Développement asymptotique de la relativité générale avec covariance galiléenne

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Getting started

As model, we take the Relativity

- The space-time $\mathcal{M}$ is **Riemannian** with 10 potentials $G_{\alpha\beta}$, solutions of the 10 **Einstein field equations** :

$$R'_{\alpha\beta} - \frac{1}{2} R \, G_{\alpha\beta} + \Lambda \, G_{\alpha\beta} = \frac{8\,\pi\,k_g}{c^4} \, T_{\alpha\beta}$$

Is this scheme transposable to classical mechanics ?

- We consider **Galileo** symmetry group :
  $\mathcal{M}$ is not **Riemannian** !
  but there is an **integrable** ***G*-structure**
- The associated connection (Galilean gravitation)
  has 2 components g, Ω
  deriving from 4 potentials ϕ, A^i
- Which are the 4 corresponding **field equations** ?

State of the Art

- **Post-Newtonian** theory [Eddington 1922, Will 1971, Ni 1972, Weinberg 1972, Misner 1973, ...]
- **Newton-Cartan** theory [Cartan 1923, Trautman 1963, Souriau 1970, Küntzle 1972, Duval 1985, Horváthy 1991, Bergshoeff 2011,...]
  Trautmann : Ricci tensor is : $\boldsymbol{R}' = -div\, g\, \boldsymbol{\tau} \otimes \boldsymbol{\tau}$
  flaw : as Poisson's equation $-div\, g = \Delta\phi = 4\,\pi\, k_g\, \rho$,
  it has not the expected Galilean covariance
- **Merging both approaches** [Dautcourt 1997, Tichy & Flanagan 2011, Van den Bleeken 2017]

Present approach : based on a variational version of General Relativity and the use of the G-structure
[GERG de Saxcé 2020 Asymptotic expansion of GR with Galilean covariance]

Galileo's group and geometry

- **Chart :** $\mathbb{R}^4 \ni X = \begin{pmatrix} t \\ x \end{pmatrix} \mapsto \boldsymbol{X} \in \mathcal{M}$ space-time

- **Galileo's group** $\mathbb{GAL}$ is the one of the **Galilean transformations** :

$$X = P\,X' + C \qquad \text{with} \qquad P = \begin{pmatrix} 1 & 0 \\ u & R \end{pmatrix}, \qquad C = \begin{pmatrix} \tau_0 \\ k \end{pmatrix}$$

  where $u \in \mathbb{R}^3$ is the **Galilean boost** or **velocity of transport** and R is a rotation

- $\mathbb{GAL}_0$ is the subgroup of *linear* Galilean transformation P

Galileo's group and geometry

Theorem

$X \mapsto X'$ is such that $\frac{\partial X'}{\partial X} = P^{-1}$ is Galilean
iff $x' = (R(t))^T (x - x_0(t)), \qquad t' = t + \tau_0$

- **Proof :** find the compatibility conditions of $\frac{\partial X'}{\partial X} = P^{-1}$ (Frobenius)
- This defines an **equivalence class** of coordinate charts
  $X \mapsto X'$ is called a **galileomorphism**,
  X and X' are called **Galilean coordinate systems** or **Galilean charts**
  In a physical point of view, they are **the frames in which the durations and distances are measured**
- Example of **integrable** G**-structure** L_G [Kobayashi 1963]

Galileo's group and geometry

Theorem

A **Galilean gravitation** is a symmetric connection Γ which,
in a Galilean chart,
takes its values in the Lie algebra $\mathfrak{gal}_0$
of the group of linear Galilean transformations :

$$\Gamma(dX) = \begin{pmatrix} 0 & 0 \\ \Omega \times dx - g\, dt & j(\Omega)\, dt \end{pmatrix},$$

where $j(\Omega)$ is the unique skew-symmetric matrix such that $j(\Omega)v = \Omega \times v$

