z\,\overline{z}}\,F_{\zeta\,\overline{\zeta}} - F_{z\,\overline{\zeta}}\,F_{\zeta\,\overline{z}} = O(\delta-1)$.

The last condition may look strange, but it is reasonable, as shows the

Proposition 1.9. *A polynomial* $F(z,\zeta,\overline{z},\overline{\zeta}) = \sum_{n=2}^{\delta} \sum_{a+b+c+d=n} \frac{F_{a,b,c,d}}{a!\,b!\,c!\,d!}\, z^a\,\zeta^b\,\overline{z}^c\,\overline{\zeta}^d$ *is a degree* δ *truncation of a formal power series* $\tilde{F}(z,\zeta,\overline{z},\overline{\zeta})$ *with* $\tilde{F}_{z\,\overline{z}}\,\tilde{F}_{\zeta\,\overline{\zeta}} - \tilde{F}_{z\,\overline{\zeta}}\,\tilde{F}_{\zeta\,\overline{z}} = 0$ *if and only if* $F_{z\,\overline{z}}\,F_{\zeta\,\overline{\zeta}} - F_{z\,\overline{\zeta}}\,F_{\zeta\,\overline{z}} = O(\delta-1)$.

Proof. (only if) When calculating the complex Hessian of a power series

$$\tilde{F}(z,\zeta,\overline{z},\overline{\zeta}) = \sum_{n=2}^{\infty} \sum_{a+b+c+d=n} \frac{\tilde{F}_{a,b,c,d}}{a!\,b!\,c!\,d!}\, z^a\,\zeta^b\,\overline{z}^c\,\overline{\zeta}^d,$$

the $\delta-2$ degree terms of $\tilde{F}_{z\,\overline{z}}\,\tilde{F}_{\zeta\,\overline{\zeta}} - \tilde{F}_{z\,\overline{\zeta}}\,\tilde{F}_{\zeta\,\overline{z}}$ involve only coefficients $\tilde{F}_{a,b,c,d}$ with $a+b+c+d \leqslant \delta$.

Let $F(z,\zeta,\overline{z},\overline{\zeta})$ be its degree δ truncation

$$F(z,\zeta,\overline{z},\overline{\zeta}) := \sum_{n=2}^{\delta} \sum_{a+b+c+d=n} \frac{\tilde{F}_{a,b,c,d}}{a!\,b!\,c!\,d!}\, z^a\,\zeta^b\,\overline{z}^c\,\overline{\zeta}^d.$$

Then $F_{z\,\overline{z}}\,F_{\zeta\,\overline{\zeta}} - F_{z\,\overline{\zeta}}\,F_{\zeta\,\overline{z}} = \tilde{F}_{z\,\overline{z}}\,\tilde{F}_{\zeta\,\overline{\zeta}} - \tilde{F}_{z\,\overline{\zeta}}\,\tilde{F}_{\zeta\,\overline{z}} + O(\delta-1) = O(\delta-1)$.

To prove the (if) part, let us introduce dependent and independent coordinates. The manifolds $\mathscr{H}$ and $\mathscr{H}_\delta$ are covered by 3 open subsets: $\{F_{1,0,1,0} \neq 0\}$, $\{F_{1,0,0,1} = \overline{F_{0,1,1,0}} \neq 0\}$ and $\{F_{0,1,0,1} \neq 0\}$. We only treat $F_{1,0,1,0} \neq 0$ case because the other two cases can be transformed into this one by changes of coordinates $(z',\zeta') = (z+\zeta, z-\zeta)$ or $(z',\zeta') = (z,\zeta)$ preserving the Levi-rank.

When $F_{1,0,1,0} \neq 0$ we have $F_{z,\overline{z}} \neq 0$ in a neighborhood of the origin. The Levi-rank 1 condition is now equivalent to

$$F_{\zeta\,\overline{\zeta}} \equiv \frac{F_{z\,\overline{\zeta}}\,F_{\zeta\,\overline{z}}}{F_{z\,\overline{z}}}.$$

By differentiating both sides, all terms $F_{z^a\,\zeta^b\,\overline{z}^c\,\overline{\zeta}^d}$ with $b \geqslant 1$ and $d \geqslant 1$ can be uniquely expressed as rational functions of $F_{z^{a'}\,\zeta^{b'}\,\overline{z}^{c'}}$ with $a'+b'+c' \leqslant a+b+c+d$ and $F_{z^{a''}\,\overline{z}^{c''}\,\overline{\zeta}^{d''}}$ with $a''+b''+c'' \leqslant a+b+c+d$. Moreover, only powers of $F_{z\,\overline{z}}$ appears in the denominators. For example:

$$F_{z\,\zeta,\overline{\zeta}} \equiv \frac{F_{z\,\zeta\,\overline{z}}\,F_{z\,\overline{\zeta}}}{F_{z\,\overline{z}}} + \frac{F_{z^2\,\overline{\zeta}}\,F_{\zeta\,\overline{z}}}{F_{z\,\overline{z}}} - \frac{F_{z^2\,\overline{z}}\,F_{z\,\overline{\zeta}}\,F_{\zeta\,\overline{z}}}{F_{z\,\overline{z}}^2}.$$

Taking their values at the origin, the coefficients $F_{a,b,c,d}$ with $b \geqslant 1$ and $d \geqslant 1$ can be uniquely expressed as rational functions of $F_{a',b',c',0}$ with $a'+b'+c' \leqslant a+b+c+d$ and $F_{a'',0,c'',d''}$ with $a''+b''+c'' \leqslant a+b+c+d$. Moreover, only powers of $F_{1,0,1,0}$ appear in the denominators. For example:

$$F_{1,1,0,1} = \frac{F_{1,1,1,0}\,F_{1,0,0,1}}{F_{1,0,1,0}} + \frac{F_{2,0,0,1}\,F_{0,1,1,0}}{F_{1,0,1,0}} - \frac{F_{2,0,1,0}\,F_{1,0,0,1}\,F_{0,1,1,0}}{F_{1,0,1,0}^2}.$$

Definition 1.10. The coefficient $F_{a,b,c,d}$ will be called *dependent* if $b \geqslant 1$ and $d \geqslant 1$. Otherwise, it will be called *independent*.

Elements in the open subset $\{F_{1,0,1,0} \neq 0\}$ of $\mathscr{H}$ and $\mathscr{H}_\delta$ are uniquely determined by the independent coefficients $F_{a,b,c,d}$ with $b\,d = 0$. Since F is real-valued, i.e. $F_{c,d,a,b} = \overline{F_{a,b,c,d}}$, one has

$$\dim_{\mathbb{R}} \mathscr{H}_\delta = \#\big\{(a,b,c,d) | a+b \geqslant 1, c+d \geqslant 1, a+b+c+d \leqslant \delta, b\,d = 0\big\}.$$

δ	2	3	4	5	6	7	8	δ
$\dim_{\mathbb{R}} \mathscr{H}_\delta$	3	11	26	50	85	133	196	$\frac{1}{6}(2\,\delta^3 + 3\,\delta^2 - 5\,\delta)$

To prove the (if) part of Proposition 1.9, one shall construct a power series $\tilde{F}(z,\zeta,\overline{z},\overline{\zeta}) = F(z,\zeta,\overline{z},\overline{\zeta}) + \sum_{n=\delta+1}^{\infty} \sum_{a+b+c+d=n} \frac{\tilde{F}_{a,b,c,d}}{a!b!c!d!} z^a \zeta^b \overline{z}^c \overline{\zeta}^d$ with $\tilde{F}_{z\overline{z}}\,\tilde{F}_{\zeta\overline{\zeta}} - \tilde{F}_{z\overline{\zeta}}\,\tilde{F}_{\zeta\overline{z}} = 0$. This can be achieved by taking all the independent coefficients $\tilde{F}_{a,b,c,d} = 0$ with $a+b+c+d \geqslant n+1$ and $b\,d = 0$ and calculate all the dependent coefficients $\tilde{F}_{a,b,c,d}$ with $b \geqslant 1$ and $d \geqslant 1$ by their rational expressions of the independent ones. □

Proposition 1.11. *For any $\delta, \delta' \in \mathbb{Z}_+$ with $\delta > \delta'$ there is a projection $\mathscr{H}_\delta \longrightarrow \mathscr{H}_{\delta'}$ by dropping terms of degree $\geqslant \delta' + 1$. For any $\delta, \delta', \delta'' \in \mathbb{Z}_+$ with $\delta > \delta' > \delta''$ The following diagram commutes.*

$$\begin{array}{ccc} \mathscr{H}_\delta & \longrightarrow & \mathscr{H}_{\delta'} \\ & \searrow & \downarrow \\ & & \mathscr{H}_{\delta''}. \end{array}$$

These projections define a projective system $\{\mathscr{H}_\delta\}_{\delta \in \mathbb{Z}_+}$. Projections $\pi_\delta : \mathscr{H} \longrightarrow \mathscr{H}_\delta$ are compatible with this system. By the universal property of the projective limit, there is a morphism

$$\mathscr{H} \longrightarrow \varprojlim_{\delta} \mathscr{H}_\delta.$$

which is indeed an inclusion.

The manifold $\mathscr{H}_\delta$ is a finite-dimensional manifold parameterized by the independent coefficients $F_{a,b,c,d}$ with $a+b+c+d \leqslant \delta$ and $b\,d = 0$. The action of the group RT on $\mathscr{H}$ induces an action on each manifold $\mathscr{H}_\delta, \forall \delta \geqslant 0$:

$$\begin{array}{ccc} \mathscr{H} & \xrightarrow{\pi_\delta} & \mathscr{H}_\delta \\ \big\downarrow{\scriptstyle (f,g,\rho)} & & \vdots \\ \mathscr{H} & \xrightarrow{\pi_\delta} & \mathscr{H}_\delta \end{array}$$

More precisely, a polynomial $F(z,\zeta,\overline{z},\overline{\zeta}) \in \mathscr{H}_\delta$ is a degree δ truncation of a (not unique) convergent power series $\tilde{F}(z,\zeta,\overline{z},\overline{\zeta}) \in \mathscr{H}$, which is transformed to another convergent power series $\tilde{F}'(z,\zeta,\overline{z},\overline{\zeta})$ by the fundamental equation

$$\begin{aligned} \tilde{F}'(z,\zeta,\overline{z},\overline{\zeta}) &= \rho\,\tilde{F}\big(\tilde{f}(z,\zeta), \tilde{g}(z,\zeta), \overline{\tilde{f}(z,\zeta)}, \overline{\tilde{g}(z,\zeta)}\big) \\ &= \rho \sum_{n=2}^{\delta} \sum_{a+b+c+d=n} \tfrac{F_{a,b,c,d}}{a!b!c!d!} \big(\tilde{f}(z,\zeta)\big)^a \big(\tilde{g}(z,\zeta)\big)^b \big(\overline{\tilde{f}(z,\zeta)}\big)^c \big(\overline{\tilde{g}(z,\zeta)}\big)^d + O(\delta+1). \end{aligned}$$

The degree δ truncation of $\tilde{F}'(z,\zeta,\overline{z},\overline{\zeta})$, denoted by $F'(z,\zeta,\overline{z},\overline{\zeta})$, is the image of $F(z,\zeta,\overline{z},\overline{\zeta})$ after the group action. It depends on the coefficients $F_{a,b,c,d}$ with $a+b+c+d \leqslant \delta$ only, hence is independent of the choice of $\tilde{F}(z,\zeta,\overline{z},\overline{\zeta})$. The group action is well-defined.

More precisely

Proposition 1.12. *There is a group action of $RT_{\delta-1}$ on $\mathscr{H}_\delta$. The group action of RT on $\mathscr{H}_\delta$ factors through the projection $\pi_{\delta-1} : RT \longrightarrow RT_{\delta-1}$, i.e. the following diagram commutes:*

$$\begin{array}{ccc} RT \times \mathscr{H}_\delta & \longrightarrow & \mathscr{H}_\delta \\ \big\downarrow & \nearrow & \\ RT_{\delta-1} \times \mathscr{H}_\delta & & \end{array}.$$

Proof. When calculating the Taylor coefficients $F'_{a,b,c,d}$ in

$$\tilde{F}'(z,\zeta,\overline{z},\overline{\zeta}) = \sum_{n=2}^{\delta} \tfrac{F'_{a,b,c,d}}{a!b!c!d!} z^a \zeta^b \overline{z}^c \overline{\zeta}^d + O(\delta+1),$$

we are calculating coefficients of $z^a \zeta^b \overline{z}^c \overline{\zeta}^d$ with $a+b+c+d \leqslant \delta$ from

$$\rho \sum_{n=2}^{\delta} \sum_{a+b+c+d=n} \tfrac{F_{a,b,c,d}}{a!b!c!d!} \big(\tilde{f}(z,\zeta)\big)^a \big(\tilde{g}(z,\zeta)\big)^b \big(\overline{\tilde{f}(z,\zeta)}\big)^c \big(\overline{\tilde{g}(z,\zeta)}\big)^d.$$

Each monomial is a product of at least 2 terms among $\{\tilde{f}(z,\zeta), \tilde{g}(z,\zeta), \overline{\tilde{f}(z,\zeta)}, \overline{\tilde{g}(z,\zeta)}\}$. Each term

$$\tilde{f}(z,\zeta) = \sum_{n=1}^{\infty} \tfrac{\tilde{f}_{j,n-j}}{j!(n-j)!} z^j \zeta^{n-j},$$
$$\tilde{g}(z,\zeta) = \sum_{n=1}^{\infty} \tfrac{\tilde{g}_{j,n-j}}{j!(n-j)!} z^j \zeta^{n-j},$$

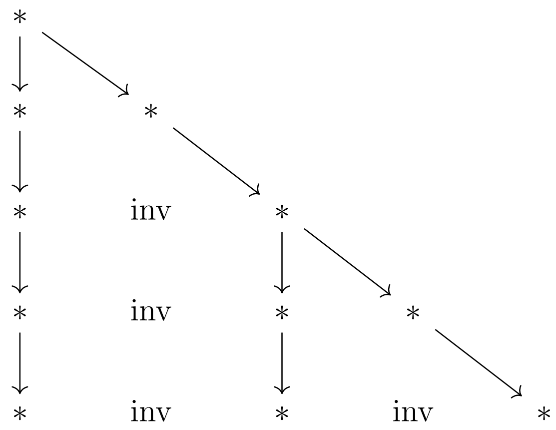
as a power series of z, ζ or $\overline{z}, \overline{\zeta}$, starts from degree 1. So only $\tilde{f}_{j,n-j}$, $\tilde{g}_{j,n-j}$ and their conjugations with $n \leqslant \delta - 1$ contribute to $F'_{a,b,c,d}$ with $a+b+c+d \leqslant \delta$. The group action of $RT_{\delta-1}$ on $\mathscr{H}_\delta$ can be well-defined and the commutative diagram is satisfied. □

Compare the two tables of dimensions:

δ	2	3	4	5	6	7	8
$\dim_{\mathbb{R}} RT_{\delta-1}$	9	21	37	57	81	109	141
$\dim_{\mathbb{R}} \mathscr{H}_\delta$	3	11	26	50	85	133	196

The theory of differential invariants of finite-dimensional Lie group actions applies: the orbit dimension of $RT_{\delta-1}$ on $\mathscr{H}_d$ is at most equal to $\dim_{\mathbb{R}} RT_{\delta-1}$ and the equality is achieved only when the action is locally free. We see immediately that the dimension of transversal, which equals to the number of linearly independent differential invariants up to order δ, is positive when $\delta \geqslant 6$.

The infinite-dimensional Lie group RT can be interpreted as an infinitely long flow of water. The space $\mathscr{H}$ can be interpreted as an infinitely high valley. At the beginning, water fills the space up. But later on as the waterfall grows wider, water cannot fill the space. Some *caves*, corresponding to the transversal dimension, or *differential invariants*, show up.



2 Normalization

Since the RT action on $\mathscr{H}_\delta$ factors through $\pi_{\delta-1} : RT \longrightarrow RT_{\delta-1}$, we have the

Proposition 2.1. *A rational function on $\mathscr{H}_\delta$ is invariant under the RT action if and only if it is invariant under the RT_δ action.* □

Thus, to calculate differential invariants of order δ under RT is equivalent to calculate those under the finite-dimensional Lie group $RT_{\delta-1}$. The algorithm goes as follows:

(1) Write down how $(f, g, \rho) \in RT_{\delta-1}$ acts on some independent parameters $F_{a,b,c,d}$.

(2) Choose certain $(f, g, \rho) \in RT_{\delta-1}$ to normalize as many independent parameters $F_{a,b,c,d}$ to 0 or 1 as possible, i.e. (f, g, ρ) send $F_{a,b,c,d}$ to $F^{(1)}_{a,b,c,d}$ and some $F^{(1)}_{a,b,c,d} = 0$ or 1.

(3) Calculate how the other independent parameters $F^{(1)}_{a,b,c,d}$ are changed under this special (f, g, ρ) action, i.e. express them as rational functions of $F_{a,b,c,d}$, $f_{j,n-j}$, $g_{j,n-j}$ and ρ.

(4) Calculate the "stabilizer", the subgroup $RT^{(1)}_{\delta-1}$ of $RT_{\delta-1}$ which preserves current normalizations.

(5) Repeat (2) (3) (4) by studying $RT^{(1)}_{\delta-1}$ actions on $F^{(1)}_{a,b,c,d}$, $RT^{(2)}_{\delta-1}$ actions on $F^{(2)}_{a,b,c,d}$ and so on, until no more terms can be normalized, i.e. $RT^{(k)}_{\delta-1}$ fixes all $F^{(k)}_{a,b,c,d}$.

(6) Express those non-constant $F^{(k)}_{a,b,c,d}$ in terms of $F_{a,b,c,d}$. They are rational functions fixed by $RT_{\delta-1}$, i.e. they are differential invariants of order $\leqslant \delta$.

