

# Flatness and Curvature of Multisymplectic Manifolds

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Overview of the material

Multisymplectic manifolds

Darboux type theorems

Transitivities of Diffeomorphism groups

# References

The talk is based on :

- ▶ **L.R. and T. Wurzbacher** An invitation to multisymplectic geometry, *Journal of Geometry and Physics*, 2019.

And :

- ▶ **G. Martin** A Darboux theorem for multi-symplectic manifolds. *Letters in Mathematical Physics*, 1988.
- ▶ **F. Turiel** Fibrations en tores et  $n$ -formes de Liouville en dimension  $2n$ . *Comptes Rendues Acad. Sci. Paris Ser. I Math.*, 2001.
- ▶ **Jürgen Moser** On the volume elements on a manifold. *Transactions of the American Mathematical Society*, 1965.
- ▶ **W. Boothby** Transitivity of the automorphisms of certain geometric structures, *Transactions of the American Mathematical Society*, 1969.

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## Reminder : Symplectic manifolds

### Definition

*Symplectic manifold* :  $(M, \omega)$ , such that  $\omega \in \Omega^2(M)$  is **closed** and **non-degenerate**.

*Non-degeneracy* : The map  $\iota_{\bullet}\omega : TM \rightarrow T^*M$ ,  $v \mapsto \iota_v\omega$  is injective.

Examples :

- ▶  $T^*Q$  with  $\omega = d\theta$ ,  $\theta_{\eta}(v) = \eta(\pi_*v)$
- ▶ 2-dimensional manifolds with volume forms
- ▶ coadjoint orbits
- ▶ Kähler manifolds

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- ▶  $k = 1$  :  $\omega$  is symplectic.
- ▶  $k = \dim(M) - 1$  :  $\omega$  is a volume form.

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# Multicotangent Bundles

**Multi-Phase spaces.** Let  $Q$  be a manifold. Consider  $M = \Lambda^k T^*Q$ . The canonical form  $\theta \in \Omega^k(M)$  is defined by

$$\theta_\eta(v_1, \dots, v_k) = \eta(\pi_*(v_1), \dots, \pi_*(v_k)).$$

The form  $\omega = d\theta$  is multisymplectic.

**Example :**  $Q = \mathbb{R}^3$ . Consider  $\mathbb{R}^6 = \Lambda^2 T^*\mathbb{R}^3$  with coordinates  $\{p^{12}, p^{13}, p^{23}, q_1, q_2, q_3\}$

$$\omega = dp^{12} \wedge dq_1 \wedge dq_2 + dp^{13} \wedge dq_1 \wedge dq_3 + dp^{23} \wedge dq_2 \wedge dq_3.$$

**Remark :** **Magnetic terms.** Let  $\mu \in \Omega_{cl}^{k+1}(Q)$ . Then  $\omega + \pi^*\mu$  is multisymplectic.

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## Product type structures

**Product type multisymplectic structures.** Let  $(M_i, \omega_i)$  be a finite collection of  $k$   $n$ -dimensional manifolds with volume forms. Then  $\sum_i \pi_i^* \omega_i$  is a multisymplectic form on  $\prod_i M_i$ .

*Remark :* When  $k = 2$  volume version of the *Lagrangian graph picture of symplectic geometry*.

*Example :*  $M = \mathbb{R}^3$ ,  $(x_1, x_2, x_3)$  and volume  $dx^{123}$   
and  $N = \mathbb{R}^3$ ,  $(y_1, y_2, y_3)$  and volume  $-dy^{123}$ .  
Then  $M \times N$  carries the multisymplectic structure

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**Complex type multisymplectic structures.** Let  $(M, J, \Omega)$  be a complex manifold of complex dimension  $n > 1$  with a holomorphic volume form  $\Omega \in \Omega^{n,0}(M)$ . Then  $(M, \omega = \text{Re}(\Omega))$  is a multisymplectic manifold (of dimension  $2n$  with a form of degree  $n$ ).

**Example :**  $\mathbb{C}^n = \mathbb{R}^n \times \mathbb{R}^n$ ,  $(z_1, \dots, z_n)$ , where  $z_k = x_k + iy_k$ . We then have  $dz_k = dx_k + idy_k$  and the holomorphic volume form  $\Omega = dz_1 \wedge \dots \wedge dz_n$ . For  $n = 3$  this means

$$\text{Re}(\Omega) = \text{Re}((dx_1 + idy_1) \wedge (dx_2 + idy_2) \wedge (dx_3 + idy_3))$$

$$= dx_1 \wedge dx_2 \wedge dx_3 - dx_1 \wedge dy_2 \wedge dy_3 - dy_1 \wedge dx_2 \wedge dy_3 - dy_1 \wedge dy_2 \wedge dx_3.$$