- g is the classical **gravity**
- Ω is a new object called **spinning**

Galileo's group and geometry

4-**velocity** $U = \dot{X} = \begin{pmatrix} 1 \\ v \end{pmatrix}$ and **linear** 4-**momentum** $T = m\, U$

Covariant law of the motion

For particles in the gravitation field

$$\nabla_U T = \dot{T} + \Gamma(U)\, T = 0$$

in a Galilean chart : $\dot{m} = 0, \qquad \dot{p} = m\,(g - 2\,\Omega \times v)$

[Souriau 1970 Structure des systèmes dynamiques]

The **inertial charts** are the ones in which the spinning Ω vanishes

Galileo's group and geometry

- The closure condition $d\omega = 0$ of the presymplectic form gives

$$curl\, g + 2\,\frac{\partial \Omega}{\partial t} = 0, \qquad \boxed{div\, \Omega = 0}$$

then there exist **potentials** ϕ, A such that

$$g = -grad\, \phi - \frac{\partial A}{\partial t}, \qquad \Omega = \frac{1}{2}\, curl\, A$$

- They are defined modulo a **gauge** : $\phi^* = \phi - \dfrac{\partial f}{\partial t}, \quad A^* = A + grad\, f$
- The corresponding Lagrangian $\mathcal{L} = \frac{1}{2}\, m \parallel v \parallel^2 + m\, A \cdot v - m\, \phi$ is covariant provided

$$\phi' = \phi - A \cdot u - \frac{1}{2} \parallel u \parallel^2, \quad A' = R^T (A + u)$$

Galilean tensors

- By restriction of the transformation law of tensors to a subgroup $G \subset \mathbb{GL}(n)$, we obtain the G-**tensors**
  - Euclidean tensors are $\mathbb{SO}(3)$-tensors
  - Galilean tensors are $\mathbb{GAL}_0$-tensors
- G-tensors may be seen as **orbits**
  for the action of G onto the tensor components
- Examples of Galilean tensors :
  - **Vectors :** $V^{\alpha'} = (P^{-1})^{\alpha'}_{\beta} V^{\beta}$ or $V' = P^{-1} V$ with $P \in \mathbb{GAL}_0$
    applied to $U = \begin{pmatrix} 1 \\ v \end{pmatrix}$ gives $v' = R^T(v - u)$
  - **Covectors :** $\Phi_{\alpha'} = \Phi^{\beta} P^{\beta}_{\alpha'}$ or $\Phi' = \Phi\, P$ applied to $\tau = (1, 0^T)$
    invariant representing the clock form $\boldsymbol{\tau} = dt$ (**absolute time**)

Galilean tensors

Examples of Galilean tensors :

- **Vectors**
- **Covectors**
- 2-**covariant tensors :** $T_{\alpha'\beta'} = P^{\mu}_{\alpha'} P^{\nu}_{\beta'} T_{\mu\nu}$ or $T' = P^T \, T \, P$
  applied to $T = \begin{pmatrix} a & w^T \\ w & M \end{pmatrix}$ symmetric gives
  $a' = a + 2\, w \cdot u + u\,(M\, u), \quad w' = R^T (w + M\, u), \quad M' = R^T M\, R$

For instance $T = \overset{(0)}{G} = \begin{pmatrix} 2\,\phi & -A^T \\ -A & -1_{\mathbb{R}^3} \end{pmatrix}$ is a Galilean 2-covariant tensor

(but is not a metrics !)

Spacetime metrics

For weak gravitational fields

- $G = \begin{pmatrix} c^2 + 2\,\phi & -A^T \\ -A & -1_{\mathbb{R}^3} \end{pmatrix} = \epsilon^{-1} \overset{(-1)}{G} + \overset{(0)}{G}$ (exact, not a, expansion !)

  where $\epsilon = c^{-2}$, $\overset{(-1)}{G} = \tau \otimes \tau$ and $\overset{(0)}{G}$ are Galilean tensors
- Truncated expansion of

$$G^{-1} \cong \begin{pmatrix} 0 & 0 \\ 0 & -1_{\mathbb{R}^3} \end{pmatrix} + \epsilon \begin{pmatrix} 1 & -A^T \\ -A & A\,A^T \end{pmatrix}$$