We fix $\delta = 5$ in this section. The goal is to show the existence of order 5 invariants and to compute their explicit expressions.

2.1 First normalization: degree 2 terms $= z\,\overline{z}$

We may assume that $F_{1,0,1,0} \neq 0$. In this case

$$\begin{aligned}F(z,\zeta,\overline{z},\overline{\zeta}) &= F_{1,0,1,0}\, z\,\overline{z} + F_{1,0,0,1}\, z\overline{\zeta} + F_{0,1,1,0}\,\zeta\,\overline{z} + \tfrac{F_{1,0,0,1}\,F_{0,1,1,0}}{F_{1,0,1,0}}\,\zeta\,\overline{\zeta} + O(3)\\ &= F_{1,0,1,0}\,\big(z + \tfrac{F_{0,1,1,0}}{F_{1,0,1,0}}\,\zeta\big)\,\big(\overline{z} + \tfrac{F_{1,0,0,1}}{F_{1,0,1,0}}\,\overline{\zeta}\big) + O(3)\\ &= \underbrace{\big(F_{1,0,1,0}^{1/2}\, z + \tfrac{F_{0,1,1,0}}{F_{1,0,1,0}^{1/2}}\,\zeta\big)}_{=:z'}\,\underbrace{\big(F_{1,0,1,0}^{1/2}\,\overline{z} + \tfrac{F_{1,0,0,1}}{F_{1,0,1,0}^{1/2}}\,\overline{\zeta}\big)}_{=:\overline{z'}}\; + O(3).\end{aligned}$$

After the rigid transformation:

$$z' = F_{1,0,1,0}^{1/2}\, z + \tfrac{F_{0,1,1,0}}{F_{1,0,1,0}^{1/2}}\,\zeta, \qquad \zeta' = \zeta, \qquad w' = w,$$

the polynomial $F(z,\zeta,\overline{z},\overline{\zeta})$ becomes $F^{(1)}(z',\zeta',\overline{z'},\overline{\zeta'}) = z'\,\overline{z'} + O(3)$. The other independent parameters $F^{(1)}_{a,b,c,d}$ with $a+b \geqslant 1, c+d \geqslant 1, b\,d = 0$ can also be uniquely expressed as rational functions of $F_{a,b,c,d}$, by the fundamental equation.

Since all the independent parameters $F^{(1)}_{a,b,c,d}$ have $b\,d = 0$ and $F^{(1)}_{c,d,a,b} = \overline{F^{(1)}_{a,b,c,d}}$, it suffices to calculate $F^{(1)}_{a,b,c,0}$ in terms of $F_{a,b,c,d}$. The inverse transformation is

$$z = \tfrac{1}{F_{1,0,1,0}^{1/2}}\, z' - \tfrac{F_{0,1,1,0}}{F_{1,0,1,0}}\,\zeta', \qquad \zeta = \zeta', \qquad w = w'.$$

In the fundamental equality

$$\begin{aligned}\sum_{a,b,c,d} \tfrac{F^{(1)}_{a,b,c,d}}{a!b!c!d!}\, z'^a\,\zeta'^b\,\overline{z'}^c\,\overline{\zeta'}^d &= \sum_{a,b,c,d} \tfrac{F_{a,b,c,d}}{a!b!c!d!}\, z^a\,\zeta^b\,\overline{z}^c\,\overline{\zeta}^d\\ &= \sum_{a,b,c,d} \tfrac{F_{a,b,c,d}}{a!b!c!d!}\,\big(\tfrac{1}{F_{1,0,1,0}^{1/2}}\, z' - \tfrac{F_{0,1,1,0}}{F_{1,0,1,0}}\,\zeta'\big)^a\,\zeta'^b\,\big(\tfrac{1}{F_{1,0,1,0}^{1/2}}\,\overline{z'} - \tfrac{F_{1,0,0,1}}{F_{1,0,1,0}}\,\overline{\zeta'}\big)^c\,\overline{\zeta'}^d,\end{aligned}$$

we calculate the coefficient of $z'^a\,\zeta'^b\,\overline{z'}^c$. On the left hand side, it is $F^{(1)}_{a,b,c,0}$. On the right hand side only $F_{j,a+b-j,c,0}$ with $a \leqslant j \leqslant a+b$ contribute. Since

$$\begin{aligned}&\tfrac{F_{j,a+b-j,c,0}}{j!(a+b-j)!c!}\,\big(\tfrac{1}{F_{1,0,1,0}^{1/2}}\, z' - \tfrac{F_{0,1,1,0}}{F_{1,0,1,0}}\,\zeta'\big)^a\,\zeta'^{a+b-j}\,\big(\tfrac{1}{F_{1,0,1,0}^{1/2}}\,\overline{z'} - \tfrac{F_{1,0,0,1}}{F_{1,0,1,0}}\,\overline{\zeta'}\big)^c\\ =&\tfrac{F_{j,a+b-j,c,0}}{j!(a+b-j)!c!}\,\tfrac{j!}{a!(j-a)!}\,\big(\tfrac{1}{F_{1,0,1,0}^{1/2}}\, z'\big)^a\,\big(-\tfrac{F_{0,1,1,0}}{F_{1,0,1,0}}\,\zeta'\big)^{j-a}\,\zeta'^{a+b-j}\,\big(\tfrac{1}{F_{1,0,1,0}^{1/2}}\,\overline{z'}\big)^c + \text{irrelevant monomials},\end{aligned}$$

We get

$$\begin{aligned}F^{(1)}_{a,b,c,0} &= \sum_{j=a}^{a+b} \tfrac{F_{j,a+b-j,c,0}}{a!(j-a)!(a+b-j)!c!}\,\big(\tfrac{1}{F_{1,0,1,0}^{1/2}}\big)^a\,\big(-\tfrac{F_{0,1,1,0}}{F_{1,0,1,0}}\big)^{j-a}\,\big(\tfrac{1}{F_{1,0,1,0}^{1/2}}\big)^c\\ &= \sum_{j=0}^{b} \tfrac{F_{a+j,b-j,c,0}}{a!j!(b-j)!c!}\,\big(\tfrac{1}{F_{1,0,1,0}^{1/2}}\big)^{a+c}\,\big(-\tfrac{F_{0,1,1,0}}{F_{1,0,1,0}}\big)^j.\end{aligned}$$

We define $\mathscr{H}_5^{(1)} := \{u := F^{(1)}(z,\zeta,\overline{z},\overline{\zeta}) = z\,\overline{z} + O(3)\}$, a codimension 3 submanifold of $\mathscr{H}_5$ since we have normalized $F^{(1)}_{1,0,1,0} = 1$ and $F^{(1)}_{1,0,0,1} = \overline{F^{(1)}_{0,1,1,0}} = 0$. So $\dim_{\mathbb{R}} \mathscr{H}_5^{(1)} = 50 - 3 = 47$.

Its stabilizer group $RT_4^{(1)}$ consists of (f, g, ρ) such that

$$f(z,\zeta) = r\,e^{i\theta}\,z + O(2), \qquad g(z,\zeta) = O(1), \qquad \rho = r^2,$$

where $r \in \mathbb{R}_+$, $\theta \in [0, 2\pi)$. It is a codimension 3 subgroup of RT_4, hence $\dim_{\mathbb{R}} RT_4^{(1)} = 57 - 3 = 54$.

2.2 Second normalization: $F^{(2)}_{a,b,1,0} = 0$ for $(a,b) \neq (1,0)$.

Now, we study the group action of $RT_4^{(1)}$ on $\mathscr{H}_5^{(1)}$. Any element in $\mathscr{H}_5^{(1)}$ has expansion:

$$\begin{aligned}F^{(1)}(z,\zeta,\overline{z},\overline{\zeta}) &= z\,\overline{z} + \overline{z}\,\Big(\sum_{2\leqslant a+b\leqslant 4} \tfrac{F^{(1)}_{a,b,1,0}}{a!b!}\,z^a\,\zeta^b\Big) + z\,\Big(\sum_{2\leqslant a+b\leqslant 4} \tfrac{\overline{F^{(1)}_{a,b,1,0}}}{a!b!}\,\overline{z}^c\,\overline{\zeta}^d\Big) + R(z,\zeta,\overline{z},\overline{\zeta})\\ &= \underbrace{\Big(z + \sum_{2\leqslant a+b\leqslant 4} \tfrac{F^{(1)}_{a,b,1,0}}{a!b!}\,z^a\,\zeta^b\Big)}_{=:z'}\,\underbrace{\Big(\overline{z} + \sum_{2\leqslant a+b\leqslant 4} \tfrac{\overline{F^{(1)}_{a,b,1,0}}}{a!b!}\,\overline{z}^a\,\overline{\zeta}^b\Big)}_{=:\overline{z'}} + R(z,\zeta,\overline{z},\overline{\zeta})\end{aligned}$$

whose the remainder $R(z,\zeta,\overline{z},\overline{\zeta})$ contains only terms $z^a\,\zeta^b\,\overline{z}^c\,\overline{z'}^d$ with either (a,b) or $(c,d) \notin \{(0,0),(1,0)\}$. After the rigid transformation in $RT_4^{(1)}$:

$$z' = z + \sum_{2\leqslant a+b\leqslant 4} \tfrac{F^{(1)}_{a,b,1,0}}{a!b!}\,z^a\,\zeta^b, \qquad \zeta' = \zeta, \qquad w' = w, \qquad (*)$$

the polynomial $F^{(1)}(z,\zeta,\overline{z},\overline{\zeta})$ becomes $F^{(2)}(z',\zeta',\overline{z'},\overline{\zeta'}) = z'\,\overline{z'} + R'(z',\zeta',\overline{z'},\overline{\zeta'})$. It remains to show that the remainder $R'(z',\zeta',\overline{z'},\overline{\zeta'})$ contains only terms $z^a\,\zeta^b\,\overline{z}^c\,\overline{z}^d$ with either (a,b) or (c,d) $\notin \{(0,0),(1,0)\}$.

Lemma 2.2. *The inverse of* $(*)$ *in* $RT_4^{(1)}$ *is of the form*

$$z = z' + \sum_{n=2}^{4}\sum_{j=0}^{n} \tfrac{\tilde{f}_{j,n-j}}{j!(n-j)!}\,z^j\,\zeta^{n-j}, \qquad \zeta = \zeta', \qquad w = w'.$$

Proof. It suffices to show that $z := \tilde{f}(z',\zeta') = z' + O_{z',\zeta'}(2)$. From $(*)$

$$z = z' - \sum_{2\leqslant a+b\leqslant 4} \tfrac{F^{(1)}_{a,b,1,0}}{a!b!}\,z^a\,\zeta^b = z' - \sum_{2\leqslant a+b\leqslant 4} \tfrac{F^{(1)}_{a,b,1,0}}{a!b!}\,\tilde{f}(z',\zeta')^a\,\zeta'^b = z' + O_{z',\zeta'}(2). \qquad \square$$

In the remainder $R(z,\zeta,\overline{z},\overline{\zeta})$, each term $z^a\,\zeta^b\,\overline{z}^c\,\overline{z}^d$ is transformed to $\big(z' + O_{z',\zeta'}(2)\big)^a\,\zeta'^b\,\big(\overline{z'} + O_{\overline{z'},\overline{\zeta'}}(2)\big)\,\overline{\zeta'}^d$, whose expansion still contains only terms $z'^a\,\zeta'^b\,\overline{z'}^c\,\overline{\zeta'}^d$ with either (a,b) or $(c,d) \notin \{(0,0),(1,0)\}$.

The terms $F^{(2)}_{a,b,c,0}$ such that $2 \leqslant a+b+c \leqslant 5$, $(a,b), (c,0) \notin \{(0,0),(1,0)\}$ can be solved in terms of $F^{(1)}_{a,b,c,d}$:

$$
\begin{aligned}
F^{(2)}_{0,1,2,0} &= F^{(1)}_{0,1,2,0},\\
F^{(2)}_{0,1,3,0} &= -3\,F^{(1)}_{0,1,2,0}\,F^{(1)}_{1,0,2,0} + F^{(1)}_{0,1,3,0},\\
F^{(2)}_{0,1,4,0} &= 15\,F^{(1)}_{0,1,2,0}\,(F^{(1)}_{1,0,2,0})^2 - 4\,F^{(1)}_{0,1,2,0}\,F^{(1)}_{1,0,3,0} - 6\,F^{(1)}_{0,1,3,0}\,F^{(1)}_{1,0,2,0} + F^{(1)}_{0,1,4,0},\\
F^{(2)}_{0,2,2,0} &= -F^{(1)}_{0,2,1,0}\,F^{(1)}_{1,0,2,0} + F^{(1)}_{0,2,2,0},\\
F^{(2)}_{0,2,3,0} &= 3\,F^{(1)}_{0,2,1,0}\,(F^{(1)}_{1,0,2,0})^2 - F^{(1)}_{0,2,1,0}\,F^{(1)}_{1,0,3,0} - 3\,F^{(1)}_{0,2,2,0}\,F^{(1)}_{1,0,2,0} + F^{(1)}_{0,2,3,0},\\
F^{(2)}_{0,3,2,0} &= 3\,F^{(1)}_{0,2,1,0}\,F^{(1)}_{1,0,2,0}\,F^{(1)}_{1,1,1,0} - 3\,F^{(1)}_{0,2,1,0}\,F^{(1)}_{1,1,2,0} - F^{(1)}_{0,3,1,0}\,F^{(1)}_{1,0,2,0} + F^{(1)}_{0,3,2,0},\\
F^{(2)}_{1,1,2,0} &= -F^{(1)}_{1,0,2,0}\,F^{(1)}_{1,1,1,0} + F^{(1)}_{1,1,2,0},\\
F^{(2)}_{1,1,3,0} &= 3\,(F^{(1)}_{1,0,2,0})^2\,F^{(1)}_{1,1,1,0} - 3\,F^{(1)}_{1,0,2,0}\,F^{(1)}_{1,1,2,0} - F^{(1)}_{1,0,3,0}\,F^{(1)}_{1,1,1,0} + F^{(1)}_{1,1,3,0},\\
F^{(2)}_{1,2,2,0} &= F^{(1)}_{0,2,1,0}\,F^{(1)}_{1,0,2,0}\,F^{(1)}_{2,0,1,0} + 2\,F^{(1)}_{1,0,2,0}\,(F^{(1)}_{1,1,1,0})^2 - F^{(1)}_{0,2,1,0}\,F^{(1)}_{2,0,2,0}\\
&\quad - F^{(1)}_{1,0,2,0}\,F^{(1)}_{1,2,1,0} - 2\,F^{(1)}_{1,1,1,0}\,F^{(1)}_{1,1,2,0} + F^{(1)}_{1,2,2,0},\\
F^{(2)}_{2,0,2,0} &= -F^{(1)}_{1,0,2,0}\,F^{(1)}_{2,0,1,0} + F^{(1)}_{2,0,2,0},\\
F^{(2)}_{2,0,3,0} &= 3\,(F^{(1)}_{1,0,2,0})^2\,F^{(1)}_{2,0,1,0} - 3\,F^{(1)}_{1,0,2,0}\,F^{(1)}_{2,0,2,0} - F^{(1)}_{1,0,3,0}\,F^{(1)}_{2,0,1,0} + F^{(1)}_{2,0,3,0},\\
F^{(2)}_{2,1,2,0} &= 3\,F^{(1)}_{1,0,2,0}\,F^{(1)}_{1,1,1,0}\,F^{(1)}_{2,0,1,0} - F^{(1)}_{1,0,2,0}\,F^{(1)}_{2,1,1,0} - 2\,F^{(1)}_{1,1,1,0}\,F^{(1)}_{2,0,2,0} - F^{(1)}_{1,1,2,0}\,F^{(1)}_{2,0,1,0} + F^{(1)}_{2,1,2,0},\\
F^{(2)}_{3,0,2,0} &= 3\,F^{(1)}_{1,0,2,0}\,(F^{(1)}_{2,0,1,0})^2 - F^{(1)}_{1,0,2,0}\,F^{(1)}_{3,0,1,0} - 3\,F^{(1)}_{2,0,1,0}\,F^{(1)}_{2,0,2,0} + F^{(1)}_{3,0,2,0}.
\end{aligned}
$$

We define $\mathscr{H}_5^{(2)} := \{u := F^{(2)}(z,\zeta,\overline{z},\overline{\zeta}) = z\,\overline{z} + O(3) | F^{(2)}_{a,b,1,0} = 0, \forall (a,b) \neq (1,0)\}$, a codimension 24 submanifold of $\mathscr{H}_5^{(1)}$. So $\dim_{\mathbb{R}} \mathscr{H}_5^{(2)} = 47 - 24 = 23$.

It will be a bit strange to talk about stabilizer group from this step. We in fact need to introduce a new definition of stabilizer. But after the final step, we will recover the stabilizer in the standard sense.

Definition 2.3. For any fixed element $F^{(2)}(z,\zeta,\overline{z},\overline{\zeta}) \in \mathscr{H}_5^{(2)}$, the subset of $RT^{(1)}_{0,4}$ consisting of elements f, g, ρ which send $F^{(2)}(z,\zeta,\overline{z},\overline{\zeta})$ to another element in $\mathscr{H}_5^{(2)}$, is defined as $RT^{(2)}_{0,4}(F^{(2)})$. It depends on the choice of the original element $F^{(2)}$.