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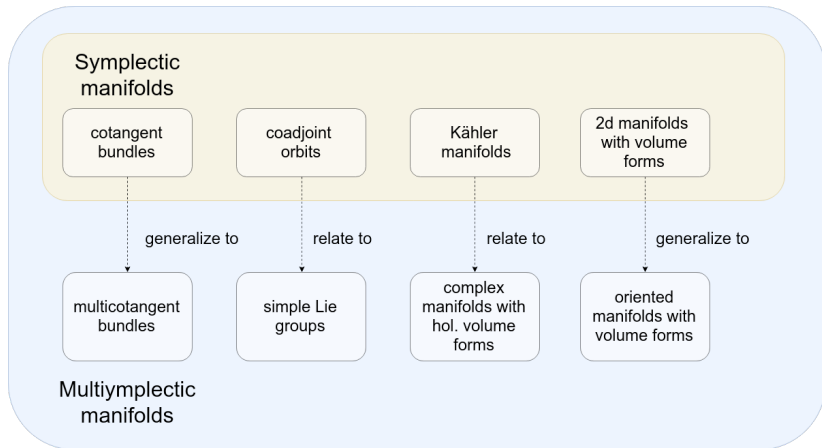
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# Symplectic vs. multisymplectic - Summary



# Darboux type theorems

# The classical Darboux-Moser-Weinstein Theorem

## Theorem

Let  $(M, \omega)$  be symplectic with  $\omega \in \Omega^2(M)$  and  $p \in M$ . Then around  $p$  there exist local coordinates  $(x_1, \dots, x_{2n})$  such that  $\omega = dx_1^2 + \dots + dx_{2n-1, n}$ .

Let  $(M, \omega)$  be volume with  $\omega \in \Omega^n(M)$  and  $p \in M$ . Then around  $p$  there exist local coordinates  $(x_1, \dots, x_n)$  such that  $\omega = dx_1 \dots dx_n$ .

## Idea of proof :

- ▶  $\omega$  is equivalent to a constant coefficient form, Moser path trick :

$$\frac{\partial}{\partial t} ((\phi^t)^* \omega_t) = (\phi^t)^* \left( \iota_{X_t} d\omega_t + d\iota_{X_t} \omega_t + \frac{d\omega_t}{dt} \right).$$

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# Multisymplectic Darboux theorems?

## Question

Do other multisymplectic manifolds have a “standard” constant coefficient representation?

## Definition

A form is called flat, if near any point it admits a constant coefficient representation.

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## Linear algebra problems

### Theorem (Mar70-Cap72-Wes81-Djo83-Ryv16)

The numbers  $\Sigma_k^n$  of non-degenerate forms in  $\Lambda^k(\mathbb{R}^n)^*$  up to elements in  $GL(n, \mathbb{R})$  are

- ▶  $\Sigma_n^n = 1, \Sigma_1^n = 0,$   
$$\Sigma_2^n = \begin{cases} 1 & n \text{ even} \\ 0 & n \text{ odd} \end{cases}, \Sigma_{n-2}^n = \begin{cases} \frac{n}{2} & n \bmod 4 = 2 \\ \lfloor \frac{n}{2} \rfloor - 1 & \text{else} \end{cases},$$
- ▶  $\Sigma_3^6 = 3, \Sigma_3^7 = 8, \Sigma_3^8 = 21, \Sigma_4^7 = 15, \Sigma_5^8 = 31.$
- ▶  $\Sigma_k^n = \infty$  in all other cases.

The form  $dx^{1234} + dx^{1256} + x_3 dx^{3456}$  is multisymplectic. Moreover it has different types for  $x_3 > 0, x_3 = 0$  and  $x_3 < 0$ . Especially, it is not flat.

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## More subtle problems

We have  $\Sigma_3^5 = 1$ . The type is  $dx^{123} + dx^{145}$ .

Theorem (Turiel 84)

$\omega = (dx^1 + x_2 dx^4) \wedge (dx^{23} + dx^{45})$  is not flat

proof :

$$D = \{v \in TM \mid \forall \alpha \in \Omega^1(\mathbb{R}^5) \text{ such that } \alpha \wedge \omega = 0 : \alpha(v) = 0\}$$

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## Product type theorem

### Theorem (RW19)

Let  $k \geq 2$ ,  $m > 2$ ,  $\omega$  multisymplectic, pointwise of type

$$de^{1,2,\dots,m} + de^{m+1,\dots,2m} + \dots + de^{(k-1)m+1,\dots,km}$$

Then  $\omega = \omega_1 + \dots + \omega_k$ , where  $\omega_1, \dots, \omega_k$  are decomposable.

Furthermore,  $\omega$  is flat if and only if  $d\omega_i = 0$  for all  $i$ .

Improvement : If  $m > 3$ ,  $d\omega_i = 0$  is automatic.