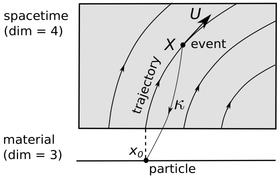
  Each term is a Galilean 2-contravariant tensor
- $\sqrt{-\det\ G} \cong c\,(1 + \epsilon\,(\phi + \frac{\|A\|^2}{2})) = c\ (1 + \epsilon\ l_0)$

  Galilean invariant

Hilbert-Einstein functional $\mathcal{A} = \int (p_M + p_G)\sqrt{-\det G}\, d^4X$

Functional due to the matter

Its motion is described by a line bundle of projection
$x_0 = \kappa(X) = \kappa(t, x)$



- **Conformation tensor** [Souriau 1964 Géométrie & Relativité]
  $\mathcal{D} = -\frac{\partial x_0}{\partial X}\left(\frac{\partial x_0}{\partial X}\right)^* = -\frac{\partial x_0}{\partial X} G^{-1}\left(\frac{\partial x_0}{\partial X}\right)^T$
  where $\frac{\partial x_0}{\partial X} = F^{-1}(-v, 1_{\mathbb{R}^3})$ with $F = \frac{\partial x}{\partial x_0}$
- mass density $\rho = \rho_0(x_0)/\det F$ which is a Galilean invariant

$$p_M\sqrt{-\det G} = c^2\rho_0(x_0)\sqrt{\det\mathcal{D}(-\det G)} \cong c[\rho\, c^2 - \rho\left(\frac{\|v\|^2}{2} + A\cdot v - \phi\right)]$$

energy at rest — Lagrangian density

Curvature tensor

We are working now with two structures

- **The Galilean structure :** we work in the charts of the G-structure
- **The Riemannian structure :** we equip the manifold with a metrics

Both structures are compatible because we use an asymptotic expansion of the metrics of which the terms are Galilean tensors

Curvature tensor

Levi-Civita connection $\Gamma(dX) = (\Gamma^{\mu}_{\alpha\beta} dX^{\alpha}) \cong \overset{(0)}{\Gamma} + \epsilon \overset{(1)}{\Gamma}$

- **Order 0** (Galilean gravitation)

  $\overset{(0)}{\Gamma}{}^{i}_{00} = -g^{i}$ gravity

  $\overset{(0)}{\Gamma}{}^{j}_{i0} = \Omega^{j}_{i}$ spinning

- **Order 1**

  $a = \overset{(1)}{\Gamma}{}^{0}_{00} = \frac{\partial \phi}{\partial t} - A \cdot g$

  $B_i = \overset{(1)}{\Gamma}{}^{0}_{0i} = \delta_{ik} \left(grad\, \phi - \Omega \times A \right)^{k}$

  $D_{ij} = \overset{(1)}{\Gamma}{}^{0}_{ij} = -\delta_{ik} \left(grad_s A \right)^{k}_{j}$

May be also obtained from the Bargmannian connection : $\overset{(1)}{\Gamma}{}^{0}_{\alpha\beta} = \Gamma^{4}_{\alpha\beta}$

Curvature tensor

Ricci tensor $R'_{\beta\gamma} = R^{\alpha}_{\alpha\beta\gamma}$

expanded as $R' \cong \overset{(0)}{R}{}' + \epsilon \overset{(1)}{R}{}'$

- **Order 0**

$$\overset{(0)}{R'} = \begin{pmatrix} 2 \parallel \Omega \parallel^2 - div\, g & (curl\, \Omega)^T \\ curl\, \Omega & 0 \end{pmatrix}$$