The stabilizer $RT^{(2)}_4(F^{(2)})$ is a codimension 24 subgroup of $RT^{(1)}_4$ hence $\dim_{\mathbb{R}} RT^{(2)}_4(F^{(2)}) = 54 - 24 = 30$. It contains elements $(f,g,\rho) = \big(r\,e^{i\,\theta}\,z + O(2), g, r^2\big) \in RT^{(1)}_4$ such that

$$
\begin{aligned}
f_{2,0} &= -r\,e^{i\,\theta}\,F^{(2)}_{2,0,0,1}\,\overline{g_{1,0}}\,\overline{g_{0,1}}^{-1},\\
f_{3,0} &= -r\,e^{i\,\theta}\,F^{(2)}_{3,0,0,1}\,\overline{g_{1,0}}\,\overline{g_{0,1}}^{-1},\\
f_{4,0} &= -r\,e^{i\,\theta}\,F^{(2)}_{4,0,0,1}\,\overline{g_{1,0}}\,\overline{g_{0,1}}^{-1},\\
f_{0,2} &= 0, f_{1,1} = 0, f_{0,3} = 0, f_{1,2} = 0, f_{2,1} = 0, f_{0,4} = 0, f_{1,3} = 0, f_{2,2} = 0, f_{3,1} = 0,
\end{aligned}
$$

which are in total 12 conditions on complex coefficients.

2.3 Third normalization: $F^{(3)}_{2,0,0,1} = \overline{F^{(3)}_{0,1,2,0}} = 1$

Any element in $\mathscr{H}_5^{(2)}$ has expansion:

$$F^{(2)}(z,\zeta,\overline{z},\overline{\zeta}) = z\,\overline{z} + \tfrac{F^{(2)}_{2,0,0,1}}{2} z^2\,\overline{\zeta} + \tfrac{\overline{F^{(2)}_{2,0,0,1}}}{2}\overline{z}^2\,\zeta + O(4).$$

By 2-non-degeneracy $F^{(2)}_{2,0,0,1} \neq 0$. So after the rigid transformation:

$$z' = z, \qquad \zeta' = \overline{F^{(2)}_{2,0,0,1}}\,\zeta = F^{(2)}_{0,1,2,0}\,\zeta, \qquad w' = w,$$

it becomes a graph $u = F^{(3)}(z,\zeta,\overline{z},\overline{\zeta}) = z\,\overline{z} + \frac{1}{2} z^2\,\overline{\zeta} + \frac{1}{2}\overline{z}^2\,\zeta + O(4)$.

The relations are $F^{(3)}_{a,b,c,0} = F^{(2)}_{a,b,c,0}\,(F^{(2)}_{0,1,2,0})^{-b}$.

We define $\mathscr{H}_5^{(3)} := \{u := F^{(3)}(z,\zeta,\overline{z},\overline{\zeta}) = z\,\overline{z} + \frac{1}{2}z^2\,\overline{\zeta} + \frac{1}{2}\overline{z}^2\,\zeta + O(4) | F^{(3)}_{a,b,1,0} = 0, \forall (a,b) \neq (1,0)\}$, a codimension 2 submanifold of $\mathscr{H}_5^{(2)}$. So $\dim_{\mathbb{R}} \mathscr{H}_5^{(3)} = 23 - 2 = 21$.

For any fixed element $F^{(3)} \in \mathscr{H}_5^{(3)}$, there exists some $F^{(2)} \in \mathscr{H}_5^{(2)}$ whose third normalization is equal to $F^{(3)}$. For example, we can take $F^{(2)} = F^{(3)}$. The stabilizer $RT_4^{(3)}(F^{(3)})$ is a codimension 2 subgroup of $RT_4^{(2)}(F^{(3)})$. Hence $\dim_{\mathbb{R}} RT_4^{(3)}(F^{(3)}) = 30 - 2 = 28$. It contains elements $(f,g,\rho) \in RT_4^{(2)}(F^{(3)})$ satisfying $g_{0,1} = e^{2\,i\,\theta}$, i.e.

$$\begin{aligned} f(z,\zeta) &= r\,e^{i\theta}\,z - \tfrac{1}{2}\,r\,e^{3\,i\,\theta}\,\overline{g_{1,0}}\,z^2 - \tfrac{1}{6}\,r\,e^{3\,i\,\theta}\,F^{(3)}_{3,0,0,1}\,\overline{g_{1,0}}\,z^3 - \tfrac{1}{24}\,r\,e^{3\,i\,\theta}\,F^{(3)}_{4,0,0,1}\,\overline{g_{1,0}}\,z^4 \\ g(z,\zeta) &= g_{1,0}\,z + e^{2\,i\,\theta}\,\zeta + O(2), \qquad \rho = r^2. \end{aligned}$$

2.4 Fourth normalization: $F^{(4)}_{2,0,2,0} = 0$

Any element in $\mathscr{H}_5^{(3)}$ has expansion:

$$\begin{aligned} F^{(3)}(z,\zeta,\overline{z},\overline{\zeta}) &= z\,\overline{z} + \tfrac{1}{2}z^2\,\overline{\zeta} + \tfrac{1}{2}\overline{z}^2\,\zeta + \tfrac{1}{4}F^{(3)}_{2,0,2,0}z^2\,\overline{z}^2 + R(z,\zeta,\overline{z},\overline{\zeta}) \\ &= z\,\overline{z} + \tfrac{1}{2}z^2\,\underbrace{\left(\overline{\zeta} + \tfrac{1}{4}F^{(3)}_{2,0,2,0}\,\overline{z}^2\right)}_{=:\overline{\zeta'}} + \tfrac{1}{2}\overline{z}^2\,\underbrace{\left(\zeta + \tfrac{1}{4}F^{(3)}_{2,0,2,0}\,z^2\right)}_{=:\zeta'} + R(z,\zeta,\overline{z},\overline{\zeta}), \end{aligned}$$

whose remainder $R(z,\zeta,\overline{z},\overline{\zeta}) = O(4)$ contains no $z^2\,\overline{z}^2$ term. After the rigid transformation in $RT_4^{(3)}$:

$$z' = z, \qquad \zeta' = \zeta + \tfrac{1}{4}F^{(3)}_{2,0,2,0}\,z^2, \qquad w' = w, \qquad (**)$$

the polynomial $F^{(3)}(z,\zeta,\overline{z},\overline{\zeta})$ becomes $F^{(4)}(z',\zeta',\overline{z'},\overline{\zeta'}) = z'\,\overline{z'} + \frac{1}{2}z'^2\,\overline{\zeta'} + \frac{1}{2}\overline{z'}^2\,\zeta' + R'(z',\zeta',\overline{z'},\overline{\zeta'})$. The inverse of $(**)$ is

$$z = z', \qquad \zeta = \zeta' - \tfrac{1}{4}F^{(3)}_{2,0,2,0}\,z'^2, \qquad w = w'.$$

So $R'(z',\zeta',\overline{z'},\overline{\zeta'}) = R\big(z', \zeta' - \frac{1}{4}F^{(3)}_{2,0,2,0}\,z'^2, \overline{z'}, \overline{\zeta'} - \frac{1}{4}F^{(3)}_{2,0,2,0}\,\overline{z'}^2\big) = O(4)$ without $z'^2\,\overline{z'}^2$ term.

The relations are

$$
\begin{aligned}
&F^{(4)}_{0,1,3,0} = F^{(3)}_{0,1,3,0}, \quad F^{(4)}_{0,2,2,0} = F^{(3)}_{0,2,2,0}, \quad F^{(4)}_{1,1,2,0} = F^{(3)}_{1,1,2,0}, \quad F^{(4)}_{0,1,4,0} = F^{(3)}_{0,1,4,0},\\
&F^{(4)}_{0,2,3,0} = F^{(3)}_{0,2,3,0}, \quad F^{(4)}_{0,3,2,0} = F^{(3)}_{0,3,2,0}, \quad F^{(4)}_{1,2,2,0} = F^{(3)}_{1,2,2,0},\\
&F^{(4)}_{2,1,2,0} = -\tfrac{1}{2} F^{(3)}_{0,2,2,0}\, F^{(3)}_{2,0,2,0} + F^{(3)}_{2,1,2,0},\\
&F^{(4)}_{3,0,2,0} = -\tfrac{3}{2} F^{(3)}_{1,1,2,0}\, F^{(3)}_{2,0,2,0} - \tfrac{1}{2} F^{(3)}_{3,0,0,1}\, F^{(3)}_{2,0,2,0} + F^{(3)}_{3,0,2,0},\\
&F^{(4)}_{1,1,3,0} = -\tfrac{3}{2} F^{(3)}_{2,0,2,0} + F^{(3)}_{1,1,3,0},\\
&F^{(4)}_{2,0,3,0} = -\tfrac{1}{2} F^{(3)}_{0,1,3,0}\, F^{(3)}_{2,0,2,0} - \tfrac{3}{2} F^{(3)}_{2,0,1,1}\, F^{(3)}_{2,0,2,0} + F^{(3)}_{2,0,3,0}.
\end{aligned}
$$

We define $\mathscr{H}_5^{(4)}$, a codimension 1 submanifold of $\mathscr{H}_5^{(3)}$ by requiring $F^{(4)}_{2,0,2,0} = 0$. So $\dim_{\mathbb{R}} \mathscr{H}_5^{(2)} = 21 - 1 = 20$.

For any fixed element $F^{(4)} \in \mathscr{H}_5^{(4)}$, the stabilizer $RT_4^{(4)}(F^{(4)})$ is a codimension 1 subgroup of some $RT_4^{(3)}(F^{(4)})$. Hence $\dim_{\mathbb{R}} RT_4^{(4)}(F^{(3)}) = 28 - 1 = 27$. It contains elements $(f, g, \rho) \in RT_4^{(3)}(F^{(4)})$ satisfying

$$
\begin{aligned}
g_{2,0} = {} & e^{-2i\theta}\, F^{(4)}_{0,2,2,0}\, g_{1,0}^2 + e^{6i\theta}\, F^{(4)}_{2,0,0,2}\, \overline{g_{1,0}}^2 - e^{-4i\theta}\, g_{0,2}\, g_{1,0}^2 - e^{8i\theta}\, \overline{g_{0,2}}\, \overline{g_{1,0}}^2\\
& - 2\, F^{(4)}_{1,1,2,0}\, g_{1,0} - 2\, e^{4i\theta}\, F^{(4)}_{2,0,1,1}\, \overline{g_{1,0}} + 2\, e^{-2i\theta}\, g_{1,0}\, g_{1,1} + 2\, e^{6i\theta}\, \overline{g_{1,0}}\, \overline{g_{1,1}}\\
& + 3\, e^{2i\theta}\, g_{1,0}\, \overline{g_{1,0}} - e^{4i\theta}\, \overline{g_{2,0}}.
\end{aligned}
$$

In other words

$$
\begin{aligned}
\mathsf{Re}\big(e^{-2i\theta}\, g_{2,0}\big) = \mathsf{Re}\big\{ & - e^{-4i\theta}\, F^{(4)}_{0,2,2,0}\, g_{1,0}^2 - e^{-6i\theta}\, g_{0,2}\, g_{1,0}^2\\
& - 2\, e^{-2i\theta}\, F^{(4)}_{1,1,2,0}\, g_{1,0} + 2\, e^{-4i\theta}\, g_{1,0}\, g_{1,1} + \tfrac{3}{2}\, g_{1,0}\, \overline{g_{1,0}} \big\}.
\end{aligned}
$$

2.5 Fifth normalization: $F^{(5)}_{a,b,2,0} = 0$ for $2 \leqslant a+b \leqslant 3$ and $(a,b) \neq (2,0)$

Any element in $\mathscr{H}_5^{(4)}$ has expansion:

$$
F^{(4)}(z,\zeta,\overline{z},\overline{\zeta}) = z\,\overline{z} + \tfrac{1}{2} z^2 \underbrace{\Big(\overline{\zeta} + \sum_{2 \leqslant a+b \leqslant 3} \tfrac{F^{(4)}_{2,0,a,b}}{a!b!}\, \overline{z}^a\, \overline{\zeta}^b\Big)}_{=:\overline{\zeta'}} + \tfrac{1}{2} \overline{z}^2 \underbrace{\Big(\zeta + \sum_{2 \leqslant a+b \leqslant 3} \tfrac{F^{(4)}_{a,b,2,0}}{a!b!}\, z^a\, \zeta^b\Big)}_{=:\overline{\zeta}} + R(z,\zeta,\overline{z},\overline{\zeta}),
$$

whose remainder $R(z,\zeta,\overline{z},\overline{\zeta}) = O(4)$ contains no $z^a\, \zeta^b\, \overline{z}^2$ term for any $2 \leqslant a+b \leqslant 3$. After the rigid transformation in $RT_4^{(4)}(F^{(4)})$:

$$
z' = z, \qquad \zeta' = \zeta + \sum_{2 \leqslant a+b \leqslant 4} \frac{F^{(4)}_{a,b,2,0}}{a!b!}\, z^a\, \zeta^b, \qquad w' = w, \qquad (***)
$$

the polynomial $F^{(4)}(z,\zeta,\overline{z},\overline{\zeta})$ becomes $F^{(5)}(z',\zeta',\overline{z'},\overline{\zeta'}) = z'\,\overline{z'} + \frac{1}{2} z'^2\, \overline{\zeta'} + \frac{1}{2} \overline{z'}^2\, \zeta' + R'(z',\zeta',\overline{z'},\overline{\zeta'})$. The inverse of $(***)$ is

$$
z = z', \qquad \zeta = \zeta' + O_{z',\zeta'}(2), \qquad w = w'.
$$

So $R'(z',\zeta',\overline{z'},\overline{\zeta'}) = R\big(z',\zeta' + O_{z',\zeta'}(2),\overline{z'},\overline{\zeta'} + O_{\overline{z'},\overline{\zeta'}}(2)\big) = O(4)$ without $z'^a\,\zeta'^b\,\overline{z'}^2$ terms for any $2 \leqslant a+b \leqslant 3$.

The relations are

$$F^{(5)}_{0,1,3,0} = F^{(4)}_{0,1,3,0}, \qquad F^{(5)}_{0,1,4,0} = F^{(4)}_{0,1,4,0},$$
$$F^{(5)}_{0,2,3,0} = -2\,F^{(4)}_{0,1,3,0}\,F^{(4)}_{0,2,2,0} + F^{(4)}_{0,2,3,0}, \qquad F^{(5)}_{1,1,3,0} = -2\,F^{(4)}_{0,1,3,0}\,F^{(4)}_{1,1,2,0} + F^{(4)}_{1,1,3,0}.$$

We define $\mathscr{H}^{(5)}_5$ a codimension 12 submanifold of $\mathscr{H}^{(4)}_5$ where $F^{(5)}_{a,b,2,0} = 0$ for

$$(a,b) \in \{(1,1),(0,2),(3,0),(2,1),(1,2),(0,3)\}.$$

So $\dim_{\mathbb{R}} \mathscr{H}^{(5)}_5 = 20 - 12 = 8$.

For any fixed element $F^{(5)} \in \mathscr{H}^{(5)}_5$, the stabilizer $RT^{(5)}_4(F^{(5)})$ is a codimension 12 subgroup of some $RT^{(4)}_4(F^{(5)})$. Hence $\dim_{\mathbb{R}} RT^{(5)}_4(F^{(5)}) = 27-12 = 15$. It contains element $(f,g,\rho) \in RT^{(4)}_4(F^{(5)})$ satisfying

$$\begin{aligned}
&g_{0,2} = 0, \quad g_{1,1} = -2\,e^{4\,i\,\theta}\,\overline{g_{1,0}}, \quad g_{0,3} = 0, \quad g_{1,2} = 0,\\
&g_{2,1} = 2\,e^{6\,i\,\theta}\,\overline{g_{1,0}}^2 - 2\,e^{4\,i\,\theta}\,F^{(5)}_{3,0,0,1}\,\overline{g_{1,0}},\\
&g_{3,0} = -5\,e^{2\,i\,\theta}\,F^{(5)}_{3,0,0,1}\,\overline{g_{1,0}}\,g_{1,0} + e^{6\,i\,\theta}\,F^{(5)}_{3,0,0,2}\,\overline{g_{1,0}}^2 - 2\,e^{4\,i\,\theta}\,F^{(5)}_{3,0,1,1}\,\overline{g_{1,0}} - e^{4\,i\,\theta}\,F^{(5)}_{3,0,0,1}\,\overline{g_{2,0}}.
\end{aligned}$$

Since $(f,g,\rho) \in RT^{(4)}_4(F^{(5)})$ we have

$$\mathsf{Re}\big(e^{-2\,i\,\theta}\,g_{2,0}\big) = \mathsf{Re}\big(-\tfrac{5}{2}\,g_{1,0}\,\overline{g_{1,0}}\big).$$

Thus $e^{-2\,i\,\theta}\,g_{2,0} = -\frac{5}{2}\,g_{1,0}\,\overline{g_{1,0}} + i\,b_{2,0}$ for some $b_{2,0} \in \mathbb{R}$. So the last equation becomes

$$g_{3,0} = -\tfrac{5}{2}\,e^{2\,i\,\theta}\,F^{(5)}_{3,0,0,1}\,\overline{g_{1,0}}\,g_{1,0} + e^{6\,i\,\theta}\,F^{(5)}_{3,0,0,2}\,\overline{g_{1,0}}^2 - 2\,e^{4\,i\,\theta}\,F^{(5)}_{3,0,1,1}\,\overline{g_{1,0}} - i\,e^{2\,i\,\theta}\,F^{(5)}_{3,0,0,1}\,b_{2,0}.$$

The stabilizer $RT^{(5)}_4(F^{(5)})$ is parametrized by 3 real variables $b_{2,0}, r, \theta$ and 6 complex variables $g_{1,0}, g_{j,4-j}$ for $0 \leqslant j \leqslant 4$.

