Analytical solutions of plane Timoshenko beam under large transformation

Marwan Hariz

Joint work with Loïc Le Marrec and Jean Lerbet

Rencontre du GDR GDM à Paris Jussieu
4-6 November, 2020

Outline

Outline

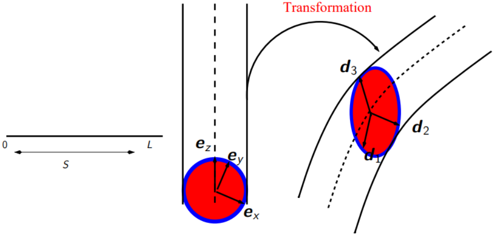
Timoshenko beam model: 1-D Cosserat Body

Cosserat beam model

- Material curve $\mathcal{C}$.
- Moving director frame $\{\boldsymbol{d_i}\} := (\boldsymbol{d_1}, \boldsymbol{d}_2, \boldsymbol{d}_3)$
  $\boldsymbol{d_3}$ normal to the section.
- S curvilinear coordinate of $\mathcal{C}$.
- $\varphi(S)$ placement of $\mathcal{C}$.
- $\boldsymbol{d_i}(S) = \boldsymbol{R}(S)\boldsymbol{e_i}$ and $\frac{\partial \boldsymbol{d_i}}{\partial S} = \boldsymbol{\kappa} \times \boldsymbol{d_i}$.

Plane transformation

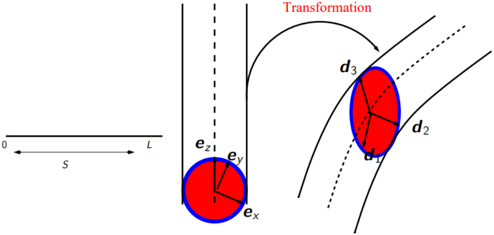
- $\varphi(S) = \varphi_1 \boldsymbol{d_1} + \varphi_3 \boldsymbol{d_3}$.
- θ rotation around $\boldsymbol{d}_2$.
- $\kappa_2(S) = \dfrac{\partial \theta}{\partial S}$



Timoshenko beam model: 1-D Cosserat Body

Cosserat beam model

- Material curve $\mathcal{C}$.
- Moving director frame
  $\{\boldsymbol{d_i}\} := (\boldsymbol{d_1}, \boldsymbol{d}_2, \boldsymbol{d}_3)$
  $\boldsymbol{d_3}$ normal to the section.
- S curvilinear coordinate of $\mathcal{C}$.
- $\varphi(S)$ placement of $\mathcal{C}$.
- $\boldsymbol{d_i}(S) = \boldsymbol{R}(S)\boldsymbol{e_i}$ and $\frac{\partial \boldsymbol{d_i}}{\partial S} = \boldsymbol{\kappa} \times \boldsymbol{d_i}$.



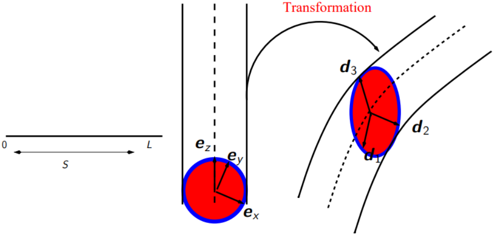
Plane transformation

- $\varphi(S) = \varphi_1 \boldsymbol{d_1} + \varphi_3 \boldsymbol{d_3}$.
- θ rotation around $\boldsymbol{d}_2$.
- $\kappa_2(S) = \dfrac{\partial \theta}{\partial S}$

Timoshenko beam model: 1-D Cosserat Body

Cosserat beam model

- Material curve $\mathcal{C}$.
- Moving director frame $\{\boldsymbol{d_i}\} := (\boldsymbol{d_1}, \boldsymbol{d}_2, \boldsymbol{d}_3)$ $\boldsymbol{d_3}$ normal to the section.
- S curvilinear coordinate of $\mathcal{C}$.
- $\boldsymbol{\varphi}(S)$ placement of $\mathcal{C}$.
- $\boldsymbol{d_i}(S) = \boldsymbol{R}(S)\boldsymbol{e_i}$ and $\frac{\partial \boldsymbol{d_i}}{\partial S} = \boldsymbol{\kappa} \times \boldsymbol{d_i}$.



Plane transformation

- $\boldsymbol{\varphi}(S) = \varphi_1 \boldsymbol{d_1} + \varphi_3 \boldsymbol{d_3}$.
- θ rotation around $\boldsymbol{d}_2$.
- $\kappa_2(S) = \dfrac{\partial \theta}{\partial S}$

Internal forces/Equilibrium relations

Non linear strains

$$\frac{d\boldsymbol{\varphi}}{dS} = \boldsymbol{\varepsilon} \quad \Longrightarrow \quad \varepsilon_1(S) = \frac{\partial \varphi_1}{\partial S} + \varphi_3 \kappa_2 \qquad \varepsilon_3(S) = \frac{\partial \varphi_3}{\partial S} - \varphi_1 \kappa_2$$

Internal forces and moment

$$\begin{aligned} \boldsymbol{N} &= N_1 \boldsymbol{d_1} + N_3 \boldsymbol{d_3} \\ \boldsymbol{M} &= M_2 \boldsymbol{d_2} \end{aligned}$$

N_1 shear force
N_3 normal force
M_2 bending moment

Linear constitutive laws

$$N_1 = GA\varepsilon_1 \qquad N_3 = EA(\varepsilon_3 - 1) \qquad M_2 = EI\kappa_2$$

Equilibrium equations [L.Le Marrec, J.Lerbet, L.R.Rakotomanana 2017]

$$\frac{\partial \boldsymbol{N}}{\partial S} = 0 \qquad \frac{\partial \boldsymbol{M}}{\partial S} + \boldsymbol{\varepsilon} \times \boldsymbol{N} = 0$$

Internal forces/Equilibrium relations

Non linear strains

$$\frac{d\boldsymbol{\varphi}}{dS} = \boldsymbol{\varepsilon} \qquad \Longrightarrow \qquad \varepsilon_1(S) = \frac{\partial \varphi_1}{\partial S} + \varphi_3 \kappa_2 \qquad \varepsilon_3(S) = \frac{\partial \varphi_3}{\partial S} - \varphi_1 \kappa_2$$