- **Order 1** for instance :

$$\overset{(1)}{R'_{lm}} = D_{lj}\Omega^j_m + D_{jm}\Omega^j_l + \frac{\partial}{\partial x^j}(A^j D_{lm}) - \frac{\partial D_{lm}}{\partial t} - \frac{\partial}{\partial x^l}(B_m + D_{jm}A^j)$$

$$\cdots$$

As expected, symmetric because $\frac{\partial}{\partial x}(B + D\,A)$ is the Hessian of I_0

Functional due to the geometry

$$p_G \sqrt{-\det G} = -\frac{c^4}{8\,\pi\,k_g}(2\,\Lambda\,\epsilon + R)\,\sqrt{-\det G}\ ,$$

where the **scalar curvature** is $R = G^{\alpha\beta} R'_{\alpha\beta} = Tr\,(G^{-1} R')$

that leads to $R \cong -2\,\epsilon\,(I + I_1)$ with the **Galilean invariants** of

- the metrics and the curvature : $I = div\, g - 2\ \parallel \Omega \parallel^2 + 2\, A \cdot curl\, \Omega$
- only the curvature : $I_1 = -\frac{1}{2}\ Tr\ \left[(D - (Tr\ D)\, 1_{\mathbb{R}^3})^2\right]$

Functional due to the geometry

Complete functional

$$c \int \left[\frac{\epsilon^{-1}}{4\,\pi\,k_g} (I + I_1 - \Lambda)\,(1 + \epsilon\, I_0) + \rho\,\epsilon^{-1} - \rho\,(\frac{\parallel v \parallel^2}{2} + A \cdot v - \phi) \right] d^4X$$

where $I = \overbrace{div\, g - 2 \parallel \Omega \parallel^2 + 2\, A \cdot curl\, \Omega}$, $\quad I_0 = \overbrace{(\phi + \frac{\|A\|^2}{2})}$

4 field equations (not 10 !)

- Variation *wrt* ϕ : $I + I_1 - \Lambda \quad = -4\,\pi\,k_g\,\rho$
- Variation *wrt* A : $-2\,(\epsilon^{-1} + I_0)\, curl\,\Omega - (I + I_1 - \Lambda)\, A = -4\,\pi\,k_g\,\rho\, v$

Consistency of the equations :

- The *rhs* $-4\,\pi\,k_g\,\rho\,U$ is a Galilean vector (because ρ is invariant)
- The *lhs* is also a Galilean vector

Asymptotic expansion of the solution

Reminder :

Variation *wrt* ϕ : $$I + I_1 - \Lambda = -4\,\pi\,k_g\,\rho \qquad \text{(t)}$$

Variation *wrt* A : $$-2\,(\epsilon^{-1} + I_0)\,curl\,\Omega - (I + I_1 - \Lambda)\,A = -4\,\pi\,k_g\,\rho\,v \qquad \text{(s)}$$

Solution in 3 steps :

1. (s) at order -1 : $curl\,\overset{(0)}{\Omega} = 0$

   remember : $div\,\overset{(0)}{\Omega} = 0$, then $\overset{(0)}{\Omega}$ is harmonic

   hypothesis : $\overset{(0)}{\Omega}$ is bounded, then $\boxed{\overset{(0)}{\Omega} = \overset{(0)}{\Omega}(t)}$

   in agreement with observations but justified by GR

   Next $\boxed{\overset{(0)}{A} = -u}$ then $\overset{(0)}{D} = 0$ and $\overset{(0)}{g} = -grad\,\overset{(0)}{\phi}$

   where occurs the velocity of transport

Asymptotic expansion of the solution

Variation *wrt* ϕ : $\quad l + l_1 - \Lambda = -4\,\pi\,k_g\,\rho \qquad$ (t)

Variation *wrt* A : $-2\,(\epsilon^{-1} + l_0)\,curl\,\Omega - (l + l_1 - \Lambda)\,A = -4\,\pi\,k_g\,\rho\,v \qquad$ (s)

1. (s) at order -1 : $curl\,\overset{(0)}{\Omega} = 0$ then $\overset{(0)}{\Omega} = \overset{(0)}{\Omega}(t)$

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Asymptotic expansion of the solution

