Courbure des rayons sonores, tenseur impulsion-energie, et calcul de Regge en acoustique des salles

Jean-Dominique Polack

Sorbonne Université, UMR CNRS 7190, Institut D'Alembert

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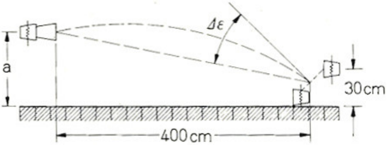
Content

Introduction

- Janowsky and Spandöck showed curvature of rays above absorbing surfaces in 1937



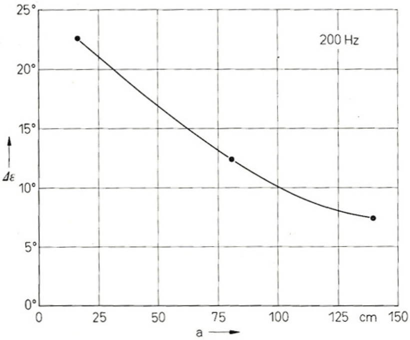


Figure: Bending of wave front above absorbing surface.

Cremer and Müller confirmed it in 1978

- pressure above surface ($x = 0$) of reduced admittance $\beta = \xi - i\sigma$:
  - $p(x, y) = 2p_0 e^{-i(k_y y - \arctan \frac{k\xi x}{1 - k\sigma x})}$
- wavefront equation: $y - y_0 = k\xi x$ for *small* x

Dujourdy et al. (2017, 2019) compute energy in corridors and open spaces:

- using energy-stress tensor formalism
- *ad hoc* boundary conditions:
  - absorption: incident intensity proportional to total energy
  - scattering: grazing intensity reduced by tangential stress

⇒ find the *natural* boundary conditions

Riemann space

Consider a 4-dimensional time-space with metric tensor g_{ij}

- distance element: $ds^2 = g_{ij} dx^i dx^j$
- volume element: $dV = \sqrt{|g|} dx^0 \dots dx^3$ with $g = \det g_{ij} < 0$
- wave equation: $\boxed{\Box \Phi = \nabla_i g^{ij} \nabla_j \Phi = 0}$
    - Φ velocity potential
    - g^{ij} inverse matrix of g_{ij}
    - ∇_i covariant derivation with respect to x^i

Covariant derivation

Covariant derivation differs from partial derivation

- takes in account changes of basis vector orientation
- depends on tensor rank
  - for function Φ: $\nabla_j \Phi = \partial_j \Phi = \Phi_j$
  - for covariant tensor (vector) Φ_i: $\nabla_j \Phi_i = \partial_j \Phi_i - \Gamma^k_{ji} \Phi_k$
  - for contravariant tensor g^{ik}: $\nabla_j g^{ik} = \partial_j g^{ik} + \Gamma^i_{jl} g^{lk} + \Gamma^k_{jl} g^{il}$
- Γ^k_{ji} are Christoffel symbols: $\Gamma^k_{ji} = \frac{1}{2} g^{kl} \left(\partial_j g_{il} + \partial_i g_{lj} - \partial_l g_{ji} \right)$

- covariant derivations do not commute
  - but $\left(\nabla_i \nabla_j - \nabla_j \nabla_i \right) \Phi = \left(\nabla_i \partial_j - \nabla_j \partial_i \right) \Phi = 0$
- covariant derivations commute with metric tensor
- contravariant derivation equals: $\nabla^i = g^{ij} \nabla_j = \nabla_j g^{ij}$

Conservation of stress-energy tensor

Consider product $\nabla_k \Phi^* \Box \Phi$. Differentiation rules lead to:

$$\begin{aligned}
\nabla_k \Phi^* \Box \Phi = \nabla_k \Phi^* \nabla_i g^{ij} \nabla_j \Phi &= \nabla_i g^{ij} \left[\nabla_k \Phi^* \nabla_j \Phi\right] - \left[\nabla_i \nabla_k \Phi^*\right] g^{ij} \nabla_j \Phi \\
&= \nabla_i g^{ij} \left[\nabla_k \Phi^* \nabla_j \Phi\right] - \left[\nabla_k \nabla_i \Phi^*\right] g^{ij} \nabla_j \Phi \\
&= \nabla^j \left[\nabla_k \Phi^* \nabla_j \Phi\right] - \left[\nabla_k \nabla_i \Phi^*\right] g^{ij} \nabla_j \Phi = 0
\end{aligned}$$