2.6 Sixth normalization: $F^{(6)}_{0,1,3,0} = 0$ and $\mathsf{Im}(F^{(6)}_{1,1,3,0}) = 0$

Any element in $\mathscr{H}^{(5)}_5$ has expansion:

$$\begin{aligned}
F^{(5)}(z,\zeta,\overline{z},\overline{\zeta}) = {} & z\,\overline{z} + \tfrac{1}{2}\,z^2\,\overline{\zeta} + \tfrac{1}{2}\,\zeta\,\overline{z}^2\\
& + \tfrac{1}{6}\,F^{(5)}_{0,1,3,0}\,\zeta\,\overline{z}^3 + \tfrac{1}{6}\,F^{(5)}_{3,0,0,1}\,z^3\,\overline{\zeta}\\
& + \tfrac{1}{6}\,F^{(5)}_{1,1,3,0}\,z\,\zeta\,\overline{z}^3 + \tfrac{1}{6}\,F^{(5)}_{3,0,1,1}\,z^3\,\overline{z}\,\overline{\zeta} + \tfrac{1}{24}\,F^{(5)}_{0,1,4,0}\,\zeta\,\overline{z}^4 + \tfrac{1}{24}\,F^{(5)}_{4,0,0,1}\,z^4\,\overline{\zeta}\\
& + \tfrac{1}{12}\,F^{(5)}_{0,2,3,0}\,\zeta^2\,\overline{z}^3 + \tfrac{1}{12}\,F^{(5)}_{3,0,0,2}\,z^3\,\overline{\zeta}^2\\
& + \zeta\,\overline{\zeta}\,(\dots).
\end{aligned}$$

We study how $g_{1,0}$ and $b_{2,0}$ act on this object, i.e. we consider an arbitrary $(f,g,\rho) \in RT^{(5)}_4(F^{(5)})$ with $r = 1$ and $\theta = g_{j,4-j} = 0$. They have the form

$$\begin{aligned}
f(z,\zeta) &= z - \tfrac{1}{2}\,\overline{g_{1,0}}\,z^2 + O(3),\\
g(z,\zeta) &= g_{1,0}\,z + \zeta + \tfrac{1}{2}\,(-\tfrac{5}{2}\,g_{1,0}\,\overline{g_{1,0}} + i\,b_{2,0})\,z^2 + O(3),\\
\rho &= 1.
\end{aligned}$$

This transformation sends $F^{(5)}$ to $F'^{(5)} \in \mathscr{H}_5^{(5)}$ such that

$$\begin{aligned}
F'^{(5)}_{3,0,0,1} &= F^{(5)}_{3,0,0,1} + 3\,\overline{g_{1,0}},\\
F'^{(5)}_{3,0,1,1} &= F^{(5)}_{3,0,1,1} - 3\,F^{(5)}_{3,0,0,1}\,g_{1,0} - F^{(5)}_{3,0,0,2}\,\overline{g_{1,0}} + \tfrac{15}{2}\,g_{1,0}\,\overline{g_{1,0}} - 3\,i\,b_{2,0}.
\end{aligned}$$

So by a unique choice of $g_{1,0}$ and $b_{2,0}$, namely

$$g_{1,0} = -\tfrac{1}{3}F^{(5)}_{0,1,3,0}, \qquad b_{2,0} = \tfrac{i}{18}\left(F^{(5)}_{0,2,3,0}\,F^{(5)}_{0,1,3,0} - F^{(5)}_{3,0,0,2}\,F^{(5)}_{3,0,0,1} + 3\,F^{(5)}_{1,1,3,0} - 3\,F^{(5)}_{3,0,1,1}\right),$$

we can normalize $F'^{(5)}_{3,0,0,1}$ to 0 and $F'^{(5)}_{3,0,1,1}$ to a real number. The polynomial $F^{(5)}(z,\zeta,\overline{z},\overline{\zeta})$ becomes

$$\begin{aligned}
F^{(6)}(z',\zeta',\overline{z'},\overline{\zeta'}) &= z'\,\overline{z'} + \tfrac{1}{2}\,z'^2\,\overline{\zeta'} + \tfrac{1}{2}\,\zeta'\,\overline{z'}^2\\
&\quad + \tfrac{1}{6}\,F^{(6)}_{1,1,3,0}\,z'\,\zeta'\,\overline{z'}^3 + \tfrac{1}{6}\,F^{(6)}_{3,0,1,1}\,z'^3\,\overline{z'}\,\overline{\zeta'} + \tfrac{1}{24}\,F^{(6)}_{0,1,4,0}\,\zeta'\,\overline{z'}^4 + \tfrac{1}{24}\,F^{(6)}_{4,0,0,1}\,z'^4\,\overline{\zeta'}\\
&\quad + \tfrac{1}{12}\,F^{(6)}_{0,2,3,0}\,\zeta'^2\,\overline{z'}^3 + \tfrac{1}{12}\,F^{(6)}_{3,0,0,2}\,z'^3\,\overline{\zeta'}^2\\
&\quad + \zeta'\,\overline{\zeta'}\,(\dots)\\
&= z'\,\overline{z'} + \tfrac{1}{2}\,z'^2\,\overline{\zeta'} + \tfrac{1}{2}\,\zeta'\,\overline{z'}^2\\
&\quad + \tfrac{1}{6}\,Q_0\,z'\,\zeta'\,\overline{z'}^3 + \tfrac{1}{6}\,Q_0\,z'^3\,\overline{z'}\,\overline{\zeta'} + \tfrac{1}{24}\,V_0\,\zeta'\,\overline{z'}^4 + \tfrac{1}{24}\,\overline{V_0}\,z'^4\,\overline{\zeta'}\\
&\quad + \tfrac{1}{12}\,I_0\,\zeta'^2\,\overline{z'}^3 + \tfrac{1}{12}\,\overline{I_0}\,z'^3\,\overline{\zeta'}^2\\
&\quad + \zeta'\,\overline{\zeta'}\,(\dots),
\end{aligned}$$

where $I_0 := F^{(6)}_{0,2,3,0} \in \mathbb{C}$, $V_0 := F^{(6)}_{0,1,4,0} \in \mathbb{C}$, $Q_0 := F^{(6)}_{1,1,3,0} \in \mathbb{R}$.

The relations are

$$\begin{aligned}
I_0 &= F^{(5)}_{0,2,3,0} + 2\,F^{(5)}_{3,0,0,1},\\
V_0 &= -\tfrac{5}{3}\,(F^{(5)}_{0,1,3,0})^2 + F^{(5)}_{0,1,4,0},\\
Q_0 &= \tfrac{1}{6}\,F^{(5)}_{0,2,3,0}\,F^{(5)}_{0,1,3,0} + \tfrac{1}{2}\,F^{(5)}_{3,0,0,1}\,F^{(5)}_{0,1,3,0} + \tfrac{1}{6}\,F^{(5)}_{3,0,0,2}\,F^{(5)}_{3,0,0,1} + \tfrac{1}{2}\,F^{(5)}_{1,1,3,0} + \tfrac{1}{2}\,F^{(5)}_{3,0,1,1}.
\end{aligned}$$

Each $F^{(t)}_{a,b,c,d}$ is a rational function of $F^{(t-1)}_{a',b',c',d'}$ for $t = 5,4,3,2$ and each $F^{(1)}_{a,b,c,d}$ is a rational function of $F_{a',b',c',d'}$. By composing these rational functions, one can express I_0, V_0, Q_0 in terms of original coordinates $F_{a,b,c,d}$:

$$\begin{aligned}
I_0 &= \frac{\text{52 terms in degree 9}}{F^{3/2}_{1,0,1,0}\,(F_{0,1,1,0}\,F_{1,0,2,0} - F_{0,1,2,0}\,F_{1,0,1,0})^3\,(F_{1,0,0,1}\,F_{2,0,1,0} - F_{1,0,1,0}\,F_{2,0,0,1})},\\
V_0 &= \frac{\text{11 terms in degree 4}}{3\,F_{1,0,1,0}\,(F_{0,1,1,0}\,F_{1,0,2,0} - F_{0,1,2,0}\,F_{1,0,1,0})^2},\\
Q_0 &= \frac{\text{824 terms in degree 18}}{6\,F^{3}_{1,0,1,0}\,(F_{0,1,1,0}\,F_{1,0,2,0} - F_{0,1,2,0}\,F_{1,0,1,0})^4\,(F_{1,0,0,1}\,F_{2,0,1,0} - F_{1,0,1,0}\,F_{2,0,0,1})^4}.
\end{aligned}$$

The numerator of I_0 is

$$
\begin{aligned}
&F_{0,1,1,0}^{3}\, F_{1,0,0,1}\, F_{1,0,1,0}^{2}\, F_{1,0,2,0}\, F_{2,0,1,0}\, F_{2,0,3,0} - F_{0,1,1,0}^{3}\, F_{1,0,0,1}\, F_{1,0,1,0}^{2}\, F_{1,0,3,0}\, F_{2,0,1,0}\, F_{2,0,2,0}\\
&+ 2\, F_{0,1,1,0}^{3}\, F_{1,0,0,1}\, F_{1,0,1,0}\, F_{1,0,2,0}^{3}\, F_{3,0,1,0} - 6\, F_{0,1,1,0}^{3}\, F_{1,0,0,1}\, F_{1,0,2,0}^{3}\, F_{2,0,1,0}^{2}\\
&- F_{0,1,1,0}^{3}\, F_{1,0,1,0}^{3}\, F_{1,0,2,0}\, F_{2,0,0,1}\, F_{2,0,3,0} + F_{0,1,1,0}^{3}\, F_{1,0,1,0}^{3}\, F_{1,0,3,0}\, F_{2,0,0,1}\, F_{2,0,2,0}\\
&- 2\, F_{0,1,1,0}^{3}\, F_{1,0,1,0}^{2}\, F_{1,0,2,0}^{3}\, F_{3,0,0,1} + 6\, F_{0,1,1,0}^{3}\, F_{1,0,1,0}\, F_{1,0,2,0}^{3}\, F_{2,0,0,1}\, F_{2,0,1,0}\\
&- F_{0,1,1,0}^{2}\, F_{0,1,2,0}\, F_{1,0,0,1}\, F_{1,0,1,0}^{3}\, F_{2,0,1,0}\, F_{2,0,3,0} - 6\, F_{0,1,1,0}^{2}\, F_{0,1,2,0}\, F_{1,0,0,1}\, F_{1,0,1,0}^{2}\, F_{1,0,2,0}^{2}\, F_{3,0,1,0}\\
&+ F_{0,1,1,0}^{2}\, F_{0,1,2,0}\, F_{1,0,0,1}\, F_{1,0,1,0}^{2}\, F_{1,0,3,0}\, F_{2,0,1,0}^{2} + 18\, F_{0,1,1,0}^{2}\, F_{0,1,2,0}\, F_{1,0,0,1}\, F_{1,0,1,0}\, F_{1,0,2,0}^{2}\, F_{2,0,1,0}^{2}\\
&+ F_{0,1,1,0}^{2}\, F_{0,1,2,0}\, F_{1,0,1,0}^{4}\, F_{2,0,0,1}\, F_{2,0,3,0} + 6\, F_{0,1,1,0}^{2}\, F_{0,1,2,0}\, F_{1,0,1,0}^{3}\, F_{1,0,2,0}^{2}\, F_{3,0,0,1}\\
&- F_{0,1,1,0}^{2}\, F_{0,1,2,0}\, F_{1,0,1,0}^{3}\, F_{1,0,3,0}\, F_{2,0,0,1}\, F_{2,0,1,0} - 18\, F_{0,1,1,0}^{2}\, F_{0,1,2,0}\, F_{1,0,1,0}^{2}\, F_{1,0,2,0}^{2}\, F_{2,0,0,1}\, F_{2,0,1,0}\\
&+ F_{0,1,1,0}^{2}\, F_{0,1,3,0}\, F_{1,0,0,1}\, F_{1,0,1,0}^{3}\, F_{2,0,1,0}\, F_{2,0,2,0} - F_{0,1,1,0}^{2}\, F_{0,1,3,0}\, F_{1,0,0,1}\, F_{1,0,1,0}^{2}\, F_{1,0,2,0}\, F_{2,0,1,0}^{2}\\
&- F_{0,1,1,0}^{2}\, F_{0,1,3,0}\, F_{1,0,1,0}^{4}\, F_{2,0,0,1}\, F_{2,0,2,0} + F_{0,1,1,0}^{2}\, F_{0,1,3,0}\, F_{1,0,1,0}^{3}\, F_{1,0,2,0}\, F_{2,0,0,1}\, F_{2,0,1,0}\\
&- 2\, F_{0,1,1,0}^{2}\, F_{1,0,0,1}\, F_{1,0,1,0}^{3}\, F_{1,0,2,0}\, F_{1,1,3,0}\, F_{2,0,1,0} + 2\, F_{0,1,1,0}^{2}\, F_{1,0,0,1}\, F_{1,0,1,0}^{3}\, F_{1,0,3,0}\, F_{1,1,2,0}\, F_{2,0,1,0}\\
&+ 2\, F_{0,1,1,0}^{2}\, F_{1,0,1,0}^{4}\, F_{1,0,2,0}\, F_{1,1,3,0}\, F_{2,0,0,1} - 2\, F_{0,1,1,0}^{2}\, F_{1,0,1,0}^{4}\, F_{1,0,3,0}\, F_{1,1,2,0}\, F_{2,0,0,1}\\
&+ 6\, F_{0,1,1,0}\, F_{0,1,2,0}^{2}\, F_{1,0,0,1}\, F_{1,0,1,0}^{3}\, F_{1,0,2,0}\, F_{3,0,1,0} - 18\, F_{0,1,1,0}\, F_{0,1,2,0}^{2}\, F_{1,0,0,1}\, F_{1,0,1,0}^{2}\, F_{1,0,2,0}\, F_{2,0,1,0}^{2}\\
&- 6\, F_{0,1,1,0}\, F_{0,1,2,0}^{2}\, F_{1,0,1,0}^{4}\, F_{1,0,2,0}\, F_{3,0,0,1} + 18\, F_{0,1,1,0}\, F_{0,1,2,0}^{2}\, F_{1,0,1,0}^{3}\, F_{1,0,2,0}\, F_{2,0,0,1}\, F_{2,0,1,0}\\
&+ 2\, F_{0,1,1,0}\, F_{0,1,2,0}\, F_{1,0,0,1}\, F_{1,0,1,0}^{4}\, F_{1,1,3,0}\, F_{2,0,1,0} - 2\, F_{0,1,1,0}\, F_{0,1,2,0}\, F_{1,0,0,1}\, F_{1,0,1,0}^{3}\, F_{1,0,3,0}\, F_{1,1,1,0}\, F_{2,0,1,0}\\
&- 2\, F_{0,1,1,0}\, F_{0,1,2,0}\, F_{1,0,1,0}^{5}\, F_{1,1,3,0}\, F_{2,0,0,1} + 2\, F_{0,1,1,0}\, F_{0,1,2,0}\, F_{1,0,1,0}^{4}\, F_{1,0,3,0}\, F_{1,1,1,0}\, F_{2,0,0,1}\\
&- 2\, F_{0,1,1,0}\, F_{0,1,3,0}\, F_{1,0,0,1}\, F_{1,0,1,0}^{4}\, F_{1,1,2,0}\, F_{2,0,1,0} + 2\, F_{0,1,1,0}\, F_{0,1,3,0}\, F_{1,0,0,1}\, F_{1,0,1,0}^{3}\, F_{1,0,2,0}\, F_{1,1,1,0}\, F_{2,0,1,0}\\
&+ 2\, F_{0,1,1,0}\, F_{0,1,3,0}\, F_{1,0,1,0}^{5}\, F_{1,1,2,0}\, F_{2,0,0,1} - 2\, F_{0,1,1,0}\, F_{0,1,3,0}\, F_{1,0,1,0}^{4}\, F_{1,0,2,0}\, F_{1,1,1,0}\, F_{2,0,0,1}\\
&- F_{0,1,1,0}\, F_{0,2,2,0}\, F_{1,0,0,1}\, F_{1,0,1,0}^{4}\, F_{1,0,3,0}\, F_{2,0,1,0} + F_{0,1,1,0}\, F_{0,2,2,0}\, F_{1,0,1,0}^{5}\, F_{1,0,3,0}\, F_{2,0,0,1}\\
&+ F_{0,1,1,0}\, F_{0,2,3,0}\, F_{1,0,0,1}\, F_{1,0,1,0}^{4}\, F_{1,0,2,0}\, F_{2,0,1,0} - F_{0,1,1,0}\, F_{0,2,3,0}\, F_{1,0,1,0}^{5}\, F_{1,0,2,0}\, F_{2,0,0,1}\\
&- 2\, F_{0,1,2,0}^{3}\, F_{1,0,0,1}\, F_{1,0,1,0}^{4}\, F_{3,0,1,0} + 6\, F_{0,1,2,0}^{3}\, F_{1,0,0,1}\, F_{1,0,1,0}^{3}\, F_{2,0,1,0}^{2}\\
&+ 2\, F_{0,1,2,0}^{3}\, F_{1,0,1,0}^{5}\, F_{3,0,0,1} - 6\, F_{0,1,2,0}^{3}\, F_{1,0,1,0}^{4}\, F_{2,0,0,1}\, F_{2,0,1,0}\\
&+ F_{0,1,2,0}\, F_{0,2,1,0}\, F_{1,0,0,1}\, F_{1,0,1,0}^{4}\, F_{1,0,3,0}\, F_{2,0,1,0} - F_{0,1,2,0}\, F_{0,2,1,0}\, F_{1,0,1,0}^{5}\, F_{1,0,3,0}\, F_{2,0,0,1}\\
&- F_{0,1,2,0}\, F_{0,2,3,0}\, F_{1,0,0,1}\, F_{1,0,1,0}^{5}\, F_{2,0,1,0} + F_{0,1,2,0}\, F_{0,2,3,0}\, F_{1,0,1,0}^{6}\, F_{2,0,0,1}\\
&- F_{0,1,3,0}\, F_{0,2,1,0}\, F_{1,0,0,1}\, F_{1,0,1,0}^{4}\, F_{1,0,2,0}\, F_{2,0,1,0} + F_{0,1,3,0}\, F_{0,2,1,0}\, F_{1,0,1,0}^{5}\, F_{1,0,2,0}\, F_{2,0,0,1}\\
&+ F_{0,1,3,0}\, F_{0,2,2,0}\, F_{1,0,0,1}\, F_{1,0,1,0}^{5}\, F_{2,0,1,0} - F_{0,1,3,0}\, F_{0,2,2,0}\, F_{1,0,1,0}^{6}\, F_{2,0,0,1}.
\end{aligned}
$$

The numerator of V_0 is

$$
\begin{aligned}
&3\, F_{0,1,1,0}^{2}\, F_{1,0,2,0}\, F_{1,0,4,0} - 5\, F_{0,1,1,0}^{2}\, F_{1,0,3,0}^{2} - 3\, F_{0,1,1,0}\, F_{0,1,2,0}\, F_{1,0,1,0}\, F_{1,0,4,0}\\
&+ 12\, F_{0,1,1,0}\, F_{0,1,2,0}\, F_{1,0,2,0}\, F_{1,0,3,0} + 10\, F_{0,1,1,0}\, F_{0,1,3,0}\, F_{1,0,1,0}\, F_{1,0,3,0} - 12\, F_{0,1,1,0}\, F_{0,1,3,0}\, F_{1,0,2,0}^{2}\\
&- 3\, F_{0,1,1,0}\, F_{0,1,4,0}\, F_{1,0,1,0}\, F_{1,0,2,0} - 12\, F_{0,1,2,0}^{2}\, F_{1,0,1,0}\, F_{1,0,3,0} + 12\, F_{0,1,2,0}\, F_{0,1,3,0}\, F_{1,0,1,0}\, F_{1,0,2,0}\\
&+ 3\, F_{0,1,2,0}\, F_{0,1,4,0}\, F_{1,0,1,0}^{2} - 5\, F_{0,1,3,0}^{2}\, F_{1,0,1,0}^{2}.
\end{aligned}
$$

We define $\mathscr{H}_5^{(6)}$ a codimension 3 submanifold of $\mathscr{H}_5^{(5)}$ by requiring $F_{0,3,1,0}^{(6)} = 0$ and $Im(F_{1,1,3,0}^{(6)}) = 0$.