Variation *wrt* ϕ : $\quad l + l_1 - \Lambda \quad = -4\,\pi\,k_g\,\rho \qquad$ (t)

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1. (s) at order -1 : $curl\,\overset{(0)}{\Omega} = 0$ then $\overset{(0)}{\Omega} = \overset{(0)}{\Omega}(t)$

   Next $\boxed{\overset{(0)}{A} = -u}$ then $\overset{(0)}{D} = 0$ and $\overset{(0)}{g} = -grad\,\overset{(0)}{\phi}$

2. (t) at order 0 : $\overset{(0)}{l} + \overset{(0)}{l}_1 = -\triangle\,\overset{(0)}{\phi} - 2\,\|\,\overset{(0)}{\Omega}\,\|^2 = -4\,\pi\,k_N\,\rho + \Lambda$

   $$\boxed{\overset{(0)}{\phi} = -\int \frac{k_N\,\rho(x',t)}{\|x-x'\|}\,dV(x') - \frac{1}{2}\,\|\,u\,\|^2 - \frac{\Lambda}{6}\,\|\,x\,\|^2}$$

Asymptotic expansion of the solution

Variation *wrt* ϕ : $\qquad l + l_1 - \Lambda \quad = -4\,\pi\,k_g\,\rho \qquad$ (t)

Variation *wrt* A : $-2\,(\epsilon^{-1} + l_0)\,curl\,\Omega - (l + l_1 - \Lambda)\,A = -4\,\pi\,k_g\,\rho\,v \qquad$ (s)

1. (s) at order -1 : $curl\,\overset{(0)}{\Omega} = 0$ then $\overset{(0)}{\Omega} = \overset{(0)}{\Omega}(t)$

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3. (s) at order 0 : $-2\,curl\,\overset{(1)}{\Omega} = -grad\,(div\,\overset{(1)}{A}) + \triangle\,\overset{(1)}{A} = -4\,\pi\,k_N\,\rho\,v$

   choice of a gauge *st* $\quad = 0$

   $$\boxed{\overset{(1)}{A} = \int \frac{k_N\,\rho(x',t)\,v(x',t)}{\|x-x'\|}\,dV(x')}$$

   relativistic effect

Conclusions and perspectives

Conclusions

- The scheme of the General Relativity is transposable to the classical mechanics. Instead of expressing the identity between two tensors of rank 2, the field equations identify two 4-vectors. **They allow to determine not only the gravity but also the spinning**

- Inspired from both Newton-Cartan and Post-Newtonian theories, the present approach leads to covariant field equations given in the charts of the G-structure, that is **the frames in which the durations and distances are measured**

Conclusions and perspectives

Perspectives

- **Closed analytical** and **numerical solutions** of the (non linear) field equations

$$g = -grad\,\phi - \frac{\partial A}{\partial t},\ \Omega = \frac{1}{2}\,curl\,A,\ I_1 = -\frac{1}{2}\,Tr\,\left[(D - (Tr\,D)\,1_{\mathbb{R}^3})^2\right]$$
$$div\,g - 2\,\parallel \Omega \parallel^2 + 2\,A \cdot curl\,\Omega + I_1 + \Lambda = -4\,\pi\,k_N \rho$$
$$-2\,(c^2 + \phi + \frac{\|A\|^2}{2})\,curl\,\Omega$$
$$-(div\,g - 2\,\parallel \Omega \parallel^2 + 2\,A \cdot curl\,\Omega + I_1 + \Lambda)\,A = -4\,\pi\,k_N \rho\,v$$

- **Extension to order** 2

- **Towards the General Relativity**

with 5 potentials $G = \begin{pmatrix} c^2 + 2\,\phi & -A^T \\ -A & -\psi\,1_{\mathbb{R}^3} \end{pmatrix}$

with 10 potentials $G = \begin{pmatrix} c^2 + 2\,\phi & -A^T \\ -A & -M \end{pmatrix}$

Thank you for your attention !

Any questions ?