Internal forces and moment

$$\begin{aligned} \boldsymbol{N} &= N_1 \boldsymbol{d_1} + N_3 \boldsymbol{d_3} \\ \boldsymbol{M} &= M_2 \boldsymbol{d_2} \end{aligned}$$

N_1 shear force

N_3 normal force

M_2 bending moment

Linear constitutive laws

$$N_1 = GA\varepsilon_1 \qquad N_3 = EA(\varepsilon_3 - 1) \qquad M_2 = EI\kappa_2$$

Equilibrium equations [L.Le Marrec, J.Lerbet, L.R.Rakotomanana 2017]

$$\frac{\partial \boldsymbol{N}}{\partial S} = 0 \qquad \frac{\partial \boldsymbol{M}}{\partial S} + \boldsymbol{\varepsilon} \times \boldsymbol{N} = 0$$

Internal forces/Equilibrium relations

Non linear strains

$$\frac{d\boldsymbol{\varphi}}{dS} = \boldsymbol{\varepsilon} \qquad \Longrightarrow \qquad \varepsilon_1(S) = \frac{\partial\varphi_1}{\partial S} + \varphi_3\kappa_2 \qquad \varepsilon_3(S) = \frac{\partial\varphi_3}{\partial S} - \varphi_1\kappa_2$$

Internal forces and moment

$$\begin{aligned} \boldsymbol{N} &= N_1\boldsymbol{d_1} + N_3\boldsymbol{d_3} \\ \boldsymbol{M} &= M_2\boldsymbol{d_2} \end{aligned}$$

N_1 shear force
N_3 normal force
M_2 bending moment

Linear constitutive laws

$$N_1 = GA\varepsilon_1 \qquad N_3 = EA(\varepsilon_3 - 1) \qquad M_2 = EI\kappa_2$$

Equilibrium equations [L.Le Marrec, J.Lerbet, L.R.Rakotomanana 2017]

$$\frac{\partial \boldsymbol{N}}{\partial S} = 0 \qquad \frac{\partial \boldsymbol{M}}{\partial S} + \boldsymbol{\varepsilon} \times \boldsymbol{N} = 0$$

Dimensionless procedure

Dimensionless parameters and kinematical variables

$$\varrho = \sqrt{\frac{I}{A}} \qquad g = \frac{E}{G} \qquad s = \frac{S}{\varrho} \qquad \ell = \frac{L}{\varrho}$$

$$\varepsilon_i(s) = \underline{\varepsilon}_i(S) \quad \kappa_i(s) = \varrho\,\underline{\kappa}_i(S) \quad \varphi_i(s) = \frac{1}{\varrho}\,\underline{\varphi}_i(S) \quad \theta(s) = \underline{\theta}(S)$$

Dimensionless equilibrium relations

$$\begin{aligned} \boldsymbol{N}' &= 0 \\ \boldsymbol{M}' + \boldsymbol{\varepsilon} \times \boldsymbol{N} &= 0 \end{aligned} \qquad \begin{aligned} N_1 &= \varepsilon_1 \\ N_3 &= g(\varepsilon_3 - 1) \\ M_2 &= g\kappa_2 \end{aligned}$$

Projection along directors

$$\begin{aligned} \varepsilon_1' + g(\varepsilon_3 - 1)\kappa_2 &= 0 \\ g\varepsilon_3' - \varepsilon_1\kappa_2 &= 0 \\ g\kappa_2' - g\varepsilon_1(\varepsilon_3 - 1) + \varepsilon_1\varepsilon_3 &= 0 \end{aligned}$$

Non linear first order 3D ODE

Three boundary conditions

Dimensionless procedure

Dimensionless parameters and kinematical variables

$$\varrho = \sqrt{\frac{I}{A}} \qquad g = \frac{E}{G} \qquad s = \frac{S}{\varrho} \qquad \ell = \frac{L}{\varrho}$$

$$\varepsilon_i(s) = \underline{\varepsilon}_i(S) \quad \kappa_i(s) = \varrho\,\underline{\kappa}_i(S) \quad \varphi_i(s) = \frac{1}{\varrho}\,\underline{\varphi}_i(S) \quad \theta(s) = \underline{\theta}(S)$$

Dimensionless equilibrium relations

$$\begin{aligned} \boldsymbol{N}' &= 0 \\ \boldsymbol{M}' + \boldsymbol{\varepsilon} \times \boldsymbol{N} &= 0 \end{aligned} \qquad \begin{aligned} N_1 &= \varepsilon_1 \\ N_3 &= g(\varepsilon_3 - 1) \\ M_2 &= g\kappa_2 \end{aligned}$$

Projection along directors

$$\begin{aligned} \varepsilon_1' + g(\varepsilon_3 - 1)\kappa_2 &= 0 \\ g\varepsilon_3' - \varepsilon_1\kappa_2 &= 0 \\ g\kappa_2' - g\varepsilon_1(\varepsilon_3 - 1) + \varepsilon_1\varepsilon_3 &= 0 \end{aligned}$$

Non linear first order 3D ODE

Three boundary conditions

Dimensionless procedure

Dimensionless parameters and kinematical variables

$$\varrho = \sqrt{\frac{I}{A}} \qquad g = \frac{E}{G} \qquad s = \frac{S}{\varrho} \qquad \ell = \frac{L}{\varrho}$$

$$\varepsilon_i(s) = \underline{\varepsilon}_i(S) \quad \kappa_i(s) = \varrho\, \underline{\kappa}_i(S) \quad \varphi_i(s) = \frac{1}{\varrho}\, \underline{\varphi}_i(S) \quad \theta(s) = \underline{\theta}(S)$$

Dimensionless equilibrium relations

$$\begin{aligned} \boldsymbol{N}' &= 0 \\ \boldsymbol{M}' + \boldsymbol{\varepsilon} \times \boldsymbol{N} &= 0 \end{aligned} \qquad \begin{aligned} N_1 &= \varepsilon_1 \\ N_3 &= g(\varepsilon_3 - 1) \\ M_2 &= g\kappa_2 \end{aligned}$$

Projection along directors

$$\begin{aligned} \varepsilon_1' + g(\varepsilon_3 - 1)\kappa_2 &= 0 \\ g\varepsilon_3' - \varepsilon_1\kappa_2 &= 0 \\ g\kappa_2' - g\varepsilon_1(\varepsilon_3 - 1) + \varepsilon_1\varepsilon_3 &= 0 \end{aligned}$$