For any fixed element $F^{(6)} \in \mathscr{H}_5^{(6)}$, the stabilizer $RT_4^{(6)}(F^{(6)})$ is a codimension 3 subgroup of some $RT_4^{(5)}(F^{(6)})$. Hence $\dim_{\mathbb{R}} RT_4^{(6)}(F^{(6)}) = 15-3 = 12$. It contains elements $(f, g, \rho) \in RT_4^{(5)}(F^{(6)})$ of the form

$$f(z,\zeta) = r\, e^{i\,\theta}\, z, \qquad g(z,\zeta) = e^{2\,i\,\theta}\, \zeta + O(4), \qquad \rho = r^2.$$

Note that this stabilizer group no longer depends on the choice of $F^{(6)} \in \mathscr{H}_5^{(6)}$. We simply write it as $RT_4^{(6)}$.

2.7 Passing to the infinite dimension

After these six normalizations, we killed $f_{0,1}$ and $g_{1,0}$. It is a miracle that now we can work directly on the infinite dimensional objects. We define $\mathscr{H}^{(7)}$ be the subspace of $\mathscr{H}$ consisting of all power series $u = F^{(7)}(z, \zeta, \overline{z}, \overline{\zeta}) = \frac{F^{(7)}_{a,b,c,d}}{a!b!c!d!} z^a\, \zeta^b\, \overline{z}^c\, \overline{\zeta}^d$ such that

- $F^{(7)}_{a,b,1,0} = 0$, $\forall (a,b) \neq (1,0)$; $F^{(7)}_{1,0,1,0} = 1$;
- $F^{(7)}_{a,b,2,0} = 0$, $\forall (a,b) \neq (0,1)$; $F^{(7)}_{0,1,2,0} = 1$;
- $F^{(7)}_{3,0,0,1} = 0$, $F^{(7)}_{3,0,1,1} = F^{(7)}_{1,1,3,0}$.

It is both infinitely-dimensional and infinitely-codimensional in $\mathscr{H}$. But it has a finitely-dimensional stabilizer.

By definition, any element in $\mathscr{H}^{(7)}$ has its degree 5 truncation in $\mathscr{H}_5^{(6)}$.

Theorem 2.4. *Any element $u = F(z, \zeta, \overline{z}, \overline{\zeta})$ in $\mathscr{H}$ can be sent to some element $u = F^{(7)}(z, \zeta, \overline{z}, \overline{\zeta})$ in $\mathscr{H}^{(7)}$ by some (but not unique) element in RT. The ambiguity can be controlled in the following sense: any element $(f, g, \rho) \in RT$ sending one element $F^{(7)} \in \mathscr{H}^{(7)}$ to another $F'^{(7)} \in \mathscr{H}^{(7)}$ has the form $f(z,\zeta) = r\, e^{i\,\theta}\, z$, $g(z,\zeta) = e^{2\,i\,\theta}\, \zeta$ and $\rho = r^2$.*

Proof. One shall simply use the six normalizations above with a bit modification: in the second (killing $F_{a,b,1,0}$) and the fifth (killing $F_{a,b,2,0}$) normalization, we normalize for infinitely many (a,b). More precisely, we start from $u = F(z, \zeta, \overline{z}, \overline{\zeta})$ in $\mathscr{H}$. After the six normalizations above we get $u = F^{(6)}(z, \zeta, \overline{z}, \overline{\zeta})$ whose degree 5 truncation $\pi_5\big(F^{(6)}(z, \zeta, \overline{z}, \overline{\zeta})\big)$ is in $\mathscr{H}_5^{(6)}$, i.e.

- $F^{(6)}_{a,b,1,0} = 0$, $\forall 2 \leqslant a+b \leqslant 4$; $F^{(6)}_{1,0,1,0} = 1$;
- $F^{(6)}_{a,b,2,0} = 0$, $\forall 2 \leqslant a+b \leqslant 4$; $F^{(6)}_{0,1,2,0} = 1$;
- $F^{(6)}_{3,0,0,1} = 0$, $F^{(6)}_{3,0,1,1} = F^{(6)}_{1,1,3,0}$.

Then we do 2 more normalizations. First

$$z' = z + \sum_{a+b \geqslant 5} \frac{F^{(6)}_{a,b,1,0}}{a!b!} z^a\, \zeta^b, \qquad \zeta' = \zeta, \qquad w' = w,$$

gives us $u' = F'(z', \zeta', \overline{z'}, \overline{\zeta'})$ with

- $F'_{a,b,1,0} = 0$, $\forall a+b \geqslant 2$; $F'_{1,0,1,0} = 1$;
- $F'_{a,b,2,0} = 0$, $\forall 2 \leqslant a+b \leqslant 4$; $F'_{0,1,2,0} = 1$;
- $F'_{3,0,0,1} = 0$, $F'_{3,0,1,1} = F'_{1,1,3,0}$.

Then

$$z'' = z', \qquad \zeta'' = \zeta' + \sum_{a+b \geqslant 5} \tfrac{F'_{a,b,2,0}}{a!b!}\, z^a\, \zeta^b, \qquad w' = w,$$

gives us $u'' = F''(z'', \zeta'', \overline{z''}, \overline{\zeta''})$ with

- $F''_{a,b,1,0} = 0$, $\forall a+b \geqslant 2$; $F''_{1,0,1,0} = 1$;
- $F''_{a,b,2,0} = 0$, $\forall a+b \geqslant 2$; $F''_{0,1,2,0} = 1$;
- $F''_{3,0,0,1} = 0$, $F''_{3,0,1,1} = F''_{1,1,3,0}$.

So $u'' = F''(z'', \zeta'', \overline{z''}, \overline{\zeta''})$ is in $\mathscr{H}^{(7)}$. It is the form we want.

Now suppose that $(f, g, \rho) \in RT$ sends one element $F^{(7)} \in \mathscr{H}^{(7)}$ to another $F'^{(7)} \in \mathscr{H}^{(7)}$. In the truncated setting, $\pi_4(f, g, \rho) \in RT_4$ sends $\pi_5(F^{(7)}) \in \mathscr{H}_5^{(6)}$ to $\pi_5(F'^{(7)}) \in \mathscr{H}_5^{(6)}$. So the truncated action $\pi_4(f, g, \rho)$ should be in the stabilizer $RT_4^{(6)}$. That is to say

$$f(z,\zeta) = r\, e^{i\,\theta}\, z + O(5), \qquad g(z,\zeta) = e^{2\,i\,\theta}\, \zeta + O(4), \qquad \rho = r^2.$$

Recall the fundamental equation

$$\rho\, F^{(7)}(z, \zeta, \overline{z}, \overline{\zeta}) = F'^{(7)}\big(f(z,\zeta), g(z,\zeta), \overline{f(z,\zeta)}, \overline{g(z,\zeta)}\big).$$

When we compare the coefficients of $z^j\, \zeta^{n-j}\, \overline{z}$ for any $n \geqslant 2$ and $0 \leqslant j \leqslant n$:

$$\begin{aligned} 0 &= \mathrm{Coef}_{z^j\, \zeta^{n-j}\, \overline{z}}\big\{F'^{(7)}\big(f(z,\zeta), g(z,\zeta), \overline{f(z,\zeta)}, \overline{g(z,\zeta)}\big)\big\} \\ &= \mathrm{Coef}_{z^j\, \zeta^{n-j}\, \overline{z}}\big\{f(z,\zeta)\, \overline{f(z,\zeta)}\big\} + \mathrm{Coef}_{z^j\, \zeta^{n-j}\, \overline{z}}\big\{\sum_{c=0, d=1} (\dots)\, \overline{g(z,\zeta)}^d\big\} \\ &\quad + \mathrm{Coef}_{z^j\, \zeta^{n-j}\, \overline{z}}\big\{\sum_{c+d \geqslant 2} (\dots)\, \overline{f(z,\zeta)}^c\, \overline{g(z,\zeta)}^d\big\} \end{aligned}$$

The last two terms are 0 because they only contain monomials with $\deg_{\overline{z}} = 0$ or $\deg_{\overline{z}} + \deg_{\overline{\zeta}} \geqslant 2$. The first term gives us $0 = r\, e^{-i\,\theta}\, \frac{f_{j,n-j}}{j!(n-j)!}$. Hence $f(z,\zeta) = r\, e^{i\,\theta}\, z$.

When we compare the coefficients of $z^j\, \zeta^{n-j}\, \overline{z}^2$ for any $n \geqslant 2$ and $0 \leqslant j \leqslant n$:

$$\begin{aligned} 0 &= \mathrm{Coef}_{z^j\, \zeta^{n-j}\, \overline{z}^2}\big\{F'^{(7)}\big(f(z,\zeta), g(z,\zeta), \overline{f(z,\zeta)}, \overline{g(z,\zeta)}\big)\big\} \\ &= \mathrm{Coef}_{z^j\, \zeta^{n-j}\, \overline{z}^2}\big\{f(z,\zeta)\, \overline{f(z,\zeta)}\big\} + \mathrm{Coef}_{z^j\, \zeta^{n-j}\, \overline{z}^2}\big\{\sum_{c=0, d=1} (\dots)\, \overline{g(z,\zeta)}\big\} \\ &\quad + \mathrm{Coef}_{z^j\, \zeta^{n-j}\, \overline{z}^2}\big\{\tfrac{1}{2} g(z,\zeta)\, \overline{f(z,\zeta)}^2\big\} + \mathrm{Coef}_{z^j\, \zeta^{n-j}\, \overline{z}^2}\big\{\textstyle\sum_{c=1, d=1} (\dots)\, \overline{f(z,\zeta)}\, \overline{g(z,\zeta)}\big\} \\ &\quad + \mathrm{Coef}_{z^j\, \zeta^{n-j}\, \overline{z}^2}\big\{\sum_{c=0, d=2} (\dots)\, \overline{g(z,\zeta)}^2\big\} + \mathrm{Coef}_{z^j\, \zeta^{n-j}\, \overline{z}}\big\{\sum_{c+d \geqslant 3} (\dots)\, \overline{f(z,\zeta)}^c\, \overline{g(z,\zeta)}^d\big\} \end{aligned}$$

Each term, except the third, is 0. The third term gives us $0 = \frac{1}{2} r^2\, \frac{g_{j,n-j}}{j!(n-j)!}$. Hence $g(z,\zeta) = e^{2\,i\,\theta}\, \zeta$. $\square$

2.8 Braches: $I_0 \neq 0$, $V_0 \neq 0$ and $I_0 \equiv 0 \equiv V_0$

To get a normal form under the full rigid transformation group, including rotations and dilations

$$z' = r\, e^{i\theta}\, z, \qquad \zeta' = e^{2\, i\, \theta}\, \zeta, \qquad \rho = r^2,$$

we should normalize I_0 or V_0. Such a rotation and a dilation would send (I_0, V_0, Q_0) to (I_0', V_0', Q_0') with

$$I_0' = r^{-1}\, e^{-i\,\theta}\, I_0, \qquad V_0' = r^{-2}\, e^{2\, i\, \theta}\, V_0, \qquad Q_0' = r^{-2}\, Q_0.$$

We avoid the mixed type and focus on the 3 possible branches:

- $I_0 \neq 0$;
- $I_0 \equiv 0$ but $V_0 \neq 0$;
- $I_0 \equiv 0 \equiv V_0$.

2.8.1 Brach $I_0 \neq 0$

In this brach we can normalize I_0 to 1 by choose $r\, e^{i\,\theta} = I_0$. More precisely, for any surface in $\mathscr{H}^{(7)}$ graphed by:

$$\begin{aligned} F^{(7)}(z, \zeta, \overline{z}, \overline{\zeta}) = z\, \overline{z} &+ \tfrac{1}{2}\, z^2\, \overline{\zeta} + \tfrac{1}{2}\, \zeta\, \overline{z}^2 \\ &+ \tfrac{1}{6}\, Q_0\, z\, \zeta\, \overline{z}^3 + \tfrac{1}{6}\, Q_0\, z^3\, \overline{z}\, \overline{\zeta} + \tfrac{1}{24}\, V_0\, \zeta\, \overline{z}^4 + \tfrac{1}{24}\, \overline{V_0}\, z^4\, \overline{\zeta} \\ &+ \tfrac{1}{12}\, I_0\, \zeta^2\, \overline{z}^3 + \tfrac{1}{12}\, \overline{I_0}\, z^3\, \overline{\zeta}^2 \\ &+ \zeta\, \overline{\zeta}\, (\dots) + O(6), \end{aligned}$$

where $I_0 \neq 0$, after the transformation

$$z' = I_0\, z, \qquad \zeta' = \frac{I_0^2}{|I_0|^2}\, \zeta, \qquad , \rho = |I_0|^2,$$

the polynomial $F^{(7)}(z, \zeta, \overline{z}, \overline{\zeta})$ becomes

$$\begin{aligned} F^{(8,1)}(z', \zeta', \overline{z'}, \overline{\zeta'}) = z'\, \overline{z'} &+ \tfrac{1}{2}\, z'^2\, \overline{\zeta'} + \tfrac{1}{2}\, \zeta'\, \overline{z'}^2 \\ &+ \tfrac{1}{6}\, Q\, z'\, \zeta'\, \overline{z'}^3 + \tfrac{1}{6}\, Q\, z'^3\, \overline{z'}\, \overline{\zeta'} + \tfrac{1}{24}\, V\, \zeta'\, \overline{z'}^4 + \tfrac{1}{24}\, \overline{V}\, z'^4\, \overline{\zeta'} \\ &+ \tfrac{1}{12}\, \zeta'^2\, \overline{z'}^3 + \tfrac{1}{12}\, z'^3\, \overline{\zeta'}^2 \\ &+ \zeta'\, \overline{\zeta'}\, (\dots) + O(6), \end{aligned}$$

where

$$V = \frac{V_0}{\overline{I_0}^2}, \qquad Q = \frac{Q_0}{|I_0|^2},$$

We define $\mathscr{H}^{(8,1)}$ a codimension 2 submanifold of $\mathscr{H}^{(7)}$ by requiring $I_0 = 1$.
For any fixed element $F^{(8,1)} \in \mathscr{H}^{(8,1)}$, the stabilizer $RT^{(8,1)}$ is the identity.

2.8.2 Brach $I_0 \equiv 0$ but $V_0 \neq 0$

In this brach we can normalize V_0 to 1 by choose $r^2\, e^{-2\,i\,\theta} = V_0$. This equation has two solutions: $r\, e^{i\,\theta} = \pm x$, where $x^2 = \overline{V_0}$ and $\arg(x) \in [0, \pi)$. More precisely, for any surface in $\mathscr{H}^{(7)}$ graphed by:

$$\begin{aligned} F^{(7)}(z,\zeta,\overline{z},\overline{\zeta}) = {} & z\,\overline{z} + \tfrac{1}{2}\, z^2\,\overline{\zeta} + \tfrac{1}{2}\,\zeta\,\overline{z}^2 \\ & + \tfrac{1}{6}\, Q_0\, z\,\zeta\,\overline{z}^3 + \tfrac{1}{6}\, Q_0\, z^3\,\overline{z}\,\overline{\zeta} + \tfrac{1}{24}\, V_0\,\zeta\,\overline{z}^4 + \tfrac{1}{24}\,\overline{V_0}\, z^4\,\overline{\zeta} \\ & + \underbrace{\tfrac{1}{12}\, I_0\,\zeta^2\,\overline{z}^3 + \tfrac{1}{12}\,\overline{I_0}\, z^3\,\overline{\zeta}^2}_{=0,\text{ when } I_0\equiv 0} \\ & + \zeta\,\overline{\zeta}\,(\ldots) + O(6), \end{aligned}$$

where $V_0 \neq 0$, after the transformation

$$z' = x\, z, \qquad \zeta' = \frac{\overline{V_0}}{|V_0|}\,\zeta, \qquad ,\rho = |V_0|,$$

the polynomial $F^{(7)}(z,\zeta,\overline{z},\overline{\zeta})$ becomes

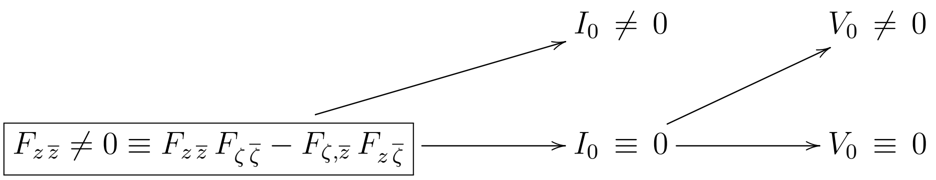
$$\begin{aligned} F^{(8,2)}(z',\zeta',\overline{z'},\overline{\zeta'}) = {} & z'\,\overline{z'} + \tfrac{1}{2}\, z'^2\,\overline{\zeta'} + \tfrac{1}{2}\,\zeta'\,\overline{z'}^2 + \tfrac{1}{6}\, Q\, z'\,\zeta'\,\overline{z'}^3 + \tfrac{1}{6}\, Q\, z'^3\,\overline{z'}\,\overline{\zeta'} \\ & + \tfrac{1}{24}\,\zeta'\,\overline{z'}^4 + \tfrac{1}{24}\, z'^4\,\overline{\zeta'} + \zeta'\,\overline{\zeta'}\,(\ldots) + O(6), \end{aligned}$$

where $Q = \frac{Q_0}{|V_0|}$. We define $\mathscr{H}^{(8,2)}$ a codimension 2 submanifold of $\mathscr{H}^{(7)}$ by requiring $V_0 = 1$. For any fixed element $F^{(8,2)} \in \mathscr{H}^{(8,2)}$, the stabilizer $RT^{(8,2)}$ is a group of two elements: the identity and $(-z, \zeta, 1)$.