Nonflat example :

$$\omega = \omega_1 + \omega_2 = x_1 dx^{135} + dx^{146} + dx^{236} + dx^{245}$$

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## Multicotangent type theorem

### Theorem (Martin88)

Let  $k > 1$  and  $\omega$  be multisymplectic pointwise of type

$$\sum_{i_1 < \dots < i_k} f^{i_1, \dots, i_k} \wedge e^{i_1} \wedge \dots \wedge e^{i_k}.$$

Then

$$D = \{v \in TM \mid \iota_v \omega \text{ is decomposable}\}$$

is a  $\binom{n}{k}$ -dimensional distribution.

Moreover  $\omega$  is flat if and only if  $D$  is involutive.

## Complex type theorem

### Theorem (Turiel01-RW19)

Let  $m > 2$ ,  $\omega$  multisymplectic, pointwise of type

$$\operatorname{Re}((e^1 + ie^2) \wedge \dots \wedge (e^{2m-1} + ie^{2m})).$$

Then, there is a unique almost-complex structure  $\pm J$  such that

$$\iota_{J(w)}\iota_v\omega = \iota_w\iota_{J(v)}\omega \quad \forall v, w$$

Furthermore,  $(U, \omega)$  is flat if and only if  $J$  is integrable.

# Transitivities of Diffeomorphism groups

# The classical theorems

## Theorem (Boothby69)

Let  $M$  be connected and  $\omega$  symplectic or a volume. Then  $\text{Diff}(M, \omega)$  acts  $k$ -transitively on  $M$  for all  $k \in \mathbb{N}$ .

## Definition

$G$  acting on  $X$  is called  $k$ -transitive if for any two  $k$ -tuples  $(p_1, \dots, p_k)$ ,  $(q_1, \dots, q_k)$  of distinct points in  $X$  of there exists  $g \in G$  such that  $g(p_i) = q_i$  for  $i = 1, \dots, k$ .

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## Obstructions to 2-transitivity

**Remark :** If  $\omega$  has varying linear type or varying flatness behaviour, then its diffeomorphisms are not even 1-transitive.

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▶ ...

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## Complex type manifolds

**Lemma :** Let  $n \geq 2$ ,  $M = \mathbb{C}^n \cong \mathbb{R}^{2n}$  and  $\omega = \operatorname{Re}(dz^1 \wedge \dots \wedge dz^n)$ . Then  $\operatorname{Diff}(M, \omega)$  acts  $k$ -transitively on  $M$  for all  $k$ .

**Example :** the Multisymplectomorphisms of  $\omega = dx^{135} - dx^{146} - dx^{236} - dx^{245}$  act  $k$ -transitively for all  $k$  on  $\mathbb{R}^6$ .

**Weird fact :** Let  $M = \{z \mid \|z\| \leq 1\} \subset \mathbb{C}^n \cong \mathbb{R}^{2n}$  with the same form. Then  $\operatorname{Diff}(M, \omega)$  acts transitively but not 2-transitively.

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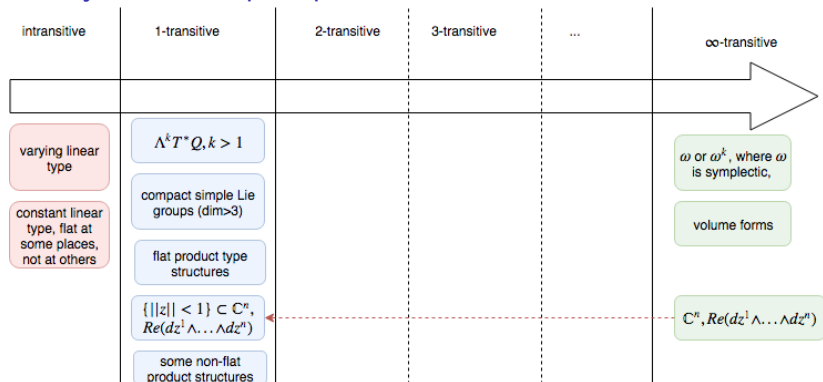
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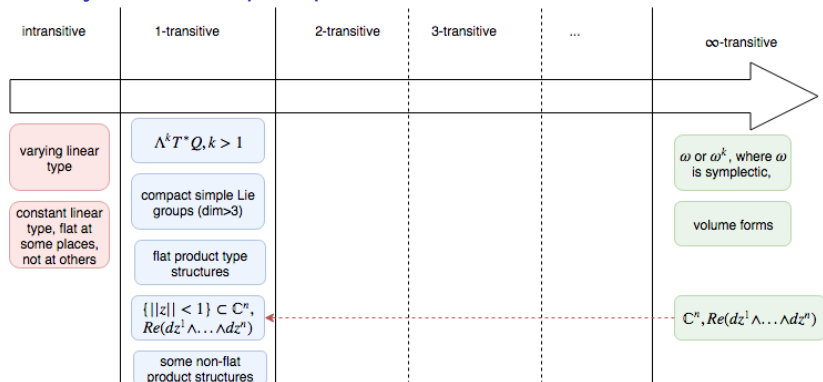
# Summary and an open problem



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Thank you for your attention !