- then: $\nabla^j \left(\nabla_j \Phi^* \nabla_k \Phi + \nabla_j \Phi \nabla_k \Phi^*\right) = \nabla_k \left(\nabla_i \Phi^* g^{ij} \nabla_j \Phi\right)$
  - is the contravariant conservation of *symmetrical* stress-energy tensor $T_{ij} = \frac{\nabla_i \Phi^* \nabla_j \Phi + \nabla_i \Phi \nabla_j \Phi^*}{2} - \frac{1}{2} g_{ij} \left(\nabla_i \Phi^* g^{ij} \nabla_j \Phi\right)$
  - that is: $\boxed{\nabla^i T_{ij} = 0}$
  - with elements (Morse & Ingard 1968):
    - reduction $T = T_{ij} g^{ij} = \left(\nabla_i \Phi^* g^{ij} \nabla_j \Phi\right) = 2L$, $\boxed{L \text{ Lagrangian}}$
    - the total energy density T_{00};
    - the active acoustic intensity T_{a0} or T_{0a}, $a \in [1, 2, 3]$;
    - the symmetrical wave-stress tensor $T_{ab}, (a, b) \in [1, 2, 3]$

Propagation above constant absorbing plane

Sound propagation above horizontal absorbing plane $x^1 = 0$

- boundary condition reduces to $\nabla_0 \Phi \beta^0 + \nabla_1 \Phi \beta^1 = 0$
  - or $\beta^0 \Phi_0 + \beta^1 \Phi_1 = 0$
  - with $\frac{\beta^0}{\beta^1} = -\xi$ *real* admittance on the absorbing plane
- outgoing 4-vector is $n_i = (0, -1, 0, 0)$
- local metric must satisfy $g^{j1} n_1 = \xi^j$
  - that is: $g^{10} = g^{01} \neq 0$

- metric tensor no longer diagonal:

$$g^{ij} = \begin{pmatrix} -c & b & 0 & 0 \\ b & a & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \; g_{ij} = \begin{pmatrix} -a & b & 0 & 0 \\ b & c & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- a, b, and c depend on coordinate x^1 *only*
- boundary condition given by $b\Phi_0 + a\Phi_1 = 0$, $\frac{b}{a} = -\xi$
- a, b, and c normalised by $ac + b^2 = 1$
  - that is: $g = \det(g^{ij}) = -1$, with $a > 0$, $b \leq 0$
- Lagrangian:
  $L = \frac{1}{2}\left[-c|\Phi_0|^2 + b(\Phi_0^*\Phi_1 + \Phi_0\Phi_1^*) + a|\Phi_1|^2 + |\Phi_2|^2 + |\Phi_3|^2\right]$
- Christoffel symbols: *all equal to* 0 *but for*
  - $\Gamma^0_{00} = \Gamma^1_{00} = -\Gamma^1_{01} = -\Gamma^1_{10} = \frac{1}{2}ba_1$
  - $\Gamma^0_{01} = \Gamma^0_{10} = -\Gamma^1_{11} = \frac{1}{2}ca_1, \Gamma^0_{11} = -cb_1 + \frac{1}{2}bc_1$

Ray equation

Sound rays follow equation $dx^i g_{ij} dx^j = 0$

- reduces to *Monge equation*:
  $$\boxed{2bx' + c(x')^2 + (y')^2 + (z')^2 = a}$$
- with notations: $x' = \frac{dx^1}{dx^0}$, $y' = \frac{dx^2}{dx^0}$, $z' = \frac{dx^3}{dx^0}$

Ray curvature:

- generalized acceleration: $\frac{dv^i}{d\tau} = -\Gamma^i_{kl} v^k v^l$
- with τ proper time defined by $d\tau^2 = -g_{ij} dx^i dx^j$
- and $v^i = \frac{dx^i}{d\tau}$

Ray curvature and acceleration

$$\frac{dv^i}{d\tau} = \frac{d}{d\tau}\frac{dx^i}{d\tau} = \frac{d^2x^i}{(dx^0)^2}(v^0)^2 - \frac{dx^i}{dx^0}\Gamma^0_{kl}v^k v^l$$

$$(x^i)'' = \frac{d^2x^i}{(dx^0)^2} = \left(\frac{dx^i}{dx^0}\Gamma^0_{kl} - \Gamma^i_{kl}\right)\frac{v^k}{v^0}\frac{v^l}{v^0}$$

All calculations done:

- $x'' = \frac{1}{2}\left[ba_1 + 3ba_1x' + 3ca_1(x')^2 - (2cb_1 - bc_1)(x')^3\right]$
- $y'' = \frac{1}{2}\left[ba_1 + 2ca_1x' + -(2cb_1 - bc_1)(x')^2\right]y'$
- $z'' = \frac{1}{2}\left[ba_1 + 2ca_1x' + -(2cb_1 - bc_1)(x')^2\right]z'$

- a increases from boundary and b negative $\Rightarrow ba_1 < 0$
- rays parallel to absorbing plane $(x' = 0)$ **bend down**

Reflection coefficient for normal incidence

Plane wave at normal incidence: $\Phi = e^{-i(k_0 x^0 - k_1 x^1)}$

- $\Phi_0 = -ik_0\Phi$, $\Phi_1 = ik_1\Phi$, $\Phi_2 = \Phi_3 = 0$
- Lagrangian reduces to:
  $L = \frac{1}{2}\left[-c|\Phi_0|^2 + 2b\Re(\Phi_0\Phi_1^*) + a|\Phi_1|^2\right] =$
  $\frac{1}{2}\left[-ck_0^2 - 2bk_0k_1 + ak_1^2\right]$
- plane wave solution: $L = 0$, that is
  - $k_{1,i} = \frac{b+1}{a}k_0$, $k_{1,r} = \frac{b-1}{a}k_0$
  - and $\Phi = e^{-ik_0x^0}\left\{e^{ik_{1,i}x^1} + re^{-ik_{1,r}x^1}\right\}$, r reflection coefficient
- at boundary $x = 0$:
  - $\Phi_0 = -ik_0(1+r)e^{-ik_0x^0}$, $\Phi_1 = ik_0\frac{(1+r)b+(1-r)}{a}e^{-ik_0x^0}$
  - and $b\Phi_0 + a\Phi_1 = 0$ *only* if $r = 1$

$\Rightarrow$ Hypothesis leads to *total* reflection, not to absorption

Lorentz transformation

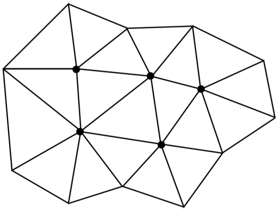
- $dx^0 = \frac{dx'^0 - \beta dx'^1}{\sqrt{1-\beta^2}}$, $dx^1 = \frac{dx'^1 - \beta dx'^0}{\sqrt{1-\beta^2}}$
- β varies with x^1 with $\beta(0) = \xi$
- boundary condition reduces to $\beta(0) = \xi$, boundary admittance
- then:
  - $\frac{\partial \Phi}{\partial x'^1} = \frac{\frac{\partial \Phi}{\partial x1} - \beta \frac{\partial \Phi}{\partial x0}}{\sqrt{1-\beta^2}}$
  - with $\frac{\partial \Phi}{\partial x'^1} = 0$ if $\frac{\partial \Phi}{\partial x^1} - \beta \frac{\partial \Phi}{\partial x^0} = 0$ on the absorbing plane
- metric tensor remains diagonal and constant

$\Rightarrow$ No ray curvature, but *elongation* of space

Conclusion: beyond curvature

- ray curvature due to non-vanishing Lagrangian
  - creates volume forces next to boundaries
- time-space does not curve at boundaries
  - except at "hinges" of co-dim 2 (Regge calculus)

- amounts to generalizing paving of space
  - to rooms of arbitrary shapes
  - beyond Euclidean geometry
- for example, triangular room
  - space is not flat
  - sum of angles $\neq 2\pi$ around hinges



Still to do:

- characterize edges with absorption and scattering properties

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