2.8.3 Branch $I_0 \equiv 0 \equiv V_0$

Since Q_0 can be generated by I_0, V_0 and their differentials, we have $Q_0 \equiv 0$. The structure equations degenerate to the model case. The surface is equivalent as the Gaussier-Merker model $u = \frac{z\,\overline{z} + \frac{1}{2}\zeta^2\,\overline{z} + \frac{1}{2} z^2\,\overline{\zeta}}{1-\zeta\,\overline{\zeta}}$.

To conclude, we draw the branches from our root assumption.



where I_0 and V_0 are relative invariants of order 5.

Theorem 2.5. *Within the branch $I_0 \equiv 0$:*

(1) When $V_0 \equiv 0$, the surface is equivalent to the Gaussier-Merker model $u = \frac{z\,\overline{z} + \frac{1}{2}\zeta^2\,\overline{z} + \frac{1}{2} z^2\,\overline{\zeta}}{1-\zeta\,\overline{\zeta}}$, and conversely;

(2) When $V_0 \neq 0$, the surface is, up to $z \mapsto -z$, equivalent to:

$$u = z\,\overline{z} + \tfrac{1}{2}\,z^2\,\overline{\zeta} + \tfrac{1}{2}\,\zeta\,\overline{z}^2 + \tfrac{1}{6}\,\tfrac{Q_0}{|V_0|}\,z\,\zeta\,\overline{z}^3 + \tfrac{1}{6}\,\tfrac{Q_0}{|V_0|}\,z^3\,\overline{z}\,\overline{\zeta} + \tfrac{1}{24}\,\zeta\,\overline{z}^4 + \tfrac{1}{24}\,z^4\,\overline{\zeta} + \zeta\,\overline{\zeta}\,(\dots) + \sum_{a+b+c+d\geqslant 6,\,b\,d=0} \tfrac{F_{a,b,c,d}}{a!b!c!d!}\,z^a\,\zeta^b\,\overline{z}^c\,\overline{\zeta}^d,$$

without any harmonic monomial $z^j\,\zeta^{n-j}$, $\forall n \geqslant 0$, $0 \leqslant j \leqslant n$ and any monomial $z^a\,\zeta^b\,\overline{z}^c$, $\forall a+b \geqslant 2$, $c \in \{1,2\}$. Pairs of collection of coefficients:

$$\frac{Q_0}{|V_0|}, \big\{F_{a,b,c,d}\big\}_{a+b+c+d\geqslant 6,\,b\,d=0}, \qquad \frac{Q_0}{|V_0|}, \big\{(-1)^{a+c}\,F_{a,b,c,d}\big\}_{a+b+c+d\geqslant 6,\,b\,d=0}$$

are in one-to-one correspondence with equivalent classes.

Within the branch $I_0 \neq 0$, the surface is, in a unique way, equivalent to:

$$u = z\,\overline{z} + \tfrac{1}{2}\,z^2\,\overline{\zeta} + \tfrac{1}{2}\,\zeta\,\overline{z}^2 + \tfrac{1}{6}\,\tfrac{Q_0}{|I_0|^2}\,z\,\zeta\,\overline{z}^3 + \tfrac{1}{6}\,\tfrac{Q_0}{|I_0|^2}\,z^3\,\overline{z}\,\overline{\zeta} + \tfrac{1}{24}\,\tfrac{V_0}{\overline{I_0}^2}\,\zeta\,\overline{z}^4 + \tfrac{1}{24}\,\tfrac{\overline{V_0}}{I_0^2}\,z^4\,\overline{\zeta} + \tfrac{1}{12}\,\zeta^2\,\overline{z}^3 + \tfrac{1}{12}\,z^3\,\overline{\zeta}^2 + \zeta\,\overline{\zeta}\,(\dots) + \sum_{a+b+c+d\geqslant 6,\,b\,d=0} \tfrac{F_{a,b,c,d}}{a!b!c!d!}\,z^a\,\zeta^b\,\overline{z}^c\,\overline{\zeta}^d,$$

without any harmonic monomial $z^j\,\zeta^{n-j}$, $\forall n \geqslant 0$, $0 \leqslant j \leqslant n$ and any monomial $z^a\,\zeta^b\,\overline{z}^c$, $\forall a+b \geqslant 2$, $c \in \{1,2\}$. Collections of coefficients: $\frac{V_0}{\overline{I_0}^2}$, $\frac{Q_0}{|I_0|^2}$ and $\big\{F_{a,b,c,d}\big\}_{a+b+c+d\geqslant 6,\,b\,d=0}$, are in one-to-one correspondence with equivalent classes.

3 Recurrence Formulas

According to Olver, by taking invariant differentials, lower order invariants generate higher order ones. The explicit equations can be obtained by Olver's recurrence formulas. The goal of the section is to find out a minimal set of generators of all differential invariants.

There are 4 cases, whether I_0, V_0 are identically 0 or nonzero. If $I_0 \equiv 0 \equiv V_0$ then the surface is equivalent to the Gaussier-Merker model hence all differential invariants are constant. The other 3 cases are

- when $I_0 \neq 0, V_0 \neq 0$.
- when $I_0 \neq 0, V_0 \equiv 0$.
- when $I_0 \equiv 0, V_0 \neq 0$.

3.1 Branch $I_0 \neq 0, V_0 \neq 0$

We normalize I_0 to 1 and there are 3 differential invariants of order 5: $V, \overline{V}$ and Q. The surface, after a unique rigid transformation, is equivalent to a graph of the normal form $u = F(z,\zeta,\overline{z},\overline{\zeta}) = \sum_{a,b,c,d} \frac{F_{a,b,c,d}}{a!b!c!d!}\,z^a\,\zeta^b\,\overline{z}^c\,\overline{\zeta}^d$ satisfying

- $F_{a,b,0,0}$=0,

- $F_{a,b,1,0}$=0, except $F_{1,0,1,0} = 1$,
- $F_{a,b,2,0}$=0, except $F_{0,1,2,0} = 1$,
- $F_{3,0,0,1}$=0,
- $F_{3,0,1,1} = F_{1,1,3,0}$.

There are 3 differential invariants of order 5: $V := F_{0,1,4,0}$, $\overline{V} := F_{4,0,0,1}$ and $Q := F_{3,0,1,1}$. Here V is complex valued while Q is real valued.

The Levi rank 1 condition, together with $F_{1,0,1,0} = 1 \neq 0$, implies that all the coefficients $F_{a,b,c,d}$ with $b \geqslant 1$ and $d \geqslant 1$ can be expressed as a rational combination of those $F_{a',b',c',d'}$ with $a' + b' + c' + d' \leqslant a + b + c + d$ and $b'\,d' = 0$, where the denominator is a power of $F_{1,0,1,0}$. So all the non-phantom invariants of order $k \geqslant 6$ can be classified into three types.

- Type 1: $I_{a,0,k-a,0} := F_{a,0,k-a,0}$ for $3 \leqslant a \leqslant k-3$;
- Type 2: $I_{l-b,b,k-l,0} := F_{l-b,b,k-l,0}$ for $1 \leqslant b \leqslant l$, $1 \leqslant l \leqslant k-3$;
- Type 3: $I_{k-l,0,l-d,d} := F_{k-l,0,l-d,d}$ for $1 \leqslant d \leqslant l$, $1 \leqslant l \leqslant k-3$, which are conjugates of Type 2.

3.1.1 Recurrence formulas at order 5

At order 5 we have

$$
\begin{aligned}
D_z V &= I_{1,1,4,0} - 2\, I_{4,0,0,2}\, V \\
D_\zeta V &= I_{0,2,4,0} - 4\, Q - 4\, V \\
D_{\overline{z}} V &= I_{0,1,5,0} - 2\, I_{3,0,1,2}\, V - 4\, V \\
D_{\overline{\zeta}} V &= -2\, I_{3,0,0,3}\, V \\
D_z Q &= \frac{1}{2}\, I_{2,1,3,0} + \frac{1}{2}\, I_{4,0,1,1} - I_{1,2,3,0}\, Q - I_{4,0,0,2}\, Q - 2\, Q\, \overline{V} + \frac{1}{6}\, Q + \frac{1}{6}\, \overline{V} \\
D_\zeta Q &= \frac{1}{2}\, I_{1,2,3,0} - I_{0,3,3,0}\, Q - 2\, Q + \frac{1}{2}\, \overline{V} + \frac{1}{6}
\end{aligned}
$$

together with their conjugates

$$
\begin{aligned}
D_z \overline{V} &= I_{5,0,0,1} - 2\, I_{1,2,3,0}\, \overline{V} - 4\, \overline{V} \\
D_\zeta \overline{V} &= -2\, I_{0,3,3,0}\, \overline{V} \\
D_{\overline{z}} \overline{V} &= I_{4,0,1,1} - 2\, I_{0,2,4,0}\, \overline{V} \\
D_{\overline{\zeta}} \overline{V} &= I_{4,0,0,2} - 4\, Q - 4\, \overline{V} \\
D_{\overline{z}} Q &= \frac{1}{2}\, I_{3,0,2,1} + \frac{1}{2}\, I_{1,1,4,0} - I_{3,0,2,1}\, Q - I_{0,2,4,0}\, Q - 2\, Q\, V + \frac{1}{6}\, Q + \frac{1}{6}\, V \\
D_{\overline{\zeta}} Q &= \frac{1}{2}\, I_{3,0,2,1} - I_{3,0,0,3}\, Q - 2\, Q + \frac{1}{2}\, V + \frac{1}{6}
\end{aligned}
$$

in total 12 equations. There are 13 non-phantom invariants of order 6:

- Type 1: $I_{3,0,3,0}$;

- Type 2: $I_{0,1,5,0}, I_{1,1,4,0}, I_{0,2,4,0}, I_{2,1,3,0}, I_{1,2,3,0}, I_{0,3,3,0}$;
- Type 3: $I_{5,0,0,1}, I_{4,0,1,1}, I_{4,0,0,2}, I_{3,0,2,1}, I_{3,0,1,2}, I_{3,0,0,3}$.

When $V \neq 0$, the 12 invariants of Type 2 and Type 3 can be solved from the 12 equations.

$$
\begin{aligned}
I_{0,1,5,0} &= 2\,V^2 + 8\,Q\,V - 2\,Q\,D_{\overline{\zeta}}V + 4\,V\,D_{\overline{\zeta}}Q - \tfrac{2}{3}\,V + D_{\overline{z}}V,\\
I_{0,2,4,0} &= D_{\zeta}V + 4\,Q + 4\,V,\\
I_{0,3,3,0} &= -\tfrac{D_{\zeta}\overline{V}}{2\,\overline{V}},\\
I_{1,1,4,0} &= 8\,Q\,V + 8\,V\,\overline{V} + 2\,V\,D_{\overline{\zeta}}\overline{V} + D_{z}V,\\
I_{1,2,3,0} &= \tfrac{1}{3\,\overline{V}}\,(12\,Q\,\overline{V} - 3\,Q\,D_{\zeta}\overline{V} - 3\,\overline{V}^2 + 6\,\overline{V}\,D_{\zeta}Q - \overline{V}),\\
I_{2,1,3,0} &= \tfrac{1}{3\,\overline{V}}\,(48\,Q^2\,\overline{V} - 6\,Q^2\,D_{\zeta}\overline{V} + 6\,Q\,\overline{V}^2 + 12\,Q\,\overline{V}\,D_{\zeta}Q + 6\,Q\,\overline{V}\,D_{\overline{\zeta}}\overline{V}\\
&\qquad - 24\,V\,\overline{V}^2 - 6\,\overline{V}^2\,D_{\zeta}V - 3\,Q\,\overline{V} - \overline{V}^2 + 6\,\overline{V}\,D_{z}Q - 3\,\overline{V}\,D_{\overline{z}}\overline{V}),\\
I_{3,0,0,3} &= -\tfrac{D_{\overline{\zeta}}V}{2\,V},\\
I_{3,0,1,2} &= \tfrac{1}{3\,V}\,(12\,Q\,V - 3\,Q\,D_{\overline{\zeta}}V - 3\,V^2 + 6\,V\,D_{\overline{\zeta}}Q - V),\\
I_{3,0,2,1} &= \tfrac{1}{3\,V}\,(48\,Q^2\,V - 6\,Q^2\,D_{\overline{\zeta}}V + 6\,Q\,V^2 + 6\,Q\,V\,D_{\zeta}V + 12\,Q\,V\,D_{\overline{\zeta}}Q\\
&\qquad - 24\,V^2\,\overline{V} - 6\,V^2\,D_{\overline{\zeta}}\overline{V} - 3\,Q\,V - V^2 - 3\,V\,D_{z}V + 6\,V\,D_{\overline{z}}Q),\\
I_{4,0,0,2} &= D_{\overline{\zeta}}\overline{V} + 4\,Q + 4\,\overline{V},\\
I_{4,0,1,1} &= 8\,Q\,\overline{V} + 8\,V\,\overline{V} + 2\,\overline{V}\,D_{\zeta}V + D_{\overline{z}}\overline{V},\\
I_{5,0,0,1} &= 2\,\overline{V}^2 + 8\,Q\,\overline{V} - 2\,Q\,D_{\zeta}\overline{V} + 4\,\overline{V}\,D_{\zeta}Q - \tfrac{2}{3}\,\overline{V} + D_{z}\overline{V}
\end{aligned}
$$

To solve $I_{3,0,3,0}$ one has to investigate recurrence formulas at order 6.

3.1.2 Recurrence formulas at order 6

There are in total 52 equations.

$D_z I_{0,1,5,0} = -5\,QV - 3\,I_{4,0,0,2} I_{0,1,5,0} + I_{1,1,5,0}$

$D_\zeta I_{0,1,5,0} = I_{0,2,5,0} - 6\,I_{0,1,5,0} - 5\,V - 5\,I_{1,1,4,0}$

$D_{\overline{z}} I_{0,1,5,0} = -5\,V^2 - 6\,V I_{0,1,5,0} - 3\,I_{3,0,1,2} I_{0,1,5,0} + I_{0,1,6,0}$

$D_{\overline{\zeta}} I_{0,1,5,0} = -3\,I_{3,0,0,3} I_{0,1,5,0}$

$D_z I_{1,1,4,0} = I_{2,1,4,0} - 14/3\,Q^2 + 1/3\,Q I_{0,2,4,0} - 2\,\overline{V}\,I_{1,1,4,0} + 2/3\,\overline{V}\,V - 4\,I_{3,0,3,0} - I_{1,2,3,0} I_{1,1,4,0} - 2\,I_{4,0,0,2} I_{1,1,4,0}$

$D_\zeta I_{1,1,4,0} = I_{1,2,4,0} - 4\,I_{1,1,4,0} - 14/3\,Q + 1/3\,I_{0,2,4,0} - 4\,I_{2,1,3,0} - I_{0,3,3,0} I_{1,1,4,0}$

$D_{\overline{z}} I_{1,1,4,0} = I_{1,1,5,0} - 4\,V I_{1,1,4,0} - 5\,QV + 1/3\,V I_{0,2,4,0} - I_{0,2,4,0} I_{1,1,4,0} - 2\,I_{3,0,1,2} I_{1,1,4,0}$

$D_{\overline{\zeta}} I_{1,1,4,0} = I_{0,1,5,0} - 2\,I_{1,1,4,0} + 2/3\,V - 2\,I_{3,0,0,3} I_{1,1,4,0}$

$D_z I_{0,2,4,0} = I_{1,2,4,0} - 8/3\,Q - 2\,I_{0,2,4,0}\overline{V} + 2/3\,\overline{V} - 2\,I_{2,1,3,0} + 2\,I_{4,0,1,1} - I_{1,2,3,0} I_{0,2,4,0} - I_{4,0,0,2} I_{0,2,4,0}$

$D_\zeta I_{0,2,4,0} = I_{0,3,4,0} - 8/3 - 2\,I_{0,2,4,0} - 2\,I_{1,2,3,0} + 2\,\overline{V} - I_{0,3,3,0} I_{0,2,4,0}$

$D_{\overline{z}} I_{0,2,4,0} = I_{0,2,5,0} - 4\,I_{1,1,4,0} - 2\,V I_{0,2,4,0} - 11/3\,V + 2/3\,Q + 2\,I_{3,0,2,1} - I_{0,2,4,0}{}^2 - I_{3,0,1,2} I_{0,2,4,0}$