Non linear first order 3D ODE

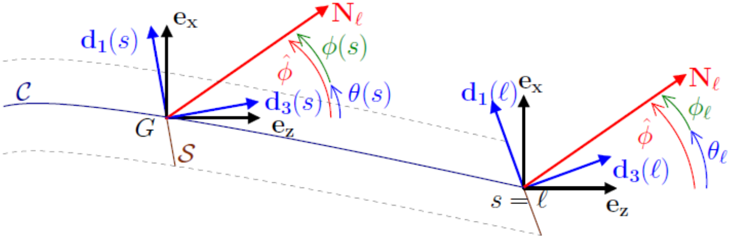
Three boundary conditions

Outline

1. Problem statement
2. **Jacobian Elliptic functions**
3. Physical Discussion
4. Conclusion

Problem statement

$$\frac{\partial \boldsymbol{N}}{\partial S} = 0 \qquad \Longrightarrow \qquad \boldsymbol{N}(s) = \boldsymbol{N}_\ell$$



$\phi(s)$ is the angle between $\boldsymbol{d}_3$ and $\boldsymbol{N}_\ell$.

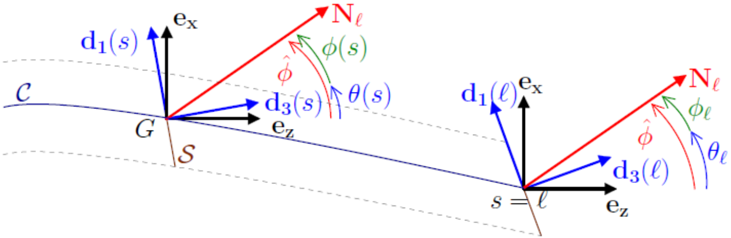
$$\begin{aligned} N_1(s) &= N_\ell \sin(\phi) \\ N_3(s) &= N_\ell \cos(\phi) \\ M_2(s) &= -g\phi' \end{aligned}$$

$$\begin{aligned} \varepsilon_1' + g(\varepsilon_3 - 1)\kappa_2 &= 0 \\ g\varepsilon_3' - \varepsilon_1\kappa_2 &= 0 \\ g\kappa_2' - g\varepsilon_1(\varepsilon_3 - 1) + \varepsilon_1\varepsilon_3 &= 0 \end{aligned}$$

$$g^2\phi'' - gN_\ell \sin(\phi) + (g-1)N_\ell^2 \sin\phi \cos(\phi) = 0$$

Problem statement

$$\frac{\partial \boldsymbol{N}}{\partial S} = 0 \qquad \Longrightarrow \qquad \boldsymbol{N}(s) = \boldsymbol{N}_\ell$$



$\phi(s)$ is the angle between $\boldsymbol{d}_3$ and $\boldsymbol{N}_\ell$.

$$\boxed{\begin{aligned} N_1(s) &= N_\ell \sin(\phi) \\ N_3(s) &= N_\ell \cos(\phi) \\ M_2(s) &= -g\phi' \end{aligned}} \longrightarrow \boxed{\begin{aligned} \varepsilon_1' + g(\varepsilon_3 - 1)\kappa_2 &= 0 \\ g\varepsilon_3' - \varepsilon_1\kappa_2 &= 0 \\ g\kappa_2' - g\varepsilon_1(\varepsilon_3 - 1) + \varepsilon_1\varepsilon_3 &= 0 \end{aligned}}$$

$$\downarrow$$

$$\boxed{g^2\phi'' - gN_\ell \sin(\phi) + (g-1)N_\ell^2 \sin\phi \cos(\phi) = 0}$$

Homogeneous solutions

Scalar second order ODE

$$g^2\phi'' = N_\ell \sin(\phi)(g - (g-1)N_\ell \cos(\phi))$$

Elastica

$N_\ell = 0$ $\longrightarrow$

- $\theta(s) = \frac{M_\ell}{g}(s - \ell) + \theta(\ell)$
- $\varphi_1(s) = \varphi_1(0) + \frac{g}{M_\ell}(\cos(\theta) - 1)$
- $\varphi_3(s) = \varphi_3(0) + \frac{g}{M_\ell}\sin(\theta)$

Pure longitudinal force

$\sin(\phi) = 0$ $\longrightarrow$

- $\varphi_1(s) = \varphi_1(0)$
- $\varphi_3(s) = \varphi_3(0) + (1 \pm \frac{N_\ell}{g})s$
- $\theta(s) = \theta(0)$

Homogeneous solutions

Scalar second order ODE

$$g^2 \phi'' = N_\ell \sin(\phi)(g - (g-1)N_\ell \cos(\phi))$$

Elastica

$$\boxed{N_\ell = 0} \longrightarrow$$

- $\theta(s) = \frac{M_\ell}{g}(s-\ell) + \theta(\ell)$
- $\varphi_1(s) = \varphi_1(0) + \frac{g}{M_\ell}(\cos(\theta) - 1)$
- $\varphi_3(s) = \varphi_3(0) + \frac{g}{M_\ell}\sin(\theta)$

Pure longitudinal force

$$\boxed{\sin(\phi) = 0} \longrightarrow$$

- $\varphi_1(s) = \varphi_1(0)$
- $\varphi_3(s) = \varphi_3(0) + (1 \pm \frac{N_\ell}{g})s$
- $\theta(s) = \theta(0)$

Homogeneous solutions

Scalar second order ODE

$$g^2\phi^{''} = N_\ell \sin(\phi)(g-(g-1)N_\ell\cos(\phi))$$

Elastica

$$\boxed{N_\ell \;=\; 0} \longrightarrow$$

- $\theta(s) = \frac{M_\ell}{g}(s-\ell)+\theta(\ell)$
- $\varphi_1(s) = \varphi_1(0) + \frac{g}{M_\ell}(\cos(\theta)-1)$
- $\varphi_3(s) = \varphi_3(0) + \frac{g}{M_\ell}\sin(\theta)$

Pure longitudinal force

$$\boxed{\sin(\phi) \;=\; 0} \longrightarrow$$

- $\varphi_1(s) = \varphi_1(0)$
- $\varphi_3(s) = \varphi_3(0) + (1 \pm \frac{N_\ell}{g})s$
- $\theta(s) = \theta(0)$

Non-homogeneous solutions

Non linear differential equation with respect to ϕ

First integral $$(g\phi')^2 + 2gN_\ell \cos\phi - (g-1)N_\ell^2 \cos^2\phi = \mu \tag{1}$$

where $$\mu = M_\ell^2 + 2gN_\ell \cos(\phi_\ell) - (g-1)N_\ell^2 \cos^2(\phi_\ell)$$

Boundary conditions analyses

$$0 \le N_\ell \le 0.1, \qquad -0.1 \le M_\ell \le 0.1, \qquad 0 \le \phi_\ell \le 2\pi$$

Remarks

(1) is a non-linear scalar first order ODE.

(1) has a unique solution depending on $(N_\ell, M_\ell, \phi_\ell)$.
$\Longrightarrow$ Cauchy initial value problem!

$(N_\ell, \phi_\ell, M_\ell)$ contributes to define

- Initial condition of the 3D system.
- Coefficients of (1).

Smooth change of N_ℓ, ϕ_ℓ or M_ℓ may affect the regularity of the solution.

Non-homogeneous solutions

Non linear differential equation with respect to ϕ

First integral $$(g\phi')^2 + 2gN_\ell \cos\phi - (g-1)N_\ell^2 \cos^2\phi = \mu \qquad (1)$$

where $$\mu = M_\ell^2 + 2gN_\ell \cos(\phi_\ell) - (g-1)N_\ell^2 \cos^2(\phi_\ell)$$

Boundary conditions analyses

$$0 \le N_\ell \le 0.1, \qquad -0.1 \le M_\ell \le 0.1, \qquad 0 \le \phi_\ell \le 2\pi$$

Remarks

(1) is a non-linear scalar first order ODE.

(1) has a unique solution depending on $(N_\ell, M_\ell, \phi_\ell)$.
$\Longrightarrow$ Cauchy initial value problem!

$(N_\ell, \phi_\ell, M_\ell)$ contributes to define

- Initial condition of the 3D system.
- Coefficients of (1).

Smooth change of N_ℓ, ϕ_ℓ or M_ℓ may affect the regularity of the solution.

Non-homogeneous solutions

Non linear differential equation with respect to ϕ

First integral $$(g\phi')^2 + 2gN_\ell \cos\phi - (g-1)N_\ell^2 \cos^2\phi = \mu \qquad (1)$$

where $$\mu = M_\ell^2 + 2gN_\ell \cos(\phi_\ell) - (g-1)N_\ell^2 \cos^2(\phi_\ell)$$

Boundary conditions analyses

$$0 \le N_\ell \le 0.1, \qquad -0.1 \le M_\ell \le 0.1, \qquad 0 \le \phi_\ell \le 2\pi$$

Remarks

(1) is a non-linear scalar first order ODE.

(1) has a unique solution depending on $(N_\ell, M_\ell, \phi_\ell)$.
$\Longrightarrow$ Cauchy initial value problem!

$(N_\ell, \phi_\ell, M_\ell)$ contributes to define

- Initial condition of the 3D system.
- Coefficients of (1).

Smooth change of N_ℓ, ϕ_ℓ or M_ℓ may affect the regularity of the solution.

Non-homogeneous solutions

Non linear differential equation with respect to ϕ

First integral
$$(g\phi')^2 + 2gN_\ell \cos\phi - (g-1)N_\ell^2 \cos^2\phi = \mu \tag{1}$$

where
$$\mu = M_\ell^2 + 2gN_\ell \cos(\phi_\ell) - (g-1)N_\ell^2 \cos^2(\phi_\ell)$$

Boundary conditions analyses

$$0 \le N_\ell \le 0.1, \qquad -0.1 \le M_\ell \le 0.1, \qquad 0 \le \phi_\ell \le 2\pi$$

Remarks

(1) is a non-linear scalar first order ODE.

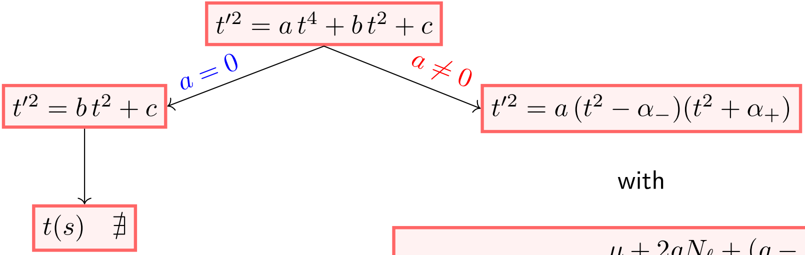
(1) has a unique solution depending on $(N_\ell, M_\ell, \phi_\ell)$.
$\Longrightarrow$ Cauchy initial value problem!

$(N_\ell, \phi_\ell, M_\ell)$ contributes to define

- Initial condition of the 3D system.
- Coefficients of (1).

Smooth change of N_ℓ, ϕ_ℓ or M_ℓ may affect the regularity of the solution.

Change of variable $t(s) = \tan\left(\frac{\phi(s)}{2}\right)$.



$$t'^2 = a\,t^4 + b\,t^2 + c$$

$a = 0$

$$t'^2 = b\,t^2 + c$$

$$t(s) \quad \nexists$$

$a \neq 0$

$$t'^2 = a\,(t^2 - \alpha_-)(t^2 + \alpha_+)$$

with

$$\begin{aligned}
a(\mu, N_\ell) &= \frac{\mu + 2gN_\ell + (g-1)N_\ell^2}{4g^2}\\
\alpha_+(\mu, N_\ell) &= \frac{\frac{g+\sqrt{g^2-(g-1)\mu}}{g-1} - N_\ell}{\frac{g+\sqrt{g^2-(g-1)\mu}}{g-1} + N_\ell}\\
\alpha_-(\mu, N_\ell) &= \frac{N_\ell - \frac{g-\sqrt{g^2-(g-1)\mu}}{g-1}}{N_\ell + \frac{g-\sqrt{g^2-(g-1)\mu}}{g-1}}
\end{aligned}$$

ODE solution

$$t(s) = \pm\sqrt{\alpha_+}\ \mathrm{cs}\left(\sqrt{a\alpha_+}(s+s_0)\,\middle|\,\frac{\alpha_+ + \alpha_-}{\alpha_+}\right)$$

Change of variable $t(s) = \tan\left(\frac{\phi(s)}{2}\right)$.

$$\begin{tikzcd}
 & \boxed{t'^2 = a\,t^4 + b\,t^2 + c} \arrow[dl, "a=0"'] \arrow[dr, "a \neq 0"] & \\
\boxed{t'^2 = b\,t^2 + c} \arrow[d] & & \boxed{t'^2 = a\,(t^2 - \alpha_-)(t^2 + \alpha_+)} \\
\boxed{t(s) \quad \nexists} & &
\end{tikzcd}$$

with

$$\boxed{\begin{aligned}
a(\mu, N_\ell) &= \frac{\mu + 2gN_\ell + (g-1)N_\ell^2}{4g^2} \\
\alpha_+(\mu, N_\ell) &= \frac{\frac{g+\sqrt{g^2-(g-1)\mu}}{g-1} - N_\ell}{\frac{g+\sqrt{g^2-(g-1)\mu}}{g-1} + N_\ell} \\
\alpha_-(\mu, N_\ell) &= \frac{N_\ell - \frac{g-\sqrt{g^2-(g-1)\mu}}{g-1}}{N_\ell + \frac{g-\sqrt{g^2-(g-1)\mu}}{g-1}}
\end{aligned}}$$

ODE solution

$$t(s) = \pm\sqrt{\alpha_+}\ \mathrm{cs}\left(\sqrt{a\alpha_+}(s+s_0)\,\Big|\,\frac{\alpha_+ + \alpha_-}{\alpha_+}\right)$$

Recovering physical quantities

Forces, moment and rotation

$$N_1(s) = N_\ell \frac{2\sqrt{\alpha_+}\,\mathrm{cs}(\zeta\,|\,m)}{1+\alpha_+\mathrm{cs}^2(\zeta\,|\,m)} \qquad N_3(s) = N_\ell \frac{1-\alpha_+\mathrm{cs}^2(\zeta\,|\,m)}{1+\alpha_+\mathrm{cs}^2(\zeta\,|\,m)}$$

$$M_2(s) = \sqrt{ag}\,\frac{2\alpha_+\,\mathrm{ns}(\zeta\,|\,m)\mathrm{ds}(\zeta\,|\,m)}{1+\alpha_+\mathrm{cs}^2(\zeta\,|\,m)} \qquad \theta(s) = \hat{\phi} - 2\arctan\Big(t(s)\Big)$$

$$\boldsymbol{M}' + \boldsymbol{\varepsilon}\times\boldsymbol{N} = 0 \qquad \Longrightarrow \qquad \boldsymbol{M}(s) - \boldsymbol{M}(0) + (\boldsymbol{\varphi}(s)-\boldsymbol{\varphi}(0))\times\boldsymbol{N}_\ell = 0$$

Placement solutions

$$\boldsymbol{\varphi}(s) = \varphi_t(s)\boldsymbol{e_t} + \varphi_n(s)\boldsymbol{e_n} \qquad \boldsymbol{e_n} = \tfrac{\boldsymbol{N}_\ell}{\|\boldsymbol{N}_\ell\|} \qquad \boldsymbol{e_t} = \boldsymbol{e_y}\times\boldsymbol{e_n}.$$

$$\varphi_t(s) = \varphi_t(0) + \frac{M_2(s)-M_2(0)}{N_\ell}$$

$$\varphi_n(s) = \varphi_n(0) + \int_0^s N_\ell\left(\frac{2t}{1+t^2}\right)^2 + \frac{N_\ell}{g}\left(\frac{1-t^2}{1+t^2}\right)^2 + \frac{1-t^2}{1+t^2}\;ds$$

Recovering physical quantities

Forces, moment and rotation

$$N_1(s) = N_\ell \frac{2\sqrt{\alpha_+}\mathrm{cs}(\zeta\,|\,m)}{1+\alpha_+\mathrm{cs}^2(\zeta\,|\,m)} \qquad N_3(s) = N_\ell \frac{1-\alpha_+\mathrm{cs}^2(\zeta\,|\,m)}{1+\alpha_+\mathrm{cs}^2(\zeta\,|\,m)}$$

$$M_2(s) = \sqrt{a}g \frac{2\alpha_+\,\mathrm{ns}(\zeta\,|\,m)\mathrm{ds}(\zeta\,|\,m)}{1+\alpha_+\mathrm{cs}^2(\zeta\,|\,m)} \qquad \theta(s) = \hat{\phi} - 2\arctan\big(t(s)\big)$$

$$\boldsymbol{M}' + \boldsymbol{\varepsilon}\times\boldsymbol{N} = 0 \qquad \Longrightarrow \qquad \boldsymbol{M}(s) - \boldsymbol{M}(0) + (\boldsymbol{\varphi}(s)-\boldsymbol{\varphi}(0))\times\boldsymbol{N}_\ell = 0$$

Placement solutions

$$\boldsymbol{\varphi}(s) = \varphi_t(s)\boldsymbol{e_t} + \varphi_n(s)\boldsymbol{e_n} \qquad \boldsymbol{e_n} = \frac{\boldsymbol{N}_\ell}{\|\boldsymbol{N}_\ell\|} \qquad \boldsymbol{e_t} = \boldsymbol{e_y}\times\boldsymbol{e_n}.$$

$$\varphi_t(s) = \varphi_t(0) + \frac{M_2(s)-M_2(0)}{N_\ell}$$

$$\varphi_n(s) = \varphi_n(0) + \int_0^s N_\ell\left(\frac{2t}{1+t^2}\right)^2 + \frac{N_\ell}{g}\left(\frac{1-t^2}{1+t^2}\right)^2 + \frac{1-t^2}{1+t^2}\;ds$$

Recovering physical quantities

Forces, moment and rotation

$$N_1(s) = N_\ell \frac{2\sqrt{\alpha_+}\mathrm{cs}(\zeta\,|\,m)}{1+\alpha_+\mathrm{cs}^2(\zeta\,|\,m)} \qquad N_3(s) = N_\ell \frac{1-\alpha_+\mathrm{cs}^2(\zeta\,|\,m)}{1+\alpha_+\mathrm{cs}^2(\zeta\,|\,m)}$$

$$M_2(s) = \sqrt{a}g \frac{2\alpha_+\,\mathrm{ns}(\zeta\,|\,m)\mathrm{ds}(\zeta\,|\,m)}{1+\alpha_+\mathrm{cs}^2(\zeta\,|\,m)} \qquad \theta(s) = \hat{\phi} - 2\arctan\Big(t(s)\Big)$$

$$\boldsymbol{M}' + \boldsymbol{\varepsilon}\times\boldsymbol{N} = 0 \qquad \Longrightarrow \qquad \boldsymbol{M}(s) - \boldsymbol{M}(0) + (\boldsymbol{\varphi}(s) - \boldsymbol{\varphi}(0)) \times \boldsymbol{N}_\ell = 0$$

Placement solutions

$$\boldsymbol{\varphi}(s) = \varphi_t(s)\boldsymbol{e_t} + \varphi_n(s)\boldsymbol{e_n} \qquad \boldsymbol{e_n} = \frac{\boldsymbol{N}_\ell}{\|\boldsymbol{N}_\ell\|} \qquad \boldsymbol{e_t} = \boldsymbol{e_y}\times\boldsymbol{e_n}.$$

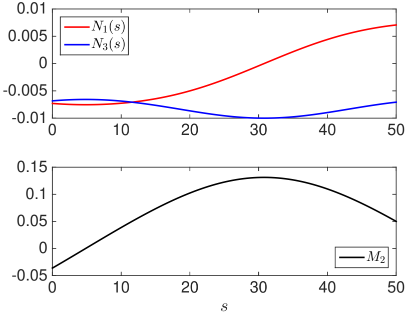
$$\varphi_t(s) = \varphi_t(0) + \frac{M_2(s) - M_2(0)}{N_\ell}$$

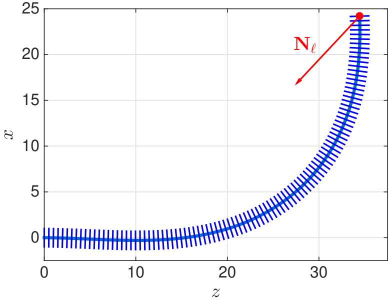
$$\varphi_n(s) = \varphi_n(0) + \int_0^s N_\ell\left(\frac{2t}{1+t^2}\right)^2 + \frac{N_\ell}{g}\left(\frac{1-t^2}{1+t^2}\right)^2 + \frac{1-t^2}{1+t^2}\;ds$$

Recovering physical quantities

Illustrating example with $g = 5/2$ and $\ell = 50$

$\phi_\ell = 3\pi/4$ $N_\ell = 0.01$ $M_\ell = 0.05$ Cauchy BC
$\varphi(0) = 0$ $\theta(0) = 0$ Kinematical BC

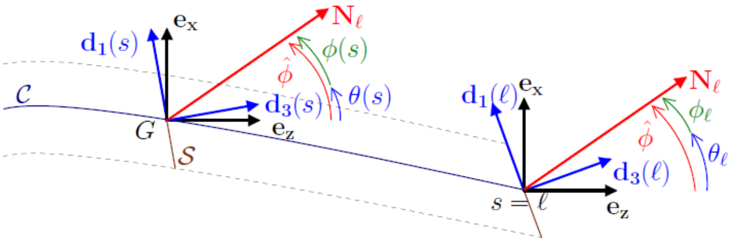


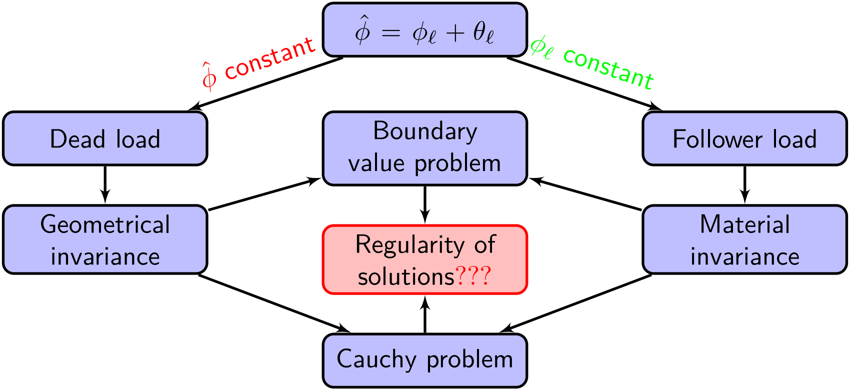


Outline

1. Problem statement
2. Jacobian Elliptic functions
3. Physical Discussion
4. Conclusion

Load control





$\hat{\phi} = \phi_\ell + \theta_\ell$

$\hat{\phi}$ constant

ϕ_ℓ constant

Dead load

Boundary value problem

Follower load

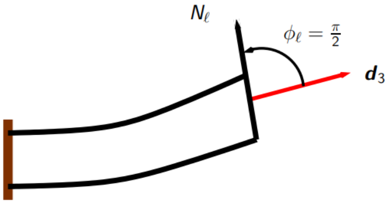
Geometrical invariance

Regularity of solutions???

Material invariance

Cauchy problem

Pure-shear load



Cauchy problem with

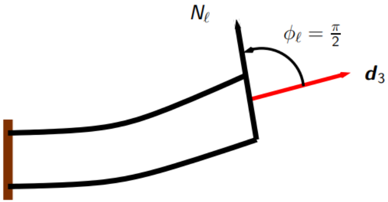
$$N_\ell \neq 0 \qquad \phi_\ell = \frac{\pi}{2} \qquad M_\ell = 0$$

Asymptotic analysis
Leading terms with respect to N_ℓ

Pure shear placement

$$\begin{aligned} \varphi_t(s) &\simeq \varphi_t(0) + \sqrt{\frac{2g}{N_\ell}}\ \mathrm{cn}\Big(\sqrt{\frac{N_\ell}{g}}(s+s_0)\,\Big|\,\frac{1}{2}\Big) \\ \varphi_n(s) &\simeq \varphi_n(0) + s - 2\sqrt{\frac{g}{N_\ell}}\mathcal{E}\Big(\sqrt{\frac{N_\ell}{g}}(s+s_0)\Big|\frac{1}{2}\Big) \end{aligned}$$

Pure-shear load



Cauchy problem with

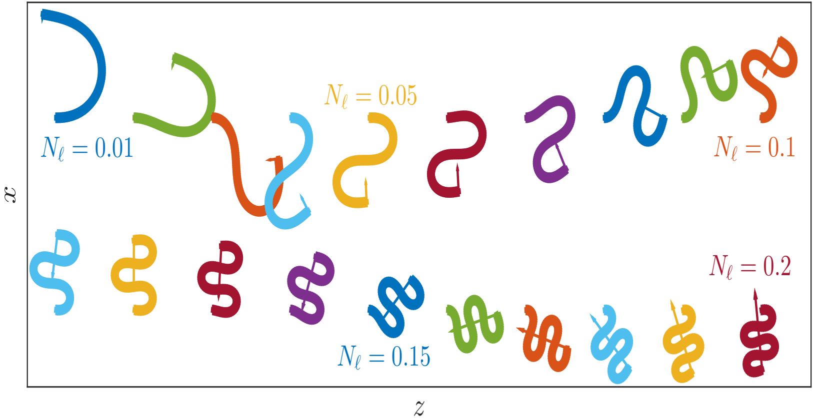
$$N_\ell \neq 0 \qquad \phi_\ell = \frac{\pi}{2} \qquad M_\ell = 0$$

Asymptotic analysis
Leading terms with respect to N_ℓ

Pure shear placement

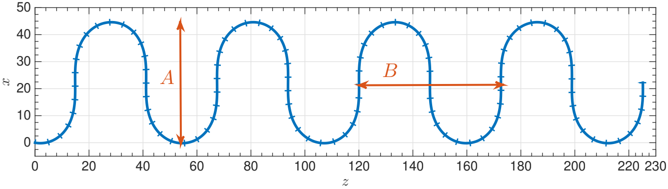
$$\begin{aligned} \varphi_t(s) &\simeq \varphi_t(0) + \sqrt{\frac{2g}{N_\ell}}\,\mathrm{cn}\Big(\sqrt{\frac{N_\ell}{g}}(s+s_0)\,\Big|\,\frac{1}{2}\Big) \\ \varphi_n(s) &\simeq \varphi_n(0) + s - 2\sqrt{\frac{g}{N_\ell}}\mathcal{E}\Big(\sqrt{\frac{N_\ell}{g}(s+s_0)}\Big|\frac{1}{2}\Big) \end{aligned}$$

Quasi static increase of N_ℓ: Follower load



Periodic pattern

Quantitative and qualitative analyses



Periodic pattern

- Material period $P = 4\sqrt{\frac{g}{N_\ell}}K(\frac{1}{2})$
- Spatial period $B = \frac{2\pi}{K(\frac{1}{2})}\sqrt{\frac{g}{N_\ell}}$
- Size $A = 2\sqrt{\frac{2g}{N_\ell}}$

$$\boxed{\frac{B}{A} = \frac{\pi}{\sqrt{2}K(\frac{1}{2})} \sim 1.2}$$

independent of N_ℓ, g and ℓ

.

Control of a kinematical boundary

Problem statement

Pure dead-load

$$\boldsymbol{N_\ell} = N_\ell \boldsymbol{e_z} \qquad M_\ell = 0$$

$\theta(0)$ varies according to a command.

Boundary value problem.

$$\boldsymbol{N_\ell} = N_\ell \boldsymbol{e_z} \qquad M_\ell = 0 \qquad \theta(0) := \theta_0$$

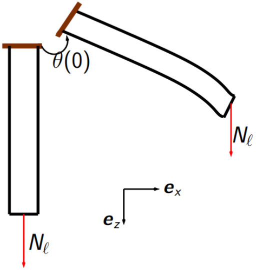
Cauchy value problem.

$$\boldsymbol{N_\ell} = N_\ell \boldsymbol{e_z} \qquad M_\ell = 0 \qquad \phi_\ell$$

What is the value of θ_0 with respect to ϕ_ℓ for a prescribed dead load ??.

$$\theta_0 = \pm 2 \arctan\left(\frac{\sqrt{\alpha_-}}{\mathrm{cn}(\sqrt{a(\alpha_- + \alpha_+)}\ell \,|\, \frac{\alpha_+}{\alpha_+ + \alpha_-})}\right)$$

for a prescribed $\phi_\ell \in [0, 2\pi]$



Control of a kinematical boundary

Problem statement

Pure dead-load

$$\boldsymbol{N_\ell} = N_\ell \boldsymbol{e_z} \qquad M_\ell = 0$$

$\theta(0)$ varies according to a command.

Boundary value problem.

$$\boldsymbol{N_\ell} = N_\ell \boldsymbol{e_z} \qquad M_\ell = 0 \qquad \theta(0) := \theta_0$$

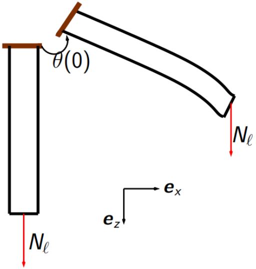
Cauchy value problem.

$$\boldsymbol{N_\ell} = N_\ell \boldsymbol{e_z} \qquad M_\ell = 0 \qquad \phi_\ell$$

What is the value of θ_0 with respect to ϕ_ℓ for a prescribed dead load ??.

$$\theta_0 = \pm 2 \arctan \left(\frac{\sqrt{\alpha_-}}{\mathrm{cn}(\sqrt{a(\alpha_- + \alpha_+)}\ell \mid \frac{\alpha_+}{\alpha_+ + \alpha_-})} \right)$$

for a prescribed $\phi_\ell \in [0, 2\pi]$



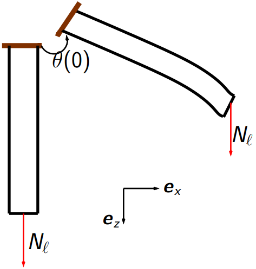
Control of a kinematical boundary

Problem statement

Pure dead-load

$$\boldsymbol{N_\ell} = N_\ell \boldsymbol{e_z} \qquad M_\ell = 0$$

$\theta(0)$ varies according to a command.



Boundary value problem.

$$\boldsymbol{N_\ell} = N_\ell \boldsymbol{e_z} \qquad M_\ell = 0 \qquad \theta(0) := \theta_0$$

Cauchy value problem.

$$\boldsymbol{N_\ell} = N_\ell \boldsymbol{e_z} \qquad M_\ell = 0 \qquad \phi_\ell$$

What is the value of θ_0 with respect to ϕ_ℓ for a prescribed dead load ??.

$$\theta_0 = \pm 2 \arctan \left(\frac{\sqrt{\alpha_-}}{\mathrm{cn}(\sqrt{a(\alpha_- + \alpha_+)}\ell \mid \frac{\alpha_+}{\alpha_+ + \alpha_-})} \right)$$

for a prescribed $\phi_\ell \in [0, 2\pi]$

Control of a kinematical boundary

Problem statement

Pure dead-load

$$\boldsymbol{N_\ell} = N_\ell \boldsymbol{e_z} \qquad M_\ell = 0$$

$\theta(0)$ varies according to a command.

Boundary value problem.

$$\boldsymbol{N_\ell} = N_\ell \boldsymbol{e_z} \qquad M_\ell = 0 \qquad \theta(0) := \theta_0$$

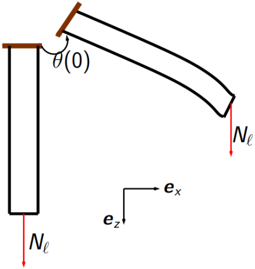
Cauchy value problem.

$$\boldsymbol{N_\ell} = N_\ell \boldsymbol{e_z} \qquad M_\ell = 0 \qquad \phi_\ell$$

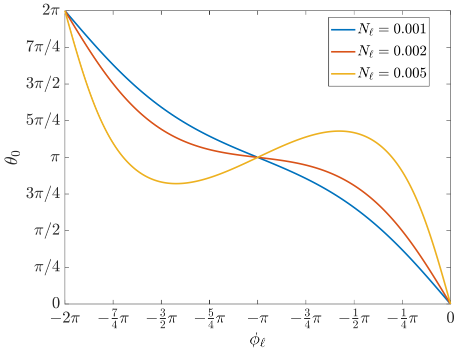
What is the value of θ_0 with respect to ϕ_ℓ for a prescribed dead load ??.

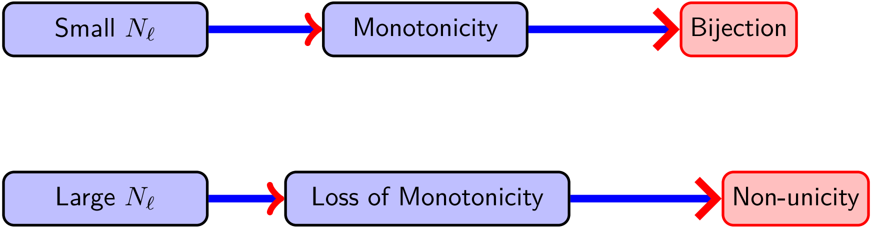
$$\theta_0 = \pm 2 \arctan \left(\frac{\sqrt{\alpha_-}}{\mathrm{cn}(\sqrt{a(\alpha_- + \alpha_+)}\ell \mid \frac{\alpha_+}{\alpha_+ + \alpha_-})} \right)$$

for a prescribed $\phi_\ell \in [0, 2\pi]$



Analysis of $\theta_0(\phi_\ell)$



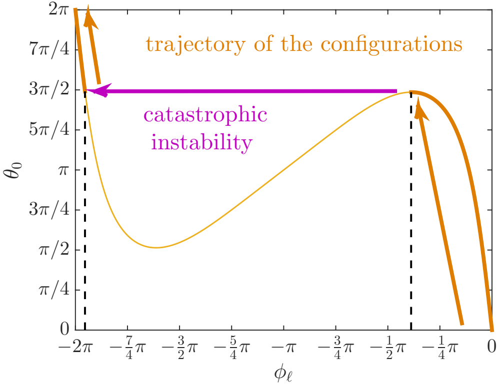


Small N_ℓ → Monotonicity → Bijection

Large N_ℓ → Loss of Monotonicity → Non-unicity

Catastrophic instabilities

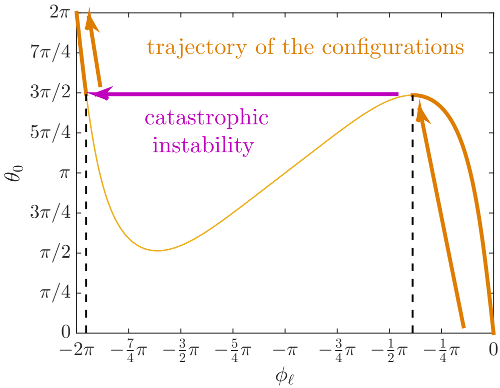
Command: "θ_0 varies from 0 to 2π"

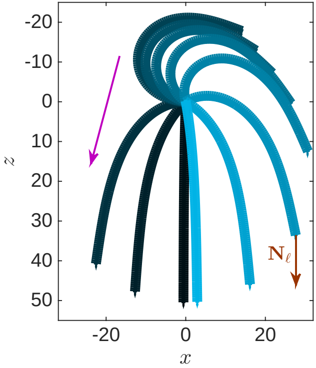


- Smooth variation of ϕ_ℓ for small θ_0.
- Large jump of ϕ_ℓ as θ_0 increases in order to maintain equilibrium position.

Catastrophic instabilities

Command: "θ_0 varies from 0 to 2π"





- Smooth variation of ϕ_ℓ for small θ_0.
- Large jump of ϕ_ℓ as θ_0 increases in order to maintain equilibrium position.
- Brutal change of the configuration of the beam.
- Catastrophic instability

Outline

1. Problem statement
2. Jacobian Elliptic functions
3. Physical Discussion
4. Conclusion

Conclusion

- Analytical study of a large transformation of a Timoshenko beam subjected to end forces and moment with linear constitutive law and large transformation.
- Pure dimensionless approach.

- Cauchy problem which depends on loads at one extremity.
- Solutions in terms of Jacobian elliptic functions.

- Pure shear load (Cauchy problem): Explicit solutions.
- Dead load (Boundary value problem): Catastrophe.

Conclusion

- Analytical study of a large transformation of a Timoshenko beam subjected to end forces and moment with linear constitutive law and large transformation.
- Pure dimensionless approach.

- Cauchy problem which depends on loads at one extremity.
- Solutions in terms of Jacobian elliptic functions.

- Pure shear load (Cauchy problem): Explicit solutions.
- Dead load (Boundary value problem): Catastrophe.

Conclusion

- Analytical study of a large transformation of a Timoshenko beam subjected to end forces and moment with linear constitutive law and large transformation.
- Pure dimensionless approach.

- Cauchy problem which depends on loads at one extremity.
- Solutions in terms of Jacobian elliptic functions.

- Pure shear load (Cauchy problem): Explicit solutions.
- Dead load (Boundary value problem): Catastrophe.