Analytical solutions of plane Timoshenko beam under large transformation

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1. Problem statement

2. Jacobian Elliptic functions

3. Physical Discussion

4. Conclusion
Outline

1. Problem statement
2. Jacobian Elliptic functions
3. Physical Discussion
4. Conclusion
Timoshenko beam model: 1-D Cosserat Body

Cosserat beam model

- Material curve $\mathcal{C}$.
- Moving director frame
  \[ \{d_i\} := (d_1, d_2, d_3) \]
  \[ d_3 \text{ normal to the section.} \]
- $S$ curvilinear coordinate of $\mathcal{C}$.
- $\varphi(S)$ placement of $\mathcal{C}$.
- $d_i(S) = R(S)e_i$ and $\frac{\partial d_i}{\partial S} = \kappa \times d_i$.

Plane transformation

- $\varphi(S) = \varphi_1 d_1 + \varphi_3 d_3$.
- $\theta$ rotation around $d_2$.
- $\kappa_2(S) = \frac{\partial \theta}{\partial S}$.
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- Material curve $C$.
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  \{d_i\} := (d_1, d_2, d_3)
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  $d_3$ normal to the section.
- $S$ curvilinear coordinate of $C$.
- $\varphi(S)$ placement of $C$.
- $d_i(S') = R(S)e_i$ and $\frac{\partial d_i}{\partial S} = \kappa \times d_i$.

Plane transformation

- $\varphi(S) = \varphi_1 d_1 + \varphi_3 d_3$.
- $\theta$ rotation around $d_2$.
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Internal forces/Equilibrium relations

Non linear strains

\[
\frac{d\varphi}{dS} = \varepsilon \quad \implies \quad \varepsilon_1(S) = \frac{\partial \varphi_1}{\partial S} + \varphi_3 \kappa_2 \quad \varepsilon_3(S) = \frac{\partial \varphi_3}{\partial S} - \varphi_1 \kappa_2
\]

Internal forces and moment

\[
\begin{align*}
N &= N_1 d_1 + N_3 d_3 \\
M &= M_2 d_2
\end{align*}
\]

- \(N_1\) shear force
- \(N_3\) normal force
- \(M_2\) bending moment

Linear constitutive laws

\[
\begin{align*}
N_1 &= GA \varepsilon_1 \\
N_3 &= EA(\varepsilon_3 - 1) \\
M_2 &= EI \kappa_2
\end{align*}
\]

Equilibrium equations [L.Le Marrec, J.Lerbet, L.R.Rakotomanana 2017]

\[
\begin{align*}
\frac{\partial N}{\partial S} &= 0 \\
\frac{\partial M}{\partial S} + \varepsilon \times N &= 0
\end{align*}
\]
### Internal forces/Equilibrium relations

#### Non linear strains

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#### Internal forces and moment

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Internal forces/Equilibrium relations

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Internal forces and moment

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N &= N_1 d_1 + N_3 d_3 \quad N_1 \text{ shear force} \\
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Dimensionless procedure

Dimensionless parameters and kinematical variables

\[ \varrho = \sqrt{\frac{I}{A}} \quad g = \frac{E}{G} \quad s = \frac{S}{\varrho} \quad \ell = \frac{L}{\varrho} \]

\[ \varepsilon_i(s) = \varepsilon_i(S) \quad \kappa_i(s) = \varrho \kappa_i(S) \quad \varphi_i(s) = \frac{1}{\varrho} \varphi_i(S) \quad \theta(s) = \theta(S) \]

Dimensionless equilibrium relations

\[ N' = 0 \]
\[ M' + \varepsilon \times N = 0 \]

\[ N_1 = \varepsilon_1 \]
\[ N_3 = g(\varepsilon_3 - 1) \]
\[ M_2 = g\kappa_2 \]

Projection along directors

\[ \varepsilon'_1 + g(\varepsilon_3 - 1)\kappa_2 = 0 \]
\[ g\varepsilon'_3 - \varepsilon_1\kappa_2 = 0 \]
\[ g\kappa'_2 - g\varepsilon_1(\varepsilon_3 - 1) + \varepsilon_1\varepsilon_3 = 0 \]

Non linear first order 3D ODE

Three boundary conditions
Dimensionless procedure

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Dimensionless equilibrium relations

\[ \dot{N}' = 0 \quad \dot{M}' + \varepsilon \times N = 0 \quad N_1 = \varepsilon_1 \quad N_3 = g(\varepsilon_3 - 1) \quad M_2 = g\kappa_2 \]

Projection along directors

\[ \varepsilon'_1 + g(\varepsilon_3 - 1)\kappa_2 = 0 \quad \text{Non linear first order 3D ODE} \]

\[ g\varepsilon'_3 - \varepsilon_1\kappa_2 = 0 \quad \text{Three boundary conditions} \]

\[ g\kappa'_2 - g\varepsilon_1(\varepsilon_3 - 1) + \varepsilon_1\varepsilon_3 = 0 \]
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Problem statement

\[ \frac{\partial N}{\partial S} = 0 \quad \Rightarrow \quad N(s) = N_\ell \]

\[ \phi(s) \] is the angle between \( d_3 \) and \( N_\ell \).

\[
\begin{align*}
N_1(s) &= N_\ell \sin(\phi) \\
N_3(s) &= N_\ell \cos(\phi) \\
M_2(s) &= -g \phi'
\end{align*}
\]

\[
\begin{align*}
\varepsilon'_1 + g(\varepsilon_3 - 1)\kappa_2 &= 0 \\
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g\kappa'_2 - g\varepsilon_1(\varepsilon_3 - 1) + \varepsilon_1\varepsilon_3 &= 0
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\[ g^2 \phi'' - gN_\ell \sin(\phi) + (g - 1)N_\ell^2 \sin \phi \cos(\phi) = 0 \]
Problem statement

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\[ g^2\phi'' - gN_\ell \sin(\phi) + (g - 1)N_\ell^2 \sin \phi \cos(\phi) = 0 \]
Homogeneous solutions

Scalar second order ODE

\[ g^2 \phi'' = N_\ell \sin(\phi)(g - (g - 1)N_\ell \cos(\phi)) \]

Elastica

- \[ N_\ell = 0 \]
- \[ \theta(s) = \frac{M_\ell}{g}(s - \ell) + \theta(\ell) \]
- \[ \varphi_1(s) = \varphi_1(0) + \frac{g}{M_\ell}(\cos(\theta) - 1) \]
- \[ \varphi_3(s) = \varphi_3(0) + \frac{g}{M_\ell} \sin(\theta) \]

Pure longitudinal force

- \[ \sin(\phi) = 0 \]
- \[ \varphi_1(s) = \varphi_1(0) \]
- \[ \varphi_3(s) = \varphi_3(0) + (1 \pm \frac{N_\ell}{g})s \]
- \[ \theta(s) = \theta(0) \]
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Non-homogeneous solutions

Non-linear differential equation with respect to $\phi$

First integral

$$(g\phi')^2 + 2gN_\ell \cos \phi - (g - 1)N_\ell^2 \cos^2 \phi = \mu$$

(1)

where

$$\mu = M_\ell^2 + 2gN_\ell \cos(\phi_\ell) - (g - 1)N_\ell^2 \cos^2(\phi_\ell)$$

Boundary conditions analyses

$$0 \leq N_\ell \leq 0.1, \quad -0.1 \leq M_\ell \leq 0.1, \quad 0 \leq \phi_\ell \leq 2\pi$$

Remarks

(1) is a non-linear scalar first order ODE.

(1) has a unique solution depending on $(N_\ell, M_\ell, \phi_\ell)$.

$\implies$ Cauchy initial value problem!

$(N_\ell, \phi_\ell, M_\ell)$ contributes to define

- Initial condition of the 3D system.
- Coefficients of (1).

Smooth change of $N_\ell, \phi_\ell$ or $M_\ell$ may affect the regularity of the solution.
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Non linear differential equation with respect to $\phi$

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Smooth change of $N_\ell$, $\phi_\ell$ or $M_\ell$ may affect the regularity of the solution.
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Non linear differential equation with respect to \( \phi \)

First integral

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(g\phi')^2 + 2gN_\ell \cos \phi - (g - 1)N_\ell^2 \cos^2 \phi = \mu \tag{1}
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First integral
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(g\phi')^2 + 2gN_\ell \cos \phi - (g - 1)N_\ell^2 \cos^2 \phi = \mu
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\[\implies \text{Cauchy initial value problem!}\]

\((N_\ell, \phi_\ell, M_\ell)\) contributes to define

- Initial condition of the 3D system.
- Coefficients of (1).

Smooth change of \(N_\ell, \phi_\ell\) or \(M_\ell\) may affect the regularity of the solution.
Change of variable \( t(s) = \tan \left( \frac{\phi(s)}{2} \right) \).

\[
t'(s)^2 = a t^4 + b t^2 + c
\]

- \( a = 0 \):
  - \( t'(s)^2 = b t^2 + c \)
- \( a \neq 0 \):
  - \( t'(s)^2 = a (t^2 - \alpha_-)(t^2 + \alpha_+) \)

with

\[
a(\mu, N_\ell) = \frac{\mu + 2g N_\ell + (g - 1) N_\ell^2}{4g^2}
\]
\[
\alpha_+(\mu, N_\ell) = \frac{g + \sqrt{g^2 - (g - 1) \mu}}{g - 1} - N_\ell
\]
\[
\alpha_-(\mu, N_\ell) = \frac{g - \sqrt{g^2 - (g - 1) \mu}}{g - 1} + N_\ell
\]

ODE solution

\[
t(s) = \pm \sqrt{\alpha_+} \cos \left( \sqrt{a \alpha_+} (s + s_0) \right) \left| \frac{\alpha_+ + \alpha_-}{\alpha_+} \right|
\]
Change of variable $t(s) = \tan \left( \frac{\phi(s)}{2} \right)$.

$t'^2 = a t^4 + b t^2 + c$

- $a = 0$, then $t'^2 = b t^2 + c$
- $a \neq 0$, then $t'^2 = a (t^2 - \alpha_-)(t^2 + \alpha_+)$

with

- $a(\mu, N_\ell) = \frac{\mu + 2gN_\ell + (g - 1)N_\ell^2}{4g^2}$
- $\alpha_+(\mu, N_\ell) = \frac{g + \sqrt{g^2 - (g - 1)\mu}}{g - 1} - N_\ell$
- $\alpha_-(\mu, N_\ell) = \frac{g - \sqrt{g^2 - (g - 1)\mu}}{g - 1} + N_\ell$

ODE solution

$t(s) = \pm \sqrt{\alpha_+} \cos(\sqrt{a\alpha_+}(s + s_0) \left| \frac{\alpha_+ + \alpha_-}{\alpha_+} \right)$
Forces, moment and rotation

\[
N_1(s) = N_\ell \frac{2\sqrt{\alpha_+ \cos(\zeta \ | \ m)}}{1 + \alpha_+ \cos^2(\zeta \ | \ m)} \\
N_3(s) = N_\ell \frac{1 - \alpha_+ \cos^2(\zeta \ | \ m)}{1 + \alpha_+ \cos^2(\zeta \ | \ m)}
\]

\[
M_2(s) = \sqrt{ag} \frac{2\alpha_+ \frac{\nu}{m} \cos(\zeta \ | \ m) \sin(\zeta \ | \ m)}{1 + \alpha_+ \cos^2(\zeta \ | \ m)} \\
\theta(s) = \phi - 2 \arctan\left( t(s) \right)
\]

\[
M' + \mathbf{\epsilon} \times \mathbf{N} = 0 \quad \implies \quad M(s) - M(0) + (\phi(s) - \phi(0)) \times \mathbf{N}_\ell = 0
\]

Placement solutions

\[
\phi(s) = \phi_t(s) e_t + \phi_n(s) e_n \\
\phi_t(s) = \phi_t(0) + \frac{M_2(s) - M_2(0)}{N_\ell} \\
\phi_n(s) = \phi_n(0) + \int_0^s N_\ell \left( \frac{2t}{1 + t^2} \right)^2 + N_\ell \left( \frac{1 - t^2}{1 + t^2} \right)^2 + \frac{1 - t^2}{1 + t^2} \, ds
\]

\[
e_n = \frac{N_\ell}{\|N_\ell\|} \\
e_t = e_y \times e_n.
\]
Forces, moment and rotation

\[ N_1(s) = N_\ell \frac{2\sqrt{\alpha_+ \cos(\zeta \mid m)}}{1 + \alpha_+ \cos^2(\zeta \mid m)} \]
\[ N_3(s) = N_\ell \frac{1 - \alpha_+ \cos^2(\zeta \mid m)}{1 + \alpha_+ \cos^2(\zeta \mid m)} \]
\[ M_2(s) = \sqrt{ag} \frac{2\alpha_+ \frac{\text{ns}(\zeta \mid m)}{\text{ds}(\zeta \mid m)}}{1 + \alpha_+ \cos^2(\zeta \mid m)} \]
\[ \theta(s) = \hat{\phi} - 2 \arctan \left( t(s) \right) \]

\[ M' + \varepsilon \times N = 0 \quad \Rightarrow \quad M(s) - M(0) + (\varphi(s) - \varphi(0)) \times N_\ell = 0 \]

Placement solutions

\[ \varphi(s) = \varphi_t(s) e_t + \varphi_n(s) e_n \]
\[ e_n = \frac{N_\ell}{\|N_\ell\|} \]
\[ e_t = e_y \times e_n. \]
\[ \varphi_t(s) = \varphi_t(0) + \frac{M_2(s) - M_2(0)}{N_\ell} \]
\[ \varphi_n(s) = \varphi_n(0) + \int_0^s N_\ell \left( \frac{2t}{1 + t^2} \right)^2 + \frac{N_\ell}{g} \left( \frac{1 - t^2}{1 + t^2} \right)^2 + \frac{1 - t^2}{1 + t^2} \ ds \]
Recovering physical quantities

Forces, moment and rotation

\[ N_1(s) = N_\ell \frac{2\sqrt{\alpha_+ \cos(\zeta \mid m)}}{1 + \alpha_+ \cos^2(\zeta \mid m)} \]
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\[ M_2(s) = \sqrt{ag} \frac{2\alpha_+ \mathrm{ns}(\zeta \mid m) \mathrm{ds}(\zeta \mid m)}{1 + \alpha_+ \cos^2(\zeta \mid m)} \]
\[ \theta(s) = \hat{\phi} - 2 \arctan(t(s)) \]

\[ M' + \varepsilon \times N = 0 \quad \implies \quad M(s) - M(0) + (\varphi(s) - \varphi(0)) \times N_\ell = 0 \]

Placement solutions

\[ \varphi(s) = \varphi_t(s)e_t + \varphi_n(s)e_n \]
\[ e_n = \frac{N_\ell}{\|N_\ell\|} \]
\[ e_t = e_y \times e_n. \]

\[ \varphi_t(s) = \varphi_t(0) + \frac{M_2(s) - M_2(0)}{N_\ell} \]

\[ \varphi_n(s) = \varphi_n(0) + \int_0^s N_\ell \left( \frac{2t}{1 + t^2} \right)^2 + \frac{N_\ell}{g} \left( \frac{1 - t^2}{1 + t^2} \right)^2 + \frac{1 - t^2}{1 + t^2} \, ds \]
Recovering physical quantities

Illustrating example with \( g = \frac{5}{2} \) and \( \ell = 50 \)

\[ \phi \ell = \frac{3\pi}{4} \quad N_\ell = 0.01 \quad M_\ell = 0.05 \]
\[ \varphi(0) = 0 \quad \theta(0) = 0 \]

Cauchy BC
Kinematical BC
Outline

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Load control

\[ \hat{\phi} = \phi_\ell + \theta_\ell \]

Dead load

Geometrical invariance

Boundary value problem

Material invariance

Follower load

Regularity of solutions??

Cauchy problem

\[ \hat{\phi} \text{ constant} \]

\[ \phi_\ell \text{ constant} \]
Pure-shear load

Cauchy problem with

\[ N_\ell \neq 0 \quad \phi_\ell = \frac{\pi}{2} \quad M_\ell = 0 \]

Asymptotic analysis
Leading terms with respect to \( N_\ell \)

Pure shear placement

\[
\begin{align*}
\varphi_t(s) & \approx \varphi_t(0) + \sqrt{\frac{2g}{N_\ell}} \, \text{cn} \left( \sqrt{\frac{N_\ell}{g}} \left( s + s_0 \right) \left| \frac{1}{2} \right. \right) \\
\varphi_n(s) & \approx \varphi_n(0) + s - 2 \sqrt{\frac{g}{N_\ell}} \, \mathcal{E} \left( \sqrt{\frac{N_\ell}{g}} \left( s + s_0 \right) \left| \frac{1}{2} \right. \right)
\end{align*}
\]
Pure-shear load

Cauchy problem with

\[ N_\ell \neq 0 \quad \phi_\ell = \frac{\pi}{2} \quad M_\ell = 0 \]

Asymptotic analysis
Leading terms with respect to \( N_\ell \)

Pure shear placement

\[
\varphi_t(s) \simeq \varphi_t(0) + \sqrt{\frac{2g}{N_\ell}} \, \text{cn}\left(\sqrt{\frac{N_\ell}{g}} (s + s_0) \frac{1}{2}\right)
\]

\[
\varphi_n(s) \simeq \varphi_n(0) + s - 2 \sqrt{\frac{g}{N_\ell}} \, \mathcal{E}\left(\sqrt{\frac{N_\ell}{g}} (s + s_0) \frac{1}{2}\right)
\]
Quasi static increase of $N_\ell$: Follower load

$N_\ell = 0.01$

$N_\ell = 0.05$

$N_\ell = 0.1$

$N_\ell = 0.15$

$N_\ell = 0.2$

Periodic pattern
Quantitative and qualitative analyses

Periodic pattern

- Material period $P = 4 \sqrt{\frac{g}{N_\ell}} K(\frac{1}{2})$
- Spatial period $B = \frac{2\pi}{K(\frac{1}{2})} \sqrt{\frac{g}{N_\ell}}$

\[
\frac{B}{A} = \frac{\pi}{\sqrt{2K(\frac{1}{2})}} \sim 1.2
\]

Size $A = 2 \sqrt{\frac{2g}{N_\ell}}$

independent of $N_\ell$, $g$ and $\ell$.
Control of a kinematical boundary

Problem statement

Pure dead-load

\[ \mathbf{N}_\ell = N_\ell \mathbf{e}_z \quad M_\ell = 0 \]

\( \theta(0) \) varies according to a command.

Boundary value problem.

\[ \mathbf{N}_\ell = N_\ell \mathbf{e}_z \quad M_\ell = 0 \quad \theta(0) := \theta_0 \]

Cauchy value problem.

\[ \mathbf{N}_\ell = N_\ell \mathbf{e}_z \quad M_\ell = 0 \quad \phi_\ell \]

What is the value of \( \theta_0 \) with respect to \( \phi_\ell \) for a prescribed dead load??.

\[ \theta_0 = \pm 2 \arctan \left( \frac{\sqrt{\alpha_-}}{\operatorname{cn}(\sqrt{a(\alpha_- + \alpha_+)} \ell \vert \frac{\alpha_+}{\alpha_+ + \alpha_-})} \right) \]

for a prescribed \( \phi_\ell \in [0, 2\pi] \)
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for a prescribed \( \phi_\ell \in [0, 2\pi] \)
Analysis of $\theta_0(\phi_\ell)$

- **Small $N_\ell$**
  - Monotonicity
  - Bijection

- **Large $N_\ell$**
  - Loss of Monotonicity
  - Non-unicity
Catastrophic instabilities

**Command:** \( \theta_0 \) varies from 0 to \( 2\pi \)

- Smooth variation of \( \phi_\ell \) for small \( \theta_0 \).
- Large jump of \( \phi_\ell \) as \( \theta_0 \) increases in order to maintain equilibrium position.
Catastrophic instabilities

Command: "\(\theta_0\) varies from 0 to \(2\pi\)"

- Smooth variation of \(\phi_\ell\) for small \(\theta_0\).
- Large jump of \(\phi_\ell\) as \(\theta_0\) increases in order to maintain equilibrium position.
- Brutal change of the configuration of the beam.
- Catastrophic instability
Outline

1. Problem statement
2. Jacobian Elliptic functions
3. Physical Discussion
4. Conclusion
Conclusion

- Analytical study of a large transformation of a Timoshenko beam subjected to end forces and moment with linear constitutive law and large transformation.

- Pure dimensionless approach.

- Cauchy problem which depends on loads at one extremity.

- Solutions in terms of Jacobian elliptic functions.

- Pure shear load (Cauchy problem): Explicit solutions.

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