$D_{\overline{\zeta}} I_{0,2,4,0} = 4\,V + 2/3 - 2\,I_{0,2,4,0} + 2\,I_{3,0,1,2} - I_{0,2,4,0} I_{3,0,0,3}$

$$D_z I_{3,0,3,0} = Q I_{2,1,3,0} + \overline{V}\, I_{3,0,2,1} - 4\,\overline{V}\, I_{3,0,3,0} - 2\, I_{1,2,3,0} I_{3,0,3,0} - 2\, I_{4,0,0,2} I_{3,0,3,0} + I_{4,0,3,0}$$
$$D_\zeta I_{3,0,3,0} = -2\, I_{0,3,3,0} I_{3,0,3,0} + I_{2,1,3,0} - 4\, I_{3,0,3,0} + I_{3,1,3,0}$$
$$D_{\overline{z}} I_{3,0,3,0} = Q I_{3,0,2,1} + V I_{2,1,3,0} - 4\, V I_{3,0,3,0} - 2\, I_{0,2,4,0} I_{3,0,3,0} - 2\, I_{3,0,1,2} I_{3,0,3,0} + I_{3,0,4,0}$$
$$D_{\overline{\zeta}} I_{3,0,3,0} = -2\, I_{3,0,0,3} I_{3,0,3,0} + I_{3,0,2,1} - 4\, I_{3,0,3,0} + I_{3,0,3,1}$$
$$D_z I_{2,1,3,0} = I_{3,1,3,0} + Q/18 + 2/3\, I_{1,2,3,0} Q - 4\,\overline{V}\, I_{2,1,3,0} - \overline{V}/18 + 7/3\, Q\overline{V} + 1/6\, I_{2,1,3,0} - 1/6\, I_{4,0,1,1} - 2\, I_{1,2,3,0} I_{2,1,3,0} - I_{4,0,0,2} I_{2,1,3,0}$$
$$D_\zeta I_{2,1,3,0} = I_{2,2,3,0} + 1/18 - 2\, I_{2,1,3,0} + 5/6\, I_{1,2,3,0} - \overline{V}/6 - 2\, I_{0,3,3,0} I_{2,1,3,0}$$
$$D_{\overline{z}} I_{2,1,3,0} = I_{2,1,4,0} - I_{3,0,3,0} - 2\, V I_{2,1,3,0} + V/18 + 2/3\, V I_{1,2,3,0} + 1/3\, Q^2 - Q/18 + 1/6\, I_{1,1,4,0} - 1/6\, I_{3,0,2,1} - 2\, I_{0,2,4,0} I_{2,1,3,0} - I_{3,0,1,2} I_{2,1,3,0}$$
$$D_{\overline{\zeta}} I_{2,1,3,0} = 2\, I_{1,1,4,0} + 3\, I_{3,0,2,1} - 1/18 - 4\, I_{2,1,3,0} + 7/3\, Q + V/6 - 1/6\, I_{3,0,1,2} - I_{3,0,0,3} I_{2,1,3,0}$$
$$D_z I_{1,2,3,0} = I_{2,2,3,0} + 1/3\, Q I_{0,3,3,0} - 4\, I_{1,2,3,0} \overline{V} - \overline{V}/3 - 2\, {I_{1,2,3,0}}^2$$
$$D_\zeta I_{1,2,3,0} = I_{1,3,3,0} + 1/3\, I_{0,3,3,0} - 2\, I_{1,2,3,0} I_{0,3,3,0}$$
$$D_{\overline{z}} I_{1,2,3,0} = I_{1,2,4,0} - 2\, I_{2,1,3,0} + 1/3\, V I_{0,3,3,0} - 10/3\, Q - 2\, I_{1,2,3,0} I_{0,2,4,0}$$
$$D_{\overline{\zeta}} I_{1,2,3,0} = I_{0,2,4,0} + 8\, Q - 1/3 - 4\, I_{1,2,3,0}$$
$$D_z I_{0,3,3,0} = -4\,\overline{V}\, I_{0,3,3,0} - 2\, I_{1,2,3,0} I_{0,3,3,0} + I_{4,0,0,2} I_{0,3,3,0} + I_{1,3,3,0}$$
$$D_\zeta I_{0,3,3,0} = -2\, {I_{0,3,3,0}}^2 + 2\, I_{0,3,3,0} + I_{0,4,3,0}$$
$$D_{\overline{z}} I_{0,3,3,0} = 2\, V I_{0,3,3,0} - 2\, I_{0,3,3,0} I_{0,2,4,0} + I_{0,3,3,0} I_{3,0,1,2} + I_{0,3,4,0} - 3\, I_{1,2,3,0} - 3$$
$$D_{\overline{\zeta}} I_{0,3,3,0} = I_{0,3,3,0} I_{3,0,0,3} - 4\, I_{0,3,3,0} + 3$$
$$D_z I_{3,0,2,1} = I_{4,0,2,1} - I_{3,0,3,0} - Q/18 + 1/3\, Q^2 - 2\,\overline{V}\, I_{3,0,2,1} + \overline{V}/18 + 2/3\,\overline{V}\, I_{3,0,1,2} - 1/6\, I_{2,1,3,0} + 1/6\, I_{4,0,1,1} - I_{1,2,3,0} I_{3,0,2,1} - 2\, I_{4,0,0,2} I_{3,0,2,1}$$
$$D_\zeta I_{3,0,2,1} = 3\, I_{2,1,3,0} + 2\, I_{4,0,1,1} - 1/18 - 4\, I_{3,0,2,1} + 7/3\, Q - 1/6\, I_{1,2,3,0} + \overline{V}/6 - I_{0,3,3,0} I_{3,0,2,1}$$
$$D_{\overline{z}} I_{3,0,2,1} = I_{3,0,3,1} - 4\, V I_{3,0,2,1} - V/18 + 7/3\, QV + Q/18 + 2/3\, Q I_{3,0,1,2} - 1/6\, I_{1,1,4,0} + 1/6\, I_{3,0,2,1} - I_{0,2,4,0} I_{3,0,2,1} - 2\, I_{3,0,1,2} I_{3,0,2,1}$$
$$D_{\overline{\zeta}} I_{3,0,2,1} = I_{3,0,2,2} + 1/18 - 2\, I_{3,0,2,1} + 5/6\, I_{3,0,1,2} - V/6 - 2\, I_{3,0,0,3} I_{3,0,2,1}$$
$$D_z I_{3,0,1,2} = I_{4,0,1,2} - 2\, I_{3,0,2,1} - 10/3\, Q + 1/3\,\overline{V}\, I_{3,0,0,3} - 2\, I_{4,0,0,2} I_{3,0,1,2}$$
$$D_\zeta I_{3,0,1,2} = 8\, Q + I_{4,0,0,2} - 1/3 - 4\, I_{3,0,1,2}$$
$$D_{\overline{z}} I_{3,0,1,2} = I_{3,0,2,2} - 4\, I_{3,0,1,2} V - V/3 + 1/3\, Q I_{3,0,0,3} - 2\, {I_{3,0,1,2}}^2$$
$$D_{\overline{\zeta}} I_{3,0,1,2} = I_{3,0,1,3} + 1/3\, I_{3,0,0,3} - 2\, I_{3,0,1,2} I_{3,0,0,3}$$
$$D_z I_{3,0,0,3} = 2\,\overline{V}\, I_{3,0,0,3} + I_{1,2,3,0} I_{3,0,0,3} - 2\, I_{4,0,0,2} I_{3,0,0,3} - 3\, I_{3,0,1,2} + I_{4,0,0,3} - 3$$
$$D_\zeta I_{3,0,0,3} = I_{0,3,3,0} I_{3,0,0,3} - 4\, I_{3,0,0,3} + 3$$
$$D_{\overline{z}} I_{3,0,0,3} = -4\, V I_{3,0,0,3} + I_{0,2,4,0} I_{3,0,0,3} - 2\, I_{3,0,1,2} I_{3,0,0,3} + I_{3,0,1,3}$$
$$D_{\overline{\zeta}} I_{3,0,0,3} = -2\, {I_{3,0,0,3}}^2 + 2\, I_{3,0,0,3} + I_{3,0,0,4}$$
$$D_z I_{4,0,1,1} = I_{5,0,1,1} - 5\, Q\overline{V} - 4\,\overline{V}\, I_{4,0,1,1} + 1/3\,\overline{V}\, I_{4,0,0,2} - 2\, I_{1,2,3,0} I_{4,0,1,1} - I_{4,0,0,2} I_{4,0,1,1}$$
$$D_\zeta I_{4,0,1,1} = I_{5,0,0,1} - 2\, I_{4,0,1,1} + 2/3\,\overline{V} - 2\, I_{0,3,3,0} I_{4,0,1,1}$$
$$D_{\overline{z}} I_{4,0,1,1} = I_{4,0,2,1} - 2\, V I_{4,0,1,1} + 2/3\,\overline{V}\, V - 14/3\, Q^2 + 1/3\, Q I_{4,0,0,2} - 2\, I_{0,2,4,0} I_{4,0,1,1} - 4\, I_{3,0,3,0} - I_{3,0,1,2} I_{4,0,1,1}$$
$$D_{\overline{\zeta}} I_{4,0,1,1} = I_{4,0,1,2} - 4\, I_{4,0,1,1} - 14/3\, Q + 1/3\, I_{4,0,0,2} - 4\, I_{3,0,2,1} - I_{3,0,0,3} I_{4,0,1,1}$$

$$
\begin{aligned}
D_z I_{4,0,0,2} &= I_{5,0,0,2} - 4\, I_{4,0,1,1} + 2/3\, Q - 11/3\, \overline{V} - 2\, \overline{V}\, I_{4,0,0,2} + 2\, I_{2,1,3,0} - I_{1,2,3,0} I_{4,0,0,2} - {I_{4,0,0,2}}^2 \\
D_\zeta I_{4,0,0,2} &= 4\, \overline{V} + 2/3 - 2\, I_{4,0,0,2} + 2\, I_{1,2,3,0} - I_{4,0,0,2} I_{0,3,3,0} \\
D_{\bar z} I_{4,0,0,2} &= I_{4,0,1,2} - 2\, I_{4,0,0,2} V + 2/3\, V - 8/3\, Q + 2\, I_{1,1,4,0} - 2\, I_{3,0,2,1} - I_{4,0,0,2} I_{0,2,4,0} - I_{4,0,0,2} I_{3,0,1,2} \\
D_{\bar\zeta} I_{4,0,0,2} &= I_{4,0,0,3} - 8/3 - 2\, I_{4,0,0,2} + 2\, V - 2\, I_{3,0,1,2} - I_{4,0,0,2} I_{3,0,0,3} \\
D_z I_{5,0,0,1} &= -5\, \overline{V}^2 - 6\, \overline{V}\, I_{5,0,0,1} - 3\, I_{1,2,3,0} I_{5,0,0,1} + I_{6,0,0,1} \\
D_\zeta I_{5,0,0,1} &= -3\, I_{0,3,3,0} I_{5,0,0,1} \\
D_{\bar z} I_{5,0,0,1} &= -5\, Q \overline{V} - 3\, I_{0,2,4,0} I_{5,0,0,1} + I_{5,0,1,1} \\
D_{\bar\zeta} I_{5,0,0,1} &= I_{5,0,0,2} - 6\, I_{5,0,0,1} - 5\, \overline{V} - 5\, I_{4,0,1,1}
\end{aligned}
$$

From 2 of them,

$$
\begin{aligned}
D_z I_{3,0,2,1} &= I_{4,0,2,1} - I_{3,0,3,0} - Q/18 + 1/3\, Q^2 - 2\, \overline{V}\, I_{3,0,2,1} + \overline{V}/18 + 2/3\, \overline{V}\, I_{3,0,1,2} - 1/6\, I_{2,1,3,0} \\
D_{\bar z} I_{4,0,1,1} &= I_{4,0,2,1} - 4\, I_{3,0,3,0} - 2\, V I_{4,0,1,1} + 2/3\, \overline{V}\, V - 14/3\, Q^2 + 1/3\, Q I_{4,0,0,2} - 2\, I_{0,2,4,0} I_{4,0,1,1} - I_{3,0,1,2} I_{4,0,1,1}
\end{aligned}
$$

one can solve $I_{3,0,3,0}$ and $I_{4,0,2,1}$, since all the other terms are solved before. In particular

$$
\begin{aligned}
I_{3,0,3,0} = {} & -\tfrac{5}{3}\, Q^2 + \tfrac{1}{9}\, Q\, I_{4,0,0,2} + \tfrac{2}{9}\, \overline{V}\, V - \tfrac{2}{3}\, V\, I_{4,0,1,1} - \tfrac{2}{9}\, \overline{V}\, I_{3,0,1,2} + \tfrac{2}{3}\, \overline{V}\, I_{3,0,2,1} - \tfrac{2}{3}\, I_{0,2,4,0}\, I_{4,0,1,1} \\
& + \tfrac{1}{3}\, I_{1,2,3,0}\, I_{3,0,2,1} - \tfrac{1}{3}\, I_{3,0,1,2}\, I_{4,0,1,1} + \tfrac{2}{3}\, I_{4,0,0,2}\, I_{3,0,2,1} + \tfrac{1}{54}\, Q - \tfrac{1}{54}\, \overline{V} \\
& + \tfrac{1}{3}\, D_z I_{3,0,2,1} - \tfrac{1}{3}\, D_{\bar z} I_{4,0,1,1} + \tfrac{1}{18}\, I_{2,1,3,0} - \tfrac{1}{18}\, I_{4,0,1,1}.
\end{aligned}
$$

Recall that all the invariants of order 6 on the right hand side can be generated by $V, \overline{V}, Q$ and their invariant derivatives, so is $I_{3,0,3,0}$. The full expression is a bit large.

$$
\begin{aligned}
I_{3,0,3,0} = {} & -\frac{2\, V\, D_\zeta \overline{V}\, Q^2}{3\, \overline{V}} + \tfrac{1}{27}\, V + \tfrac{10}{3}\, D_{\bar z} Q\, \overline{V} - \tfrac{4}{9}\, D_{\bar\zeta} \overline{V}\, Q - \tfrac{4}{3}\, V\, D_\zeta Q\, D_{\bar\zeta} \overline{V} - \tfrac{16}{3}\, \overline{V}\, V\, D_\zeta Q \\
& + \tfrac{8}{3}\, V\, D_\zeta \overline{V}\, Q - \frac{2\, Q^2\, D_z D_{\bar\zeta} V}{3\, V} - 24\, \overline{V}^2\, V - \frac{4\, D_\zeta Q\, D_{\bar\zeta} V\, Q^2}{3\, V} - \frac{2\, D_\zeta V\, D_\zeta \overline{V}\, Q^2}{3\, \overline{V}} \\
& + \frac{D_{\bar\zeta} V\, D_{\bar z} \overline{V}\, Q}{3\, V} + \tfrac{4}{3}\, D_z Q\, D_{\bar\zeta} Q - \frac{4\, D_\zeta \overline{V}\, D_{\bar\zeta} Q\, Q^2}{3\, \overline{V}} + \frac{2\, D_\zeta \overline{V}\, Q^2}{9\, \overline{V}} + \tfrac{1}{27}\, \overline{V} \\
& + \frac{V\, D_\zeta \overline{V}\, Q}{9\, \overline{V}} + \tfrac{8}{3}\, D_{\bar\zeta} V\, Q\, \overline{V} + \tfrac{4}{3}\, D_\zeta Q\, D_\zeta V\, Q - \tfrac{1}{3}\, D_{\bar z} D_{\bar z} \overline{V} + \tfrac{32}{3}\, D_\zeta Q\, Q^2 \\
& - \tfrac{2}{3}\, D_{\bar\zeta} Q\, D_{\bar z} \overline{V} + \tfrac{4}{3}\, Q\, D_z D_{\bar\zeta} Q + \tfrac{32}{3}\, D_{\bar\zeta} \overline{V}\, Q^2 + \tfrac{2}{3}\, D_z D_{\bar z} Q + 8\, Q^2\, V - 24\, V^2\, \overline{V} \\
& + \tfrac{4}{3}\, V\, D_\zeta Q\, Q + \frac{2\, D_{\bar\zeta} V\, Q^2}{9\, V} - \tfrac{20}{3}\, D_{\bar z} \overline{V}\, Q - \tfrac{14}{3}\, Q\, V\, (1/9) + 2\, D_z Q\, D_\zeta V \\
& + \tfrac{2}{3}\, Q\, D_z D_\zeta V - \tfrac{4}{3}\, D_z V\, D_{\bar\zeta} \overline{V} - \tfrac{4}{9}\, D_{\bar\zeta} Q\, \overline{V} - \tfrac{2}{3}\, V\, D_z D_{\bar\zeta} \overline{V} - \tfrac{8}{3}\, \overline{V}\, D_{\bar z} V \\
& - \tfrac{1}{3}\, D_z D_z V + \tfrac{2}{3}\, D_z Q\, V - \tfrac{2}{9}\, D_{\bar z} Q - \frac{16\, D_\zeta \overline{V}\, Q^3}{3\, \overline{V}} + \frac{2\, V\, D_\zeta \overline{V}\, D_{\bar\zeta} \overline{V}\, Q}{3\, \overline{V}} \\
& + \frac{2\, D_\zeta V\, D_{\bar\zeta} V\, Q\, \overline{V}}{3\, V} + 8\, D_\zeta V\, Q^2 + 16\, D_{\bar\zeta} Q\, Q^2 - \frac{8\, D_{\bar\zeta} V\, Q^3}{V} - \tfrac{2}{3}\, D_\zeta Q\, D_z V \\
& - \tfrac{2}{9}\, D_z Q + 8\, D_{\bar z} Q\, Q + 16\, \overline{V}\, Q^2 - 82\, \overline{V}\, V\, Q + 64\, Q^3 - \tfrac{4}{3}\, D_\zeta V\, D_{\bar\zeta} Q\, \overline{V} + \tfrac{8}{3}\, D_\zeta Q\, D_{\bar\zeta} Q\, Q \\
& - \tfrac{34}{3}\, \overline{V}\, V\, D_{\bar\zeta} \overline{V} - \tfrac{20}{3}\, V\, D_{\bar\zeta} \overline{V}\, Q + \tfrac{1}{3}\, \overline{V}\, V - \tfrac{10}{3}\, D_z V\, Q + \tfrac{32}{3}\, D_z Q\, Q - \tfrac{4}{9}\, D_{\bar\zeta} Q\, Q \\
& - \tfrac{2}{9}\, D_\zeta Q\, V - \tfrac{4}{3}\, D_{\bar z} \overline{V}\, D_\zeta V - \tfrac{2}{3}\, \overline{V}\, D_{\bar z} D_\zeta V - \tfrac{8}{3}\, D_z \overline{V}\, V - \tfrac{17}{3}\, \overline{V}\, D_z V \\
& - \tfrac{4}{3}\, D_\zeta V^2\, \overline{V} - \tfrac{4}{3}\, V\, D_{\bar\zeta} \overline{V}^2 + \tfrac{2}{27}\, Q - \tfrac{17}{3}\, D_{\bar z} \overline{V}\, V - \tfrac{55}{9}\, Q^2 \\
& - \tfrac{34}{3}\, D_\zeta V\, V\, \overline{V} - \tfrac{16}{3}\, V\, D_{\bar\zeta} Q\, \overline{V} + \tfrac{4}{3}\, D_\zeta V\, D_{\bar\zeta} \overline{V}\, Q + \tfrac{8}{3}\, D_{\bar\zeta} Q\, D_{\bar\zeta} \overline{V}\, Q \\
& - \frac{4\, D_{\bar\zeta} V\, D_{\bar\zeta} \overline{V}\, Q^2}{3\, V} - \frac{4\, D_{\bar\zeta} V\, D_z Q\, Q}{3\, V} + \frac{2\, D_{\bar\zeta} V\, D_z V\, Q^2}{3\, V^2} \\
& + \frac{2\, \overline{V}\, D_{\bar\zeta} V\, Q}{9\, V} - \frac{10\, \overline{V}\, D_{\bar\zeta} V\, Q^2}{3\, V} + \frac{2\, D_\zeta \overline{V}\, D_{\bar\zeta} V\, Q^3}{3\, \overline{V}\, V} + \frac{D_\zeta \overline{V}\, D_z V\, Q}{3\, \overline{V}} \\
& - \frac{2\, D_\zeta \overline{V}\, D_{\bar z} Q\, Q}{3\, \overline{V}} + \tfrac{4}{3}\, D_{\bar z} Q\, D_{\bar\zeta} \overline{V} + \tfrac{4}{3}\, D_{\bar z} Q\, D_\zeta Q - \tfrac{4}{9}\, D_\zeta Q\, Q \\
& - \tfrac{2}{9}\, D_\zeta V\, Q - \tfrac{26}{9}\, Q\, \overline{V} - \tfrac{22}{3}\, \overline{V}\, D_\zeta V\, Q + \tfrac{20}{3}\, \overline{V}\, D_{\bar\zeta} Q\, Q
\end{aligned}
$$

Theorem 3.1. *When $I_0 \neq 0$ and $V_0 \neq 0$, all the non-phantom invariants are generated by V, $\overline{V}$, Q and their invariant derivatives.*

Proof. The proof is by an induction on the order. All non-phantom invariants of order $k \leqslant 6$ are generated by V, $\overline{V}$, Q and their invariant derivatives. Suppose all non-phantom invariants of order $k = k_0 \geqslant 6$ are generated by V, $\overline{V}$, Q and their invariant derivatives, then for those of order $k = k_0+1$, recall the division of three types:

- Type 1: $I_{a,0,k_0+1-a,0}$ for $3 \leqslant a \leqslant k_0$;
- Type 2: $I_{l-b,b,k_0+1-l,0}$ for $1 \leqslant b \leqslant l$, $1 \leqslant l \leqslant k_0 - 3$;
- Type 3: $I_{k_0+1-l,0,l-d,d}$ for $1 \leqslant d \leqslant l$, $1 \leqslant l \leqslant k_0 - 3$.

They can be generated from these recurrence relations of order k_0:

$$\begin{aligned} D_{\overline{z}} I_{a,0,k_0-a} &= I_{a,0,k_0+1-a,0} + \{\text{correction terms}\} \\ D_{\overline{z}} I_{l-b,b,k_0-l,0} &= I_{l-b,b,k_0+1-l,0} + \{\text{correction terms}\} \\ D_z I_{k_0-l,0,l-d,d} &= I_{k_0+1-l,0,l-d,d} + \{\text{correction terms}\} \end{aligned}$$

where the correction terms are of order at most k_0 hence generatable, so do the three types of terms of ordre $k_0 + 1$. □

3.1.3 Commutators

Four equations

$$\begin{aligned} [D_z, D_{\overline{z}}] &= I_{0,2,4,0}\, D_z - \tfrac{1}{3}\, V\, D_\zeta - I_{4,0,0,2}\, D_{\overline{z}} + \tfrac{1}{3}\, D_{\overline{\zeta}} \\ \left[D_\zeta, D_{\overline{\zeta}}\right] &= (-I_{3,0,0,3} + 2)\, D_\zeta + (I_{0,3,3,0} - 2)\, D_{\overline{\zeta}} \\ [D_z, D_\zeta] &= I_{0,3,3,0}\, D_z + (-I_{1,2,3,0} + I_{4,0,0,2} - 2\,\overline{V} - \tfrac{1}{3})\, D_\zeta \\ \left[D_z, D_{\overline{\zeta}}\right] &= 2\, D_z - D_{\overline{z}} + (I_{1,2,3,0} - I_{4,0,0,2} + 2\,\overline{V})\, D_{\overline{\zeta}} \end{aligned}$$

plus two conjugates

$$\begin{aligned} \left[D_{\overline{z}}, D_{\overline{\zeta}}\right] &= I_{3,0,3,0}\, D_{\overline{z}} + (I_{0,2,4,0} - I_{3,0,1,2} - 2\,V - \tfrac{1}{3})\, D_{\overline{\zeta}} \\ [D_{\overline{z}}, D_\zeta] &= -D_z + 2\, D_{\overline{z}} + (I_{3,0,1,2} - I_{0,2,4,0} + 2\,V)\, D_\zeta \end{aligned}$$

Not sure if they are useful yet.

3.2 Branch $I_0 \neq 0, V_0 \equiv 0$

We normalize I_0 to 1. There is only one differential invariant of order 5: $Q := F_{1,1,3,0} = F_{3,0,1,1}$.

The Levi rank 1 condition implies that all $F_{a,b+1,c,d+1}$ can be solved in terms of $F_{a',b',c',d'}$ with $b'\,d' = 0$. The vanishing of V_0 and of $\overline{V_0}$ implies that $F_{a+4,b,c,d+1}$ and $F_{a,b+1,c+4,d}$ can be solved in terms of $F_{a',b',c',d'}$ with ($a' < 4$ or $d' < 1$) and ($b' < 1$ or $c' < 4$). So all the non-phantom independent invariants of order $k \geqslant 6$ can be classified into three types

- Type 1: $I_{a,0,k-a,0} := F_{a,0,k-a,0}$ for $3 \leqslant a \leqslant k - 3$;
- Type 2: $I_{k-3-b,b,3,0} := F_{k-3-b,b,3,0}$ for $1 \leqslant b \leqslant k - 3$;
- Type 3: $I_{3,0,k-3-d,d} := F_{3,0,k-3-d,d}$ for $1 \leqslant d \leqslant k - 3$, which are conjugates of Type 2.

3.2.1 Recurrence formulas at order 5

At order 5 we have

$$\begin{aligned}
D_z Q &= -4\,Q^2 + \tfrac{1}{6}\,Q + \tfrac{1}{2}\,I_{2,1,3,0} - I_{1,2,3,0}\,Q \\
D_\zeta Q &= \tfrac{1}{6} - 2\,Q + \tfrac{1}{2}\,I_{1,2,3,0} - I_{0,3,3,0}\,Q \\
D_{\overline{z}} Q &= \tfrac{1}{2}\,I_{3,0,2,1} - 4\,Q^2 + \tfrac{1}{6}\,Q - I_{3,0,1,2}\,Q \\
D_{\overline{\zeta}} Q &= \tfrac{1}{2}\,I_{3,0,1,2} + \tfrac{1}{6} - 2\,Q - I_{3,0,0,3}\,Q
\end{aligned}$$

in total 4 equations. There are 7 non-phantom independent invariants of order 6:

- Type 1: $I_{3,0,3,0}$;
- Type 2: $I_{2,1,3,0}, I_{1,2,3,0}, I_{0,3,3,0}$;
- Type 3: $I_{3,0,2,1}, I_{3,0,1,2}, I_{3,0,0,3}$.

One can solve $I_{2,1,3,0}, I_{1,2,3,0}, I_{3,0,2,1}$ and $I_{3,0,1,2}$ in terms of $I_{0,3,3,0}$ and $I_{3,0,0,3}$.

To solve $I_{3,0,3,0}$ one should use higher recurrence formulas.

3.2.2 Recurrence formulas at order 6

At order 6 we have a huge list

$$\begin{aligned}
D_z I_{3,0,3,0} &= Q\,I_{2,1,3,0} - 8\,Q\,I_{3,0,3,0} - 2\,I_{1,2,3,0} I_{3,0,3,0} + I_{4,0,3,0} \\
D_\zeta I_{3,0,3,0} &= -2\,I_{0,3,3,0} I_{3,0,3,0} + I_{2,1,3,0} - 4\,u_{3,0,3,0} + I_{3,1,3,0} \\
D_{\overline{z}} I_{3,0,3,0} &= Q\,I_{3,0,2,1} - 8\,Q\,I_{3,0,3,0} - 2\,I_{3,0,1,2} I_{3,0,3,0} + I_{3,0,4,0} \\
D_{\overline{\zeta}} I_{3,0,3,0} &= -2\,I_{3,0,0,3} I_{3,0,3,0} + I_{3,0,2,1} - 4\,I_{3,0,3,0} + I_{3,0,3,1} \\
D_z I_{2,1,3,0} &= I_{3,1,3,0} - 4\,Q\,I_{2,1,3,0} + Q/18 + 2/3\,I_{1,2,3,0} Q + 1/6\,I_{2,1,3,0} - 2\,I_{1,2,3,0} I_{2,1,3,0} \\
D_\zeta I_{2,1,3,0} &= I_{2,2,3,0} + 1/18 - 2\,I_{2,1,3,0} + 5/6\,I_{1,2,3,0} - 2\,I_{0,3,3,0} I_{2,1,3,0} \\
D_{\overline{z}} I_{2,1,3,0} &= 11/3\,Q^2 + 3\,I_{3,0,3,0} - 8\,Q\,I_{2,1,3,0} - Q/18 - 1/6\,I_{3,0,2,1} - I_{3,0,1,2} I_{2,1,3,0} \\
D_{\overline{\zeta}} I_{2,1,3,0} &= 3\,I_{3,0,2,1} - 1/18 - 4\,I_{2,1,3,0} + 7/3\,Q - 1/6\,I_{3,0,1,2} - I_{3,0,0,3} I_{2,1,3,0} \\
D_z I_{1,2,3,0} &= I_{2,2,3,0} + 1/3\,Q\,I_{0,3,3,0} - 2\,{I_{1,2,3,0}}^2 \\
D_\zeta I_{1,2,3,0} &= I_{1,3,3,0} + 1/3\,I_{0,3,3,0} - 2\,I_{1,2,3,0} I_{0,3,3,0} \\
D_{\overline{z}} I_{1,2,3,0} &= -8\,I_{1,2,3,0} Q + 2\,I_{2,1,3,0} \\
D_{\overline{\zeta}} I_{1,2,3,0} &= 12\,Q - 1/3 - 4\,I_{1,2,3,0} \\
D_z I_{0,3,3,0} &= 4\,Q\,I_{0,3,3,0} - 2\,I_{1,2,3,0} I_{0,3,3,0} + I_{1,3,3,0} \\
D_\zeta I_{0,3,3,0} &= -2\,{I_{0,3,3,0}}^2 + 2\,I_{0,3,3,0} + I_{0,4,3,0} \\
D_{\overline{z}} I_{0,3,3,0} &= -8\,Q\,I_{0,3,3,0} + I_{0,3,3,0} I_{3,0,1,2} + 1/3 + I_{1,2,3,0} \\
D_{\overline{\zeta}} I_{0,3,3,0} &= I_{0,3,3,0} I_{3,0,0,3} - 4\,I_{0,3,3,0} + 3
\end{aligned}$$

$$
\begin{aligned}
D_z I_{3,0,2,1} &= 11/3\, Q^2 + 3\, I_{3,0,3,0} - 8\, Q\, I_{3,0,2,1} - Q/18 - 1/6\, I_{2,1,3,0} - I_{1,2,3,0} I_{3,0,2,1} \\
D_\zeta I_{3,0,2,1} &= 3\, I_{2,1,3,0} - 1/18 - 4\, I_{3,0,2,1} + 7/3\, Q - 1/6\, I_{1,2,3,0} - I_{0,3,3,0} I_{3,0,2,1} \\
D_{\bar z} I_{3,0,2,1} &= I_{3,0,3,1} - 4\, Q\, I_{3,0,2,1} + Q/18 + 2/3\, Q\, u_{3,0,1,2} + 1/6\, I_{3,0,2,1} - 2\, I_{3,0,1,2} I_{3,0,2,1} \\
D_{\bar\zeta} I_{3,0,2,1} &= I_{3,0,2,2} + 1/18 - 2\, I_{3,0,2,1} + 5/6\, I_{3,0,1,2} - 2\, I_{3,0,0,3} I_{3,0,2,1} \\
D_z I_{3,0,1,2} &= -8\, Q\, I_{3,0,1,2} + 2\, I_{3,0,2,1} \\
D_\zeta I_{3,0,1,2} &= 12\, Q - 1/3 - 4\, I_{3,0,1,2} \\
D_{\bar z} I_{3,0,1,2} &= u_{3,0,2,2} + 1/3\, Q\, I_{3,0,0,3} - 2\, {I_{3,0,1,2}}^2 \\
D_{\bar\zeta} I_{3,0,1,2} &= I_{3,0,1,3} + 1/3\, I_{3,0,0,3} - 2\, I_{3,0,1,2} I_{3,0,0,3} \\
D_z I_{3,0,0,3} &= -8\, Q\, I_{3,0,0,3} + I_{1,2,3,0} I_{3,0,0,3} + I_{3,0,1,2} + 1/3 \\
D_\zeta I_{3,0,0,3} &= I_{0,3,3,0} I_{3,0,0,3} - 4\, I_{3,0,0,3} + 3 \\
D_{\bar z} I_{3,0,0,3} &= 4\, Q\, I_{3,0,0,3} - 2\, I_{3,0,1,2} I_{3,0,0,3} + I_{3,0,1,3} \\
D_{\bar\zeta} I_{3,0,0,3} &= -2\, {I_{3,0,0,3}}^2 + 2\, I_{3,0,0,3} + I_{3,0,0,4}
\end{aligned}
$$

3.3 Branch $I_0 \equiv 0, V_0 \neq 0$

We normalize V_0 to 1. There is only one differential invariant of order 5: $Q := F_{1,1,3,0} = F_{3,0,1,1}$.

The Levi rank 1 condition implies that all $F_{a,b+1,c,d+1}$ can be solved in terms of $F_{a',b',c',d'}$ with $b'\,d' = 0$. The vanishing of I_0 and of $\overline{I_0}$ implies that $F_{a+3,b,c,d+2}$ and $F_{a,b+2,c+3,d}$ can be solved in terms of $F_{a',b',c',d'}$ with ($a' < 3$ or $d' < 2$) and ($b' < 2$ or $c' < 3$). So all the non-phantom independent invariants of order $k \geqslant 6$ can be classified into three types

- Type 1: $I_{a,0,k-a,0} := F_{a,0,k-a,0}$ for $3 \leqslant a \leqslant k-3$;
- Type 2: $I_{k-4,1,3,0} := F_{k-4,1,3,0}$;
- Type 3: $I_{3,0,k-4,1} := F_{3,0,k-4,1}$, which is the conjugate of Type 2.

3.3.1 Recurrence formulas at order 5

At order 5 we have

$$
\begin{aligned}
D_z Q &= \frac{1}{2} I_{4,0,1,1} - 4\, Q^2 + \tfrac{1}{6}\, Q + \tfrac{1}{2}\, I_{2,1,3,0} - I_{1,2,3,0}\, Q \\
D_\zeta Q &= \tfrac{1}{6} - 2\, Q + \tfrac{1}{2}\, I_{1,2,3,0} - I_{0,3,3,0}\, Q \\
D_{\bar z} Q &= \tfrac{1}{2}\, I_{3,0,2,1} - 4\, Q^2 + \tfrac{1}{6}\, Q - I_{3,0,1,2}\, Q \\
D_{\bar\zeta} Q &= \tfrac{1}{2}\, I_{3,0,1,2} + \tfrac{1}{6} - 2\, Q - I_{3,0,0,3}\, Q
\end{aligned}
$$

in total 4 equations. There are 7 non-phantom independent invariants of order 6:

- Type 1: $I_{3,0,3,0}$;
- Type 2: $I_{2,1,3,0}, I_{1,2,3,0}, I_{0,3,3,0}$;
- Type 3: $I_{3,0,2,1}, I_{3,0,1,2}, I_{3,0,0,3}$.

One can solve $I_{2,1,3,0}, I_{1,2,3,0}, I_{3,0,2,1}$ and $I_{3,0,1,2}$ in terms of $I_{0,3,3,0}$ and $I_{3,0,0,3}$.

To solve $I_{3,0,3,0}$ one should use higher recurrence